

Location of the zeros of certain parametric families of functions of generalized Fresnel integral type

J. Lobo-Segura* M. A. Villalobos-Arias†

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Abstract

In this paper, two parametric families of functions, the so-called Complementary Fresnel Integral and the Lommel type, which are of generalized Fresnel integral type, are considered. We review the problems of existence and uniqueness of their zeros in certain determined intervals, called location intervals, which improve previous results of other authors. For the approximation error obtained, bounds, monotonicity as well as the asymptotic behavior are analyzed. The study uses results from the theory of fixed point of real functions, introducing the concept of “fixed point sequential problem” (FPSP) and the properties of certain special functions.

1 Introduction

It is considered in this work several families of functions of real variable depending on a real parameter: the family called *Complementary Fresnel Integral* (CFI), and a certain type of *functions of the Lommel type* whose precise definitions will be given later. If $f_\alpha, \alpha \in J$, for J in an interval of \mathbb{R} , is one of these families, the associated zeros are studied finding first numerable families of disjoint intervals $I_{n,\alpha}, n \in \mathbb{N}$ (*location intervals*), each interval containing one and only one zero of the function f_α , and each zero of f_α belonging to only one of them. Then, if $a_{n,\alpha}, b_{n,\alpha}$ define the lower end (respectively upper) of $I_{n,\alpha}$ and $z_{n,\alpha}$ is the only zero of f_α in that interval we look for bounds of the successions of errors $|a_{n,\alpha} - z_{n,\alpha}|, |b_{n,\alpha} - z_{n,\alpha}|, n \in \mathbb{N}, \alpha \in J$ been fixed. We establish the monotonicity of these sequences of errors and find asymptotic equivalents of them, whenever they converge to zero.

Our study constitutes a revision and at the same time an extension of the work done on the CFI family in [10] and those of the Lommel family carried out by S. Koumandos and M. Lamprecht in [7]. To carry out a joint study of the two mentioned families, we introduce in a preliminary section a basic theory of the so-called *fixed point sequential problem* (FPSP), where we enunciate the existence and uniqueness of fixed-point of the real variable. In other sections we study several families of auxiliary parametric functions defined by integrals, in several aspects: analytical properties with respect to the variable, basic identities and inequalities. All this provides a common basis for a unified approach in the study of the zeros of these families.

*CIMPA, Universidad de Costa Rica, San Pedro de Montes de Oca, San José, Costa Rica.
jaimelobosegura@hotmail.com

†same address, mario.villalobos@ucr.ac.cr

Background and basic definitions

The CFI family is the pair c_α, s_α of functions defined in \mathbb{R}_+ by:

$$c_\alpha(t) + is_\alpha(t) = \int_{(t,\infty)} t^{-\alpha} e^{it} dt$$

for $\alpha > 0$. The Lommel family (LF) considered is the one denoted by $s_{\mu-1/2,1/2}(t) = t^{-1/2}F_\mu(t)$, where:

$$F_\mu(t) = \int_{(0,t)} u^{\mu-1} \sin(t-u) du, \quad t \in \mathbb{R}_+, \quad \mu \in (0,1).$$

For each μ the zeros of this function coincide then with those of F_μ .

Regarding the CFI family, we review the following results, contained in Theorem 2.1 of [10]:

Quote 1.1. : *The intervals $(n\pi/2, (n+1)\pi/2)$, n even (respectively n odd) are location intervals of the zeros of c_α , (resp. of s_α).*

Let $z_{n,\alpha}$ be zero in the interval $(n\pi/2, (n+1)\pi/2)$, defined in the previous quote, then

Quote 1.2. : *The sequence $n \rightarrow z_{n,\alpha} - n\pi/2$ is strictly decreasing and converging to zero with $z_{n,\alpha} - n\pi/2 \sim \frac{2\alpha}{n\pi}$.*

Quote 1.3. : *The fixed point $\mathcal{T}_{n,\alpha}$ of $z \rightarrow n\pi/2 + \frac{\alpha}{z}$ in $(n\pi/2, (n+1)\pi/2)$ exists for n large enough, and it is an upper bound of the zero in this interval with $\mathcal{T}_{n,\alpha} - z_{n,\alpha} = o(\frac{1}{n})$, $n \rightarrow \infty$.*

In the previous results we used a multiplicative decomposition lemma, which will be used later and which is stated as follows (the proof can be found in lemma 3.1 in [10]):

Lemma 1.4. *For $\alpha > 0$, $t > 0$ we have:*

$$c_\alpha(t) + is_\alpha(t) = e^{it}(g_\alpha(t) + if_\alpha(t)),$$

where

$$g_\alpha(t) = \frac{1}{\Gamma(\alpha)} \int_0^\infty \frac{\exp(-ty)y^\alpha}{1+y^2} dy, \quad f_\alpha(t) = \frac{1}{\Gamma(\alpha)} \int_0^\infty \frac{\exp(-ty)y^{\alpha-1}}{1+y^2} dy. \quad (1.1)$$

For the Lommel family we have the next results (see [7]):

Quote 1.5. (*theorem 1.1 in [7]*): *the intervals*

$$I_{2n-1}(\mu) = ((2n-1+\mu/2)\pi, (2n-1+\mu)\pi], \quad I_{2n}(\mu) = (2n\pi, (2n+\mu/2)\pi)$$

are location intervals of the zeros of F_μ ,

Quote 1.6. (*theorem 1.2 in [7]*): *If $z_{n,\mu}$ is zero defined in 1.5 belonging to $I_n(\mu)$, the subsequence of $(z_{n,\mu} - n\pi - \mu/2\pi)_{n \in \mathbb{N}}$ associated to even n (resp. odd n) converges to 0 and is strictly increasing (resp converges to zero and is strictly decreasing),*

i.e. $[z_{2k,\mu} - (2k)\pi - \mu/2\pi] \nearrow 0$ and $[z_{2k+1,\mu} - (2k+1)\pi - \mu/2\pi] \searrow 0$

Plan of the article

In the sections 2, 3 of preliminaries we expose the results of fixed point problems and auxiliary functions respectively. The concept of FPSP exposed in the section 2 is original and allows to translate the concept of the zero of a function into a concept of fixed point of a function, as it is explained in the respective sections. Some of the functions addressed in the section 3 appear more or less explicitly in the literature that deals with the problem of the asymptotic development of functions defined by integrals. However, the analytical study undertaken here seems original and has its own interest. The theory of probability is used (theory of gamma distribution for example) and in a certain way it extends the results of L. Gordon on the probabilistic approach of the gamma function (see [5]).

In sections 4, 5 we develop the theory about zeros properly. The property of zeroes of the CFI family is translated in fixed point problem by a geometric argument that differs from the one used in [10] and that seems more natural. The theorem 4.1 also offers an error bound for $\mathcal{T}_{n,\alpha}$ whereby a problem raised in [10] about the asymptotic behavior of this error is solved (see section 6 in [10]).

The equivalence of zeros for the Lommel family is more direct by the decomposition lemma 1.4, and leads to two different FPSP that must be treated separately. We deduce from this translation not only the results of the quote 1.5 and 1.6, but also new location intervals (theorem 5.5) and their respective errors (theorem 5.6).

2 The fixed point sequential problem

For the study of the zeros proposed in the Introduction, we develop here a topic related to fixed point problems of real variable functions, whose interest will be revealed in later sections. We consider a general situation as the following: let $\mathcal{I}_n = [\mathbf{a}_n, \mathbf{b}_n]_{n \in \mathbb{N}}$ a sequence of intervals of \mathbb{R}_+ which is increasing: $\mathbf{b}_n < \mathbf{a}_{n+1}$ for all n , and such that the sequence $(\mathbf{a}_n)_{n \in \mathbb{N}}$ diverges to infinity.

Let \mathcal{U} be an interval of $+\infty$, J an open interval of \mathbb{R}_+ and $G_\alpha : \mathcal{U} \rightarrow \mathbb{R}_+$, function that depends on the parameter $\alpha \in J$.

It defines the sequence of functions:

$$G_{n,\alpha}(t) \equiv \mathbf{a}_n + G_\alpha(t), t \in \mathcal{I}_n.$$

Sufficient conditions are sought to guarantee the existence of a single fixed point $g_{n,\alpha}$ in each \mathcal{I}_n for each $G_{n,\alpha}(\cdot)$, as well as the most relevant properties of the sequence $(g_{n,\alpha})_{n \in \mathbb{N}}$ under different conditions. We call this “*fixed point sequential problem*” and, that we abbreviated by FPSP. We say that an FPSP has “*unique solution*” if for all n there is a single fixed point of $G_{n,\alpha}$ on \mathcal{I}_n . We first consider the particular problem of fixed points for functions defined in a single bounded interval of the real line. Thereafter, I is a closed and bounded interval $[a, b]$ of the real line.

Proposition 2.1. (*existence and uniqueness*) *Given a continuous function $G : [a, b] \rightarrow (a, b)$. Then G has a single fixed point in I in the following cases:*

- case 1): G is strictly decreasing
- case 2): G is strictly concave or convex in all I .

Proof

The hypotheses about the domain and the codomain imply that the function $Id - G$ changes sign at the ends of I and joint to the continuity property the existence of at least one fixed point in I is guaranteed in all cases.

In case 1) the uniqueness is derived from the fact that $Id - G$ is strictly increasing.

In case 2) the conditions ensure that $Id - G$ is either strictly monotonic in I or unimodal in I with global extreme inside I ; in both cases there can only be one zero of that function in I .

QED

Proposition 2.2. *(comparison of fixed points): in a given interval $I = [a, b]$, let F, G, H continuous functions defined in I and with values in $int(I)$, that meet: $F < G < H$ and that have a single fixed point in I , denoted f, g, h respectively. So:*

- $F(h) < g < h$ if F, G, H satisfy the conditions of case 1) (of proposition 2.1).
- $g < h$ if G, H satisfy the conditions of case 2) (of proposition 2.1).

Proof

In case 1) $g < h$ must be fulfilled because otherwise, using the strict decrease of G , we would have $g = G(g) \leq G(h) < H(h) = h$, that is contradictory. The relation $g < h$ implies that $F(h) < F(g) < G(g) = g$, the second relation in this case.

In case 2) it is easy to verify that $x > g \Leftrightarrow x - G(x) > 0$, $x > h \Leftrightarrow x - H(x) > 0$. And since $G(h) < H(h) = h$ it is concluded in this case. **QED**

We will consider the FPSP defined by $(\mathfrak{I}_n)_{n \in \mathbb{N}}$, an increasing sequence of intervals \mathcal{U} , with $\mathfrak{I}_n = [\mathbf{a}_n, \mathbf{b}_n]$, and by a continuous parametric function $G_\alpha : \mathcal{U} \rightarrow \mathbb{R}_+$.

In certain cases we can assure that this problem has a unique solution, for example:

Corolary 2.3. *Let l_n the lengths of the intervals \mathfrak{I}_n that satisfy $L = \inf_{n \in \mathbb{N}} \{l_n\} > 0$. We assume that the codomain of G_α is included in $(0, L)$. If G_α is strictly decreasing or it is concave or convex in all its domain, the associated FPSP has a unique solution.*

Proof

It is enough to observe that under the conditions of the proposition all the functions $G_{n,\alpha}$ defined in each \mathfrak{I}_n have codomain within $int(\mathfrak{I}_n)$. On the other hand, the other conditions imply that each $G_{n,\alpha}$ satisfies case 1) or case 2) of the proposition 2.1. The result is then derived from 2.1.

QED

It will be assumed in the rest of this subsection that the FPSP problem has a unique solution: for all n there exists a real only $g_{n,\alpha}$ which is the only fixed point of $G_{n,\alpha}$ in \mathfrak{I}_n . Since the intervals \mathfrak{I}_n “grow”, meaning $\sup \mathfrak{I}_n < \inf \mathfrak{I}_{n+1}$, the sequence $(g_{n,\alpha})_{n \in \mathbb{N}}$ is strictly increasing and diverges to infinity. We will suppose further that the intervals \mathfrak{I}_n are of constant length $c > 0$ and that the sequence $(\mathbf{a}_n)_{n \in \mathbb{N}}$ is arithmetic of common difference $A > 0$ (arithmetic conditions).

Also consider the condition that we call

$$\text{LIM} : G_\alpha \text{ is strictly decreasing in all } \mathcal{U} \text{ and converges to } 0 \text{ in } \infty \text{ for all } \alpha \in J.$$

We define a special class of functions:

Definition 2.4. A function $f : \mathbb{R}^+ \rightarrow \mathbb{R}$ satisfies condition @ if there are real K, β such that:

$$f(t) \sim Kt^{-\beta} \quad (t \rightarrow \infty),$$

" $\cdot \sim \cdot (t \rightarrow \infty)$ " denotes the relation of asymptotic equivalence in infinity.

The subindex α is omitted from now on to simplify the notation.

The following result follows from the definition of fixed point itself:

Proposition 2.5. (*monotony of the distance to extremes*): Let a FPSP of unique solution with arithmetic conditions, of constants A, c .

Then it holds that:

- If G satisfies condition LIM the sequence $(g_n - \mathbf{a}_n)_{n \in \mathbb{N}}$ converges to 0 and is strictly decreasing and
- If $c - G$ satisfies the condition LIM the sequence $(g_n - \mathbf{b}_n)_{n \in \mathbb{N}}$ converges to 0 and is strictly increasing.

If in the first case G meets the property @ then $g_n - \mathbf{a}_n \sim G(\mathbf{a}_n), n \rightarrow \infty$ and if in the second case it is met by $G - c$ then $g_n - \mathbf{b}_n \sim (G(\mathbf{a}_n) - c), n \rightarrow \infty$.

Proof

Writing $g_n - \mathbf{a}_n = G(g_n)$ in the first case and $g_n - \mathbf{b}_n = G(g_n) - c$ in the second and taking into account that $\lim_{n \rightarrow \infty} g_n = \infty$ and that $(g_n)_{n \in \mathbb{N}}$ is strictly increasing, the conclusions follow directly from the conditions of the LIM condition.

The proof of the second part is almost immediate from the relation $g_n = \mathbf{a}_n + G(g_n)$ and in the first case that property @ implies that $G(\mathbf{a}_n) \sim G(\mathbf{g}_n), n \rightarrow \infty$, given that $\mathbf{a}_n \sim g_n, n \rightarrow \infty$ by the arithmetic conditions. Analogously the second case is dealt with. **QED**

3 Auxiliary functions and gamma distribution

We study here the parametric functions f_α, g_α , introduced in Lemma 1.4 of introduction. To do this, we introduce certain special functions denoted J_α , whose definition is given in terms of the gamma probability distribution of the parameter α , denoted by $(\gamma_\alpha, \alpha > 0)$ and defined by

$$\gamma_\alpha(dx) = \frac{1}{\Gamma(\alpha)} e^{-x} x^{\alpha-1} dx.$$

Remember also the bi-parametric gamma family $(\gamma_{\alpha,t}, \alpha, t > 0)$ that is defined by:

$$\gamma_{\alpha,t}(dx) = \frac{1}{\Gamma(\alpha)} t^\alpha e^{-tx} x^{\alpha-1} dx.$$

por lo tanto $\gamma_\alpha = \gamma_{\alpha,1}$.

The parametric function:

$$J_\alpha(t) \equiv E_\alpha \left(\frac{1}{1 + (tX)^2} \right) = \int \frac{1}{1 + (tx)^2} d\gamma_\alpha(x),$$

is well defined for all positive t , E_α denoting expectation operator with respect to the gamma law of parameter α .

Note that the functions f_α, g_α defined in the introduction can be redefined as:

$$f_\alpha(t) = t^{-\alpha} J_\alpha(t^{-1}), \quad g_\alpha(t) = \alpha f_{\alpha+1}(t), \quad \text{donde } t \in \mathbb{R}_+^*.$$

This notation is the same one that is used in [13].

We define for a $\alpha \in \mathbb{R}_+$: $(\alpha)_2 \equiv \alpha(\alpha+1)$ [in general $(\alpha)_n \equiv (\alpha)(\alpha+1)\dots(\alpha+n-1)$, Pochhammer symbol].

We can find the asymptotic development of order 2 of J_α at 0:

$$J_\alpha(x) = 1 - (\alpha)_2 x^2 + o(x^2) \tag{3.1}$$

from the classical theory of asymptotic developments for parametric integrals (see Watson's lemma, for example in section 4.1 of [3]).

We study some properties of variable functions t , $J_\alpha(t), f_\alpha(t)$, with t positive real, and α positive positive:

- For fixed α , the function

$$t \rightarrow J_\alpha(t) \text{ is decreasing in } \mathbb{R}_+ \text{ and range } (0, 1) . \tag{3.2}$$

This is verified, as the quantity $J_\alpha(t)$ being the expectation with respect to γ_α of the function $h(x) = (1 + (tx)^2)^{-1}$, it suffices to use the fact that h is bounded and decreasing with respect to t , convergent to the constant function 1 when $t \rightarrow 0$, and to the constant 0 when $t \rightarrow \infty$ and then we use the classic convergence theorems.

- For all $\alpha > 0$ the function

$$t \rightarrow \frac{g_\alpha}{f_\alpha}(t) \text{ is strictly decreasing} \tag{3.3}$$

This is proved in lemma 3.4 of [10].

- We have:

$$\frac{g_\alpha}{f_\alpha}(t) < \frac{\alpha}{t} \quad \text{y} \quad \frac{t^{-\alpha}}{1 + (\alpha)_2 t^{-2}} < f_\alpha(t) < t^{-\alpha}. \tag{3.4}$$

The first inequality was demonstrated in Lemma 3.4 of [10], using Harris's inequality (see [9] Theorem 5.13, chap 4). For the second one we can use Jensen's inequality by writing

$$\frac{1}{1 + (tx)^2} = h(1 + (tx)^2), \quad \text{where } h(x) = 1/x,$$

using the convexity of h and then applying the formula for the second moment of the gamma distribution, that is $(\alpha)_2$.

- The function f_α is derivable in its domain, with

$$f'_\alpha(t) = -g_\alpha(t), \quad g'_\alpha(t) = f_\alpha(t) - t^{-\alpha}, \tag{3.5}$$

It is enough to use the Lebesgue theorem of derivation of parametric integrals, observing that integrand functions have derivatives with respect to t , which are integrable in \mathbb{R}_+ .

- We have:

$$f_\alpha(t) \sim t^{-\alpha}, t \rightarrow \infty. \quad (3.6)$$

Just use the relation $f_\alpha(t) = t^{-\alpha} J_\alpha(t^{-1})$ and the property cited in(3.2).

- For all $t \in \mathbb{R}_+, \mu \in (0, 1)$ it is fulfilled:

$$f_{1-\mu}(t) < \Gamma(\mu) \sin(\mu\pi/2), \quad (3.7)$$

$$g_{1-\mu}(t) < \Gamma(\mu) \cos(\mu\pi/2), \quad (3.8)$$

Indeed, according to the lemma 1.4 the amounts $f_{1-\mu}(0), g_{1-\mu}(0)$ exists and it is fulfilled:

$$f_{1-\mu}(0) = s_{1-\mu}(0) = \Gamma(\mu) \sin(\mu\pi/2),$$

$$g_{1-\mu}(0) = c_{1-\mu}(0) = \Gamma(\mu) \cos(\mu\pi/2),$$

by applying the known identity: $\int_0^\infty t^{\mu-1} e^{it} dt = \Gamma(\mu) e^{i\mu\pi/2}$ (see [14], page 52).

We now use the fact that the functions $t \rightarrow f_{1-\mu}(t)$ and $t \rightarrow g_{1-\mu}(t)$ are strictly decreasing in \mathbb{R}_+ and we conclude (3.7) and (3.8).

4 Zeros of the CFI family

In this section we develop several of the problems raised for the family c_α, s_α , defined as CFI, in the introduction of the article. In the first subsection we identify the zeros of the CFI family as the solution set of an FPSP, then deducting from the results of the sections 3, 2 the results 1.3, 1.1 of the Introduction. In the subsection 4.2 we establish the existence of a new lower bound, defined from the theoretical upper bound $\mathcal{T}_{n,\alpha}$ mentioned in the introduction, and a bound of the error of the new location interval.

4.1 Zeros of the CFI family as fixed points of problems FPSP with unique solution

The result quote 1.1 of the introduction refers to the existence and uniqueness of the zeros of the functions of this family and their location in intervals of the real line. We can establish this by means of the theory of the section 2, about the problems called FPSP. Indeed, thanks to the identity of the representation lemma 1.4 it can be taken $t \rightarrow t + \pi/2 - \arctan\left(\frac{g_\alpha(t)}{f_\alpha(t)}\right)$ as a parameterized function of the angular measure of the curve $t \rightarrow c_\alpha(t) + is_\alpha(t)$. Therefore in the case of the function c_α for example, since a real t is one of its zeros if and only the angular measure in the value t is an odd multiple of $\pi/2$, it follows from the above that t is a fixed point of the function $k\pi/2 + \arctan\left(\frac{g_\alpha}{f_\alpha}\right)$ for an even value of the integer k . The result is analogous for s_α taking k as an odd integer. Which is the content of corollary 3.3 in [10], where a different reasoning was followed.

Then the zeros of c_α and s_α are exactly the fixed points of the FPSP, in the sense of the section 2, defining $G_\alpha(t) = \arctan\left(\frac{g_\alpha}{f_\alpha}\right)(t)$, the intervals $\mathfrak{I}_n = [n\pi/2, (n+1)\pi/2]_{n \in \mathbb{N}}$ and taking $\mathcal{U} = \mathcal{J} = \mathbb{R}_+^*$.

Then in this case $\mathfrak{a}_n = n\pi/2$ y $\mathfrak{b}_n = \mathfrak{a}_{n+1}$. The existence and uniqueness of the fixed points within each \mathfrak{J}_n is guaranteed by the corollary 2.3 since case 1) is fulfilled (the functions G_α are all decreasing as mentioned in the property 3.3).

In the rest of the section we denote $z_{n,\alpha}$ the zero of the CFI belonging to the interval \mathfrak{J}_n and defined by the value α of the parameter.

The result of the quote 1.3, except for the asymptotic result, it is obtained from the proposition 2.2, since the function $z \rightarrow n\pi/2 + \frac{\alpha}{z}$ is bounded above by $G_{n,\alpha}$ (see first inequality of (3.4)) and it satisfies in each \mathfrak{J}_n the conditions of that proposition with condition 1) as stated above.

The results enunciated in the quote 1.2 come from the proposition 2.5. In effect, according to what has been said above, the FPSP associated with this problem has a unique solution with arithmetic conditions ($A = c = \pi/2$ in this case). The function G satisfies in this case the LIM condition, according to what has been said above about the decrease of the G and its convergence to 0 at infinity using for example the inequality in 3.4. The asymptotic behavior of $(z_{n,\alpha} - n\pi/2)_{n \in \mathbb{N}}$ is deduced from this same proposition since G fulfills the property @ with $K = \alpha$, $\beta = -1$, in virtue of the property (3.6).

It should be noted that a formulation of a FPSP like the one above appears in Macleod [12] for the case of c_1 for $\alpha = 1$.

4.2 A theoretical lower bound and a bound of the error, CFI case

We now look for theoretical lower bounds of the zeros of the CFI family, which improve the theoretical levels $\mathfrak{a}_n, n \in \mathbb{N}$ established in the previous subsection. The analysis of the error associated with these bounds allows to deduce a stronger version than that of the quote 1.3.

We define

$$F_\alpha(t) \equiv \arctan\left(\frac{\alpha}{t} \frac{1}{1 + (\alpha + 1)_2/t^2}\right). \quad (4.1)$$

Theorem 4.1. *For n large enough, $(\alpha + 2 < \mathfrak{a}_n)$, $n\pi/2 + F_\alpha(\mathcal{T}_{n,\alpha})$ is a lower bound of $z_{n,\alpha}$ in \mathfrak{J}_n . Also an estimate of the error is*

$$|(n\pi/2 + F_\alpha(\mathcal{T}_{n,\alpha})) - \mathcal{T}_{n,\alpha}| < K_\alpha \mathfrak{a}_n^{-3}, \quad (4.2)$$

where $K_\alpha = \alpha^3/3 + \alpha(\alpha + 1)(\alpha + 2)$

Proof

Thanks to the inequalities in (3.4) and the relationship $\frac{g_\alpha}{f_\alpha}(t) = \frac{\alpha f_{\alpha+1}(t)}{f_\alpha(t)}$ we deduce the inequality:

$$F_\alpha(t) < G_\alpha(t), \quad \text{for all } t \in \mathcal{U}.$$

It is verified without difficulty that F_α satisfies the conditions of the corollary 2.3 because its codomain lies in $(0, \pi/2)$ and it is strict decreasing for $t > \alpha + 2$. Then case 1) of the proposition 2.2 is fulfilled in \mathfrak{J}_n if $\alpha + 2 < \mathfrak{a}_n$, and therefore of the double inequality of this proposition we deduce that $n\pi/2 + F_\alpha(\mathcal{T}_{n,\alpha})$ is a lower bound of zero $z_{n,\alpha}$.

Given that $\mathcal{T}_{n,\alpha}$ is a fix point of $z \rightarrow n\pi/2 + \frac{\alpha}{z}$, it is obtained

$$|n\pi/2 + F_\alpha(\mathcal{T}_{n,\alpha}) - \mathcal{T}_{n,\alpha}| = \left| F_\alpha(\mathcal{T}_{n,\alpha}) - \frac{\alpha}{\mathcal{T}_{n,\alpha}} \right|.$$

Expressing now $F_\alpha(\mathcal{T}_{n,\alpha})$ as $\arctan(x)$ and using inequality $\arctan(x) > x - x^3/3$ for $|x| < 1$, we find that for $\alpha + 2 < a_n$:

$$\left| F_\alpha(\mathcal{T}_{n,\alpha}) - \frac{\alpha}{\mathcal{T}_{n,\alpha}} \right| < \alpha(\alpha + 1)_2 \mathcal{T}_{n,\alpha}^{-3} + x^3/3 \quad (4.3)$$

But in this last expression $\alpha(\alpha + 1)_2 = (\alpha)_3$ on the one hand while on the other $x < \frac{\alpha}{\mathcal{T}_{n,\alpha}}$ from where we conclude. **QED**

Note: A similar previous analysis applied to the fixed point of $t \rightarrow n\pi/2 + \arctan(\alpha/t)$ en \mathcal{J}_n , that we denote again $\mathcal{T}_{n,\alpha}$, shows that this is an upper bound of $z_{n,\alpha}$ and also an error bound is given as that of theorem 4.1 taking $K_\alpha = (\alpha)_3$.

According to the theorem 4.1 the asymptotic relation $|z_{n,\alpha} - \mathcal{T}_{n,\alpha}| = o(1/n^2)$ is in particular fulfilled, responding in this way to the conjecture of [10] formulated in the conclusions taking $p = 2$ and also finding an upper bound of the approximation error (4.2).

5 Zeros of the Lommel family

As we explained in the introduction the zeros of the Lommel family are same of the family $(F_\mu)_{\mu \in J}$. The results of this section consist first of all in the characterization of the problem of the zeroes of this last family as several fixed point sequential problems FPSP, as defined in the section 2. Once identified intervals and functions for each fixed point problem we turn to the theory developed in the previous sections to establish the results 1.5 and 1.6. We show then the existence of new theoretical bounds of the zeros and the proposition 2.2 studying the approximation error. In other sections we present the asymptotic results that result from the application of the proposition 2.5.

From the lemma 1.4 an additive decomposition is revealed for the family of functions $(F_\mu)_{\mu \in J}$:

$$F_\mu(z) = \Gamma(\mu) \sin(z - \mu\pi/2) + f_{1-\mu}(z). \quad (5.1)$$

In effect, considering the function $H : z \rightarrow \int_0^z t^{\mu-1} e^{i(z-t)} dt$. We factor in $H(z)$ the factor e^{iz} and express the other factor as the complex conjugate of $\int_0^z t^{\mu-1} e^{it} dt$ which in turn we express as $\int_0^\infty t^{\mu-1} e^{it} dt - \int_z^\infty t^{\mu-1} e^{it} dt$. Using $\int_0^\infty t^{\mu-1} e^{it} dt = \Gamma(\mu) e^{i\mu\pi/2}$ we get the expression:

$$H(z) = e^{iz} \left(\int_0^\infty t^{\mu-1} e^{-it} dt - (c_{1-\mu}(z) - is_{1-\mu}(z)) \right).$$

By the integral formula for $c_{1-\mu}(z) + is_{1-\mu}(z)$ from lemma 1.4, and matching the imaginary parts of both members of the last expression we get (5.1).

Note: The formula cited above: $\int_0^\infty t^{\mu-1} e^{it} dt = \Gamma(\mu) e^{i\mu\pi/2}$ admits a proof based on the identity of the lemma 1.4 (see [11]).

By virtue of the decomposition (5.1) the zeros of F_μ are the solutions of the equations

$$\Gamma(\mu) \sin(t - \mu\pi/2) = -f_{1-\mu}(t) \quad (5.2)$$

5.1 The zeros as FPSP problems

We will denote from now on $a_n = n\pi/2$, and for \mathcal{J}_n the interval (a_n, a_{n+1}) .

We translate the equation (5.2) in terms of fixed point sequential problems as it was proposed in section 2. For this we need to define the sequence of intervals $(\mathcal{J}_n)_{n \in \mathbb{N}}$ and the function associated with each of the FPSP. We need the following definitions of intervals and functions, for an integer n :

$$I_n = \begin{cases} (a_n, a_{n+2}) & \text{if } n = 2 \pmod{4}, \\ (a_{n-1}, a_{n+1}) & \text{if } n = 0 \pmod{4}, n \neq 0 \\ (a_0, a_1) & \text{if } n = 0, \end{cases} \quad (5.3)$$

and the functions

$$G_\mu^{(1)}(t) = \mu\pi/2 + \arcsin\left(\frac{f_{1-\mu}(t)}{\Gamma(\mu)}\right), \quad G_\mu^{(2)}(t) = \mu\pi/2 + \arccos\left(\frac{f_{1-\mu}(t)}{\Gamma(\mu)}\right), \quad (5.4)$$

whose common domains and codomains are \mathbb{R}_+ and $(0, \pi)$ respectively.

Lemma 5.1.

1) The zeros of F_μ are the fixed points of the following FPSP's:

- a) that of the intervals $(I_{4n+2})_{n \in \mathbb{N}}$ and function $G_\mu^{(1)}$,
- b) that of the intervals $(I_{4n})_{n \in \mathbb{N}}$ and function $G_\mu^{(2)}$.

where in both cases $\mathcal{U} = \mathbb{R}_+$, $\mathcal{J} = (0, 1)$, with notations of section 2.

Note: in this case the intervals $(I_{4n+k})_{n \in \mathbb{N}}, k = 0, 2$, correspond, in each case, to the intervals $(\mathcal{J}_n)_{n \in \mathbb{N}}$ from the section 2.

Proof

It is a matter of studying the equation (5.2). As the right member of this equation is negative, a zero z of this equation must satisfy $\sin(z - \mu\pi/2) \leq 0$. So we restrict the discussion to the real t satisfying $t - \mu\pi/2 \in I_n, n = 2 \pmod{4}$.

Since $\sin(a_{n+1}) = \cos(a_n) = -1$ we have $\sin(t - \mu\pi/2) = -\sin(t - \mu\pi/2 - a_n)$ y $\sin(t - \mu\pi/2) = -\cos(t - \mu\pi/2 - a_{n+1})$, so that:

$$t - \mu\pi/2 \in \mathcal{J}_n : t \text{ fulfills (5.2)} \iff t = a_n + G_\mu^{(1)}(t) \quad (5.5)$$

$$t - \mu\pi/2 \in \mathcal{J}_{n+1} : t \text{ fulfills (5.2)} \iff t = a_{n+1} + G_\mu^{(2)}(t) \quad (5.6)$$

since the sine and cosine functions are invertible in each subinterval \mathcal{J}_n . Expressions of the right member both in (5.5) as in (5.6) are well defined in all $\mu\pi/2 + I_n$ for all integers n as the quantity $\frac{f_{1-\mu}(t)}{\Gamma(\mu)}$ is positive and less than 1. Indeed, it is enough to use inequality $\frac{f_{1-\mu}(t)}{\Gamma(\mu)} < \sin(\mu\pi/2)$ (see relationship (3.7)).

Now, it have $\mu\pi/2 + \mathcal{J}_n \subset I_n$, $\mu\pi/2 + \mathcal{J}_{n+1} \subset I_{n+2}$ on the one hand, while on the other a fixed point of $a_n + G_\mu^{(1)}$ in I_n must be on $\mu\pi/2 + \mathcal{J}_n$, and a fixed point of $a_{n+1} + G_\mu^{(2)}$ in I_{n+2} must be on $\mu\pi/2 + \mathcal{J}_{n+1}$. Whence we conclude. **QED**

Lemma 5.2.

- a) The codomains of $G_\mu^{(1)}$, $G_\mu^{(2)}$ are included in $(0, \pi)$.
- b) i) The function $G_\mu^{(1)}$ is strictly decreasing and convex.
 ii) The function $G_\mu^{(2)}$ is strictly increasing and concave.
- c) $G_\mu^{(1)}(t) - \mu\pi/2 \sim t^{\mu-1}/\Gamma(\mu)$, $t \rightarrow \infty$,
 $G_\mu^{(2)}(t) - \mu\pi/2 - \pi/2 \sim -t^{\mu-1}/\Gamma(\mu)$, $t \rightarrow \infty$,

Proof

Since the sum $G_\mu^{(1)} + G_\mu^{(2)}$ is constant in the domain $\mathcal{U} = \mathbb{R}_+$, it is enough to analyze one of them. Let $G = G_\mu^{(1)}$.

Item a) is immediate from the condition $J = (0, 1)$ and that the term $\arcsin\left(\frac{f_{1-\mu}(t)}{\Gamma(\mu)}\right)$ is the arcsine evaluated in the real interval $(0, 1)$.

Item b) it follows from the relationship

$$\Gamma(\mu)G'(t) = (\arcsin)'\left(\frac{f_{1-\mu}(t)}{\Gamma(\mu)}\right) f'_{1-\mu}(t) \tag{5.7}$$

where the factor $(\arcsin)'\left(\frac{f_{1-\mu}(t)}{\Gamma(\mu)}\right)$ is positive and decreasing since arcsine is increasing and convex and $f_{1-\mu}$ is decreasing; the factor $f'_{1-\mu}(t) = -g_{1-\mu}(t)$ (see property (3.5)) is increasing and negative strict; then the product of these two factors is a strict and negative increasing function, i.e. the convexity of G and its decrease. For part c): in the first pair of limits we use the fact that $\arcsin(t) \sim t$ at 0, while $f_{1-\mu}(t) \sim t^{\mu-1}$ at infinity. For the second limit we use the expression (5.7) and the facts: $\arcsin'(t) \sim 1$ at 0, $f'_{1-\mu}(t) = -g_{1-\mu}(t) \sim -(1-\mu)t^{\mu-2}$ at infinity (see definition of $g_{1-\mu}$ and property (3.5)) to conclude. **QED**

Lemma 5.3. *The FPSP's defined in lemma 5.1 have a unique solution.*

Proof

It is enough to invoke the corollary 2.3 from the section 2 that is fulfilled with $L = \pi$, since the length of the intervals is constant L and that the other conditions of this corollary (with the condition of convexity or concavity) are fulfilled thanks to the lemma 5.2. **QED**

Finally we can establish quote 1.5, for the zeros of the Lommel functions:

Proof of quote 1.5:

The intervals $I_{2n-1}(\mu)$, $I_{2n}(\mu)$ belong to the intervals I_{4n-2} , I_{4n} respectively defined above. Under the lemma 5.3 it is enough to prove that $I_{2n-1}(\mu)$ (resp. $I_{2n}(\mu)$) contains a fixed point of $G = a_{4n-2} + G_\mu^{(1)}$, (resp. of $G = a_{4n-1} + G_\mu^{(2)}$). Actually we proved that the interiors of $I_{2n-1}(\mu)$, $I_{2n}(\mu)$ contain them.

Let z be the only fixed point of the G corresponding to the interval I_{4n-2} , then: $z = a_{4n-2} + G_\mu^{(1)}(z)$. Since $G_\mu^{(1)}(t) > \mu\pi/2$ for all $t \in \mathcal{U}$ we obtain $z > a_{4n-2} + \mu\pi/2$. On the other hand, using 3.7, we have $G_\mu^{(1)}(t) < \mu\pi/2 + \arcsin(\sin(\mu\pi/2)) = \mu\pi$ for all $t \in \mathcal{U}$ and in particular we find $z < a_{4n-2} + \mu\pi$. then $z \in I_{2n-1}(\mu)$.

If z is the only fixed point of the G corresponding to the interval I_{4n} we have: $z = a_{4n-1} + G_\mu^{(2)}(z)$. Since $G_\mu^{(2)}(t) < \mu\pi/2 + \pi/2$ we have

$$z < a_{4n-1} + \mu\pi/2 + \pi/2 = a_{4n} + \mu\pi/2.$$

On the other hand

$$z = a_{4n-1} + \mu\pi/2 + \pi/2 - \arcsin(f_{1-\mu}(z)/\Gamma(\mu)),$$

where $-\arcsin(f_{1-\mu}(z)/\Gamma(\mu)) > -\mu\pi/2$ (por (3.7)). Then $z > a_{4n}$ and we conclude that $z \in I_{2n}(\mu)$. **QED**

By virtue of the above results, the problem of zeros of the family $(F_\mu)_{\mu \in J}$ is equivalent to the following pair of FPSP's (*reduced FPSP's*):

- *reduced problem a*): defined by the intervals $I_{2n+1}(\mu), n \in \mathbb{N}$ and the function $G_\mu = G_\mu^{(1)} - \mu\pi/2$
- *reduced problem b*): defined by the intervals $I_{2n}(\mu), n \in \mathbb{N}$ and the function $G_\mu = G_\mu^{(2)} - \pi/2$

Through the reduced FPSP's above, we can prove quote 1.6 invoking the proposition 2.5.

Proof of the quote 1.6 :

Thanks to the lemma 5.2 it follows that the reduced problems are of unique solution with arithmetic conditions, $A = c = \mu\pi/2$, and that the G associated to the reduced problem a) satisfies LIM condition, while the G associated with the reduced problem b) is such that $\mu\pi/2 - G$ satisfies LIM. The application of the proposition 2.5, part b), allows to conclude the results of the quote 1.6. **QED**

We denote by $z_{n,\mu}$ the unique zero of F_μ in the interval $I_n(\mu)$. We complete the quote 1.6 by an asymptotic law for the convergent sequences mentioned there (they are denoted by $a_{n,\mu}, b_{n,\mu}$ resp.) the lower end (upper end resp.) of $I_n(\mu)$:

Proposition 5.4. *We have the following asymptotic approximations*

$$z_{2n+1,\mu} - a_{2n+1,\mu} \underset{n \rightarrow \infty}{\sim} \frac{a_{2n+1,\mu}^{\mu-1}}{\Gamma(\mu)}, \quad z_{2n,\mu} - b_{2n,\mu} \underset{n \rightarrow \infty}{\sim} -\frac{b_{2n,\mu}^{\mu-1}}{\Gamma(\mu)},$$

Proof

According to lemma 5.2, item c), the property @ is met in both reduced problems with $G(t) \sim t^{\mu-1}/\Gamma(\mu), t \rightarrow \infty$ in the reduced problem a) and $G(t) - c \sim -t^{\mu-1}/\Gamma(\mu) t \rightarrow \infty$, in the reduced problem b). It suffices to apply then the proposition 2.5 and the interpretation of the zeros as fixed points of each reduced FPSP. **QED**

5.2 New theoretical bounds and their error

The location intervals $I_n(\mu), n \in \mathbb{N}$ studied in the previous section have constant length $\mu\pi/2$ and therefore the property $(b_{n,\mu} - a_{n,\mu}) \xrightarrow{n \rightarrow \infty} 0$ is not accomplished. We look for some intervals with this property resorting to proposition 2.2, for which it is required to find a pair of functions defined in each interval $I_n(\mu)$ which bounds function G .

We remember that $z_{n,\mu}$ denotes the unique zero of F_μ in the interval $I_n(\mu)$.

We define:

1. for n odd, let be

$$\mathcal{F}_\mu(z) = \arcsin\left(\frac{z^{\mu-1}(1 + (1 - \mu)(2 - \mu)z^{-2})^{-1}}{\Gamma(\mu)}\right), \quad \mathcal{H}_\mu(z) = \arcsin\left(\frac{z^{\mu-1}}{\Gamma(\mu)}\right).$$

$$\mathcal{H}_{n,\mu} = a_{n,\mu} + \mathcal{H}_\mu, \quad \mathcal{F}_{n,\mu} = a_{n,\mu} + \mathcal{F}_\mu,$$

2. for n even, let be

$$\mathcal{F}_\mu(z) = \arccos\left(\frac{z^{\mu-1}}{\Gamma(\mu)}\right) + \mu\pi/2 - \pi/2, \quad \mathcal{F}_{n,\mu} = a_{n,\mu} + \mathcal{F}_\mu,$$

Theorem 5.5. *We assume that $(a_{n,\mu})^{\mu-1} < \Gamma(\mu + 1)$. We have the following theoretical bounds of $z_n(\mu)$:*

1. *if n is odd, the function $\mathcal{H}_{n,\mu}$ is well defined in $I_n(\mu)$ and there is a unique fixed point $h_{n,\mu}$ of this function in this interval that fulfills:*

$$\mathcal{F}_{n,\mu}(h_{n,\mu}) < z_{n,\mu} < h_{n,\mu}$$

2. *if n is even ($n \geq 2$) the function $\mathcal{F}_{n,\mu}$ is well defined in $I_n(\mu)$ and there is a unique fixed point $f_{n,\mu}$ of this function in this interval that fulfills:*

$$f_{n,\mu} < z_{n,\mu} < b_{n,\mu}$$

Proof

1. For n odd the inequalities are satisfied for $z \in I_n(\mu)$:

$$\mathcal{F}_\mu(z) < G_\mu^{(1)}(z) - \mu\pi/2 < \mathcal{H}_\mu(z)$$

as long as its members are defined. They result from the growth of the function arcsine on the one hand and on the other hand the inequalities of 3.4 for the function f_μ .

The well definition of $\mathcal{H}_{n,\mu}$ in $I_n(\mu)$ follows from $a_{n,\mu} > 1$ for n odd, so the function $z \rightarrow \frac{z^{\mu-1}}{\Gamma(\mu)}$, bounded above by $\frac{(a_{n,\mu})^{\mu-1}}{\Gamma(\mu)}$ in $I_n(\mu)$, is less than 1 and then belongs to the arcsine domain. The existence and uniqueness of the fixed point of function $\mathcal{H}_{n,\mu}$ in $I_n(\mu)$ follows from the proposition 2.1. In effect, by inequality $\arcsin(x) < \pi/2x$, $x > 0$ and the condition of the theorem, it is fulfilled that $0 < \mathcal{H}_\mu < \mu\pi/2$ for all $I_n(\mu)$. Then the codomain of $\mathcal{H}_{n,\mu}$ is in the interior of $I_n(\mu)$ and as on the other hand $\mathcal{H}_{n,\mu}$ fulfills case 1) of this proposition we conclude.

On the other hand, the function $\mathcal{F}_{n,\mu}$ is well defined in $I_n(\mu)$ if $\mathcal{H}_{n,\mu}$ is. It satisfies the conditions of the proposition 2.2 and case 1) of the proposition 2.1. For the condition of the case 1) let's prove that for n odd the function $z \rightarrow z^{\mu-1}(1 + (1 - \mu)(2 - \mu)z^{-2})^{-1}$ is decreasing in $I_n(\mu)$. It is easy to prove, using the logarithmic derivative, that this last condition is equivalent to $z^2 > (1 + \mu)_{(2)}$, $z \in I_n(\mu)$ which is equivalent to $a_{n,\mu}^2 > (1 + \mu)_{(2)}$ for n odd; since $a_{n,\mu}$ is increasing for n odd, it is enough to prove it for $n = 1$, which is almost immediate. The result is then derived from the proposition 2.2.

2. For n even the function \mathcal{F}_μ is well defined in $I_n(\mu)$ for reasons similar to those given in case 1 for the function $\mathcal{H}_{n,\mu}$. The inequality is fulfilled for $z \in I_n(\mu)$:

$$\mathcal{F}_\mu(z) < G_\mu^{(2)}(z) - \pi/2$$

resulting from the decrease of arccosine and the inequalities of (3.4). The existence and uniqueness of the fixed point of $\mathcal{F}_{n,\mu}$ in $I_n(\mu)$ is a consequence of the proposition 2.1, case 2), because it is concave with codomain in the interior of $I_n(\mu)$. As the functions of the previous inequality satisfy the conditions of the case 2) of the proposition 2.2 (see b) from lemma 5.2) the result is a consequence of this theorem. **QED**

The bounds found in the preceding theorem define new location intervals for zeros whose approximation errors are studied below:

Theorem 5.6. *We have the following error bounds:*

$$1. \text{ If } n \text{ is odd} \quad |\mathcal{F}_{n,\mu}(h_{n,\mu}) - h_{n,\mu}| < K(\mu)a_n^{\mu-3}(\mu),$$

$$\text{where } K(\mu) = \frac{(1-\mu)^2}{\Gamma(\mu)}(1 - (a_2^{\mu-1}(\mu)/\Gamma(\mu))^2)^{-1/2},$$

$$2. \text{ if } n \text{ is even:} \quad |b_{n,\mu} - f_{n,\mu}| < \frac{\pi a_n^{\mu-1}(\mu)}{2 \Gamma(\mu)}$$

Proof

From the bounds defined in the theorem 5.5:

1) An upper bound for $|\mathcal{F}_{n,\mu}(h_{n,\mu}) - h_{n,\mu}|$ is $\|\mathcal{F}_{n,\mu} - \mathcal{H}_{n,\mu}\|_{I_n(\mu)}$, where the right member denotes the uniform norm of the function $\mathcal{F}_{n,\mu} - \mathcal{H}_{n,\mu}$ in $I_n(\mu)$.

Now $\mathcal{F}_{n,\mu} - \mathcal{H}_{n,\mu} = \mathcal{F}_\mu - \mathcal{H}_\mu$, the function of the right member being a difference of the function arcsine in points of the form $\frac{z^{\mu-1}(1+(1-\mu)(2-\mu)z^{-2})^{-1}}{\Gamma(\mu)}$ and $\frac{z^{\mu-1}}{\Gamma(\mu)}$ for a real $z \in I_n(\mu)$, we can then use in this case the mean value theorem using the derivative of $\arcsin()$ and the convexity property of this function in $(0, 1)$ to obtain the sought inequality .

2) The relation

$$|b_{n,\mu} - \mathcal{F}_{n,\mu}(f_{n,\mu})| = \arcsin\left(\frac{(f_{n,\mu})^{\mu-1}}{\Gamma(\mu)}\right)$$

holds, and as $\arcsin(z) < \frac{\pi}{2}z$ for z positive and $(f_{n,\mu})^{\mu-1} < (a_{n,\mu})^{\mu-1}$ the desired relation is finally deduced. **QED**

Then as a corollary of previous result the lengths of the new location intervals in 5.5 converge to 0 when $n \rightarrow \infty$, thus improving the quality of the bounds $a_{n,\mu}, b_{n,\mu}$ considered in the subsection 5.1.

Remark:

Other bounds of the zeros are obtained by taking on new upper bound functions \mathcal{H}, \mathcal{F} . For example for n odd, you can take:

$$\mathcal{H}_\mu(z) = \frac{\pi z^{\mu-1}}{2 \Gamma(\mu)}, \quad \mathcal{F}_\mu(z) = \frac{z^{\mu-1}}{\Gamma(\mu)}(1 + (1 - \mu)(2 - \mu)z^{-2})^{-1}.$$

These result from the functions \mathcal{H}_μ and \mathcal{F}_μ defined above, and inequalities: $\arcsin(z) > z, z > 0$ y $\arcsin(z) < \frac{\pi}{2}z, z > 0$. The advantage is that it is easier to determine, at least numerically, the fixed point of the $\mathcal{H}_{n,\mu} = a_{n,\mu} + \mathcal{H}_\mu()$ that for the previous one. The fixed points $h_{n,\mu}$ of $\mathcal{H}_{n,\mu}$ in $I_n(\mu)$ are, in this case, the new theoretical upper bounds of the zeros

As in the proof of the theorem 5.6 one can obtain:

$$a_{n,\mu} + \mathcal{F}_\mu(h_{n,\mu}) < z_{n,\mu} < h_{n,\mu}$$

y

$$|h_{n,\mu} - z_{n,\mu}| < \frac{c^{\mu-1}}{\Gamma(\mu)} \left(\frac{\pi}{2} - (1 + (1 - \mu)(2 - \mu)c^{-2})^{-1} \right), c = a_{n,\mu}$$

The second inequality is almost immediate from definitions. The error bound is asymptotically larger than that of the theorem 5.6, because it is $O(a_{n,\mu}^{\mu-1})$ while that of the theorem is $O(a_{n,\mu}^{\mu-3})$, $n \rightarrow \infty$.

5.3 Conclusions and future work

The results developed in this paper about the location of the zeros of the parametric functions considered above has been based on the theory that we called *fixed point sequential problems*. In particular we have been able to find bounds of approximation error that were not evident in the previous work [10] and that are absent in [7].

We hope to continue with this approach in a next work about on the problem of parametric dependence of the zeros of the same families of functions.

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