Riemannian manifolds in noncommutative geometry

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Abstract

We present a definition of Riemannian manifold in noncommutative geometry. Using products of unbounded Kasparov modules, we show one can obtain such Riemannian manifolds from noncommutative spin^c manifolds; and conversely, in the presence of a spin^c structure. We also show how to obtain an analogue of Kasparov's fundamental class for a Riemannian manifold, and the associated notion of Poincaré duality. Along the way we clarify the bimodule and first-order conditions for spectral triples.

1 Introduction

The noncommutative geometry program of extending differential manifold structures via the concept of a spectral triple has seen several recent developments. Any closed Riemannian manifold M endowed with a spin^c structure can be recovered from a spectral triple over its algebra $C^\infty(M)$ of smooth functions, via the reconstruction theorem of Connes in its several iterations [1–3].

We present a definition of closed noncommutative Riemannian manifold. In the absence of a spin^c structure and associated Dirac operator, our definition is modelled on the Hodge–de Rham operator acting on differential forms. Classically, this construction relies only on a choice of orientation and metric.

To produce examples, and to check the compatibility of our definition with the existing definition of spin^c manifolds, we show that:

(A) given a spin^c manifold, we can obtain a Riemannian manifold;

(B) given a Riemannian manifold and a "spin^c structure", we can obtain a spin^c manifold.

These operations are mutually inverse when both are defined.

Underlying these operations is Plymen's theorem [4], which identifies spin^c structures on a closed Riemannian manifold M with Morita equivalences between the C^*-algebras $C(M)$, of continuous functions on M; and $\mathbb{C}\ell(M)$, of continuous sections of the Clifford algebra bundle. To translate this into the language of spectral triples, we reformulate such Morita equivalences, or "spin^c structures", as holding between $C^\infty(M)$ and a suitable "smooth subalgebra" of $\mathbb{C}\ell(M)$, while ensuring that the link to KK-theory is preserved.

The main tool used throughout to marry modules and spectral triples is the Kasparov product of unbounded Kasparov modules, [5, 6]. This has been recently revisited by Mesland [7]. Since we utilise the Kasparov product to produce our various spectral triples, we are also able to show that the spin^c and Riemannian notions of Poincaré duality are carried into each other by the operations (A) and (B). Another essential point in these constructions is the identification of a noncommutative analogue of Kasparov's fundamental class for a Riemannian manifold.

Several routes towards "almost commutative" spectral triples have recently been taken. The thesis of Zhang [8] introduces a spectral triple over $C^\infty(M)$ in the non-spin^c case, using twisted K-theory to overcome the obstruction. Ćaćić [9] introduces spectral triples suitable for vector bundles over M, thereby extending the reconstruction theorem to that case, at the price of weakening the "orientability" axiom. Boeijink and van Suijlekom [10] instead formulate a spectral triple over the Clifford algebra bundle, and relate it to the spin^c case using Kucerovsky's work, much as we do here.

By contrast, our approach to noncommutative Riemannian manifolds makes no assumptions of commutativity. It originated in the thesis [11] of the first author (for an earlier attempt, see [12]), and consists in replacing the "noncommutative spin^c" condition for spectral triples with a "noncommutative Riemannian" condition.

In Section 2 we prepare the ground by examining pre-Morita equivalences of hermitian modules over smooth subalgebras of C^*-algebras. Section 3 develops some tools for studying operators on finitely generated projective modules. Here we introduce bimodule connections and characterise those operators satisfying a first order condition on suitable bimodules. After reviewing basic notions of spectral triples, we lay out a bimodule twisting procedure that represents a Kasparov product between unbounded Kasparov modules.

In Section 4 we come to manifold structures on spectral triples. First we formulate such triples in the spin^c case, with a slight strengthening of the usual conditions [3, 13]. Next we replace the spin^c condition with a new Riemannian condition, that does not require any spin^c properties. We show in detail how such a Riemannian spectral triple represents a generalisation of Kasparov's fundamental class in KK-theory.

Finally, in Section 5 we state and prove precise versions of (A) and (B). We also show how Kasparov's fundamental class provides a translation between the spin^c and Riemannian Poincaré duality isomorphisms.

► We shall use the following notational conventions.

- ◇ Throughout, A and B denote separable unital C^*-algebras. Script letters $\mathcal{A}$ and $\mathcal{B}$ denote dense $*$-subalgebras $\mathcal{A} \subset A$ and $\mathcal{B} \subset B$. Often, $\mathcal{A}$ will come equipped a locally convex topology finer than that given by the C^*-norm of A; and similarly for $\mathcal{B}$ and B.
- ◇ In a C^*-algebra A, or in its dense subalgebra $\mathcal{A}$, the notation $a \geqslant 0$ means that a is a positive element of the C^*-algebra A; we write $a > 0$ when a is a nonzero positive element of A.
- ◇ For any algebra $\mathcal{A}$, its opposite algebra will be denoted $\mathcal{A}^\circ$ with elements a°, b°, etc., satisfying $a^\circ b^\circ = (ba)^\circ$.
- ◇ We deal with two kinds of "inner products": hermitian pairings with values in a $*$-algebra $\mathcal{A}$ or $\mathcal{B}$ are written with round brackets, like $(e \mid f)_{\mathcal{A}}$ or ${}_{\mathcal{B}}(e \mid f)$; while *scalar* products of

vectors in a Hilbert space have angle brackets, like $\langle \xi \mid \eta \rangle$.

- ⋄ The standard basis of $\mathbb{C}^n$ will be written $\{u_1, \dots, u_n\}$. The same notation will be used for the standard basis of "column vectors" in the right $\mathcal{A}$-module $\mathcal{A}^n$.
- ⋄ When we discuss tensor products of Fréchet algebras $\mathcal{A}$ and $\mathcal{B}$, with the projective tensor product topology, the notation $\mathcal{A} \otimes \mathcal{B}$ will refer to the *completed* tensor product, which is often written as $\mathcal{A} \mathbin{\widehat{\otimes}} \mathcal{B}$; thus our $\mathcal{A} \otimes \mathcal{B}$ is then a Fréchet space. A similar convention will be used for balanced tensor products of topological modules.
- ⋄ On a Riemannian manifold (M, g), there is a Clifford algebra bundle with base M generated by the Clifford product on the complexified cotangent bundle. The notation $\mathbb{C}\ell(M)$ will denote the (unital, assuming M to be compact) C^*-algebra of continuous *sections* of this bundle; its isomorphism class does not depend on g.
- ⋄ If T is a closed operator on a Hilbert (or Banach) space $\mathcal{H}$, its domain is $\operatorname{Dom} T$. Its *smooth domain* is

$$\operatorname{Dom}^\infty T := \bigcap_{k=1}^{\infty} \operatorname{Dom} T^k.$$

If $\mathcal{D}$ is a selfadjoint operator, its regularised modulus is the operator $\langle \mathcal{D} \rangle := (1 + \mathcal{D}^2)^{1/2}$. Note that $\langle \mathcal{D} \rangle - |\mathcal{D}|$ is positive and bounded, and $\operatorname{Dom}^\infty \langle \mathcal{D} \rangle = \operatorname{Dom}^\infty |\mathcal{D}| = \operatorname{Dom}^\infty \mathcal{D}$.

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2 Hermitian modules and Morita equivalence

We begin with a preliminary discussion of hermitian modules and bimodules over dense subalgebras of unital C^*-algebras. Much of Rieffel's theory of strong Morita equivalence remains true, provided one treads carefully when invoking spectral theory. The expected properties of finitely generated projective modules all hold, but we must spell it out.

The theory of Morita equivalence between C^*-algebras is fully laid out in the monograph [14], whose notations we follow mostly. What we require are the analogous notions for certain dense subalgebras.

Definition 2.1. Let $\mathcal{A}$ be a dense subalgebra of a C^*-algebra A. A right $\mathcal{A}$-module $\mathcal{E}$ is *hermitian* if it carries a pairing $\mathcal{E} \times \mathcal{E} \to \mathcal{A} : (e, f) \mapsto (e \mid f)_{\mathcal{A}}$, also called an $\mathcal{A}$-valued *inner product*,[1] which is linear in the *second* entry and satisfies $(e \mid f)_{\mathcal{A}}^* = (f \mid e)_{\mathcal{A}}$ and thus is antilinear in the first entry; it is also positive definite, $(e \mid e)_{\mathcal{A}} \geqslant 0$ in A with equality if and only if $e = 0$; and it satisfies

$$(eb \mid fa)_{\mathcal{A}} = b^*(e \mid f)_{\mathcal{A}}\, a \quad \text{for all} \quad a, b \in \mathcal{A}.$$

[1] In [14], $\mathcal{E}$ is called an *inner product $\mathcal{A}$-module*.

We say that $\mathcal{E}$ is *full* if $(\mathcal{E} \mid \mathcal{E})_{\mathcal{A}} := \operatorname{span}\{\, (e \mid f)_{\mathcal{A}} : e, f \in \mathcal{E} \,\}$ is dense in $\mathcal{A}$. A hermitian $\mathcal{A}$-module $\mathcal{E}$ is *projective* if it is a direct summand of a free module $\mathcal{A}^N$, and is *finitely generated* if N is finite and there exist $e_1, \dots, e_N \in \mathcal{E}$ such that every $e \in \mathcal{E}$ is of the form $e = e_1 a_1 + \dots + e_N a_N$ for some $a_1, \dots, a_N \in \mathcal{A}$.

If $\mathcal{E}_{\mathcal{A}}$ is a right hermitian $\mathcal{A}$-module, the conjugate vector space $\mathcal{E}^\sharp$ is a *left* $\mathcal{A}$-module under the left action $a \cdot e^\sharp = (ea^*)^\sharp$, where $e^\sharp$ is just $e \in \mathcal{E}$ regarded as an element of $\mathcal{E}^\sharp$. This conjugate module $\mathcal{E}^\sharp$ also carries a *left* hermitian pairing ${}_{\mathcal{A}}(\cdot \mid \cdot) : \mathcal{E}^\sharp \times \mathcal{E}^\sharp \to \mathcal{A}$, given by

$$ {}_{\mathcal{A}}(e^\sharp \mid f^\sharp) := (e \mid f)_{\mathcal{A}} \,. $$

Left hermitian pairings are linear in the *first* entry and antilinear in the second; they obey ${}_{\mathcal{A}}(e \mid f)^* = {}_{\mathcal{A}}(f \mid e)$ and are positive definite over $\mathcal{A}$; and they satisfy

$$ {}_{\mathcal{A}}(ae \mid bf) = a\,{}_{\mathcal{A}}(e \mid f) b^* \quad \text{for all} \quad a, b \in \mathcal{A}. $$

The formula $\|e\|^2 := \|(e \mid e)_{\mathcal{A}}\|_A$ defines a norm on $\mathcal{E}$; if $\mathcal{E}$ is complete in this norm and $\mathcal{A} = A$ is a C^*-algebra, then $\mathcal{E}$ is a *C^*-module* over A.

Given a right C^*-module E over a C^*-algebra A, we denote the C^*-algebra of all adjointable endomorphisms and its closed subalgebra of A-compact endomorphisms by $\operatorname{End}_A(E)$ and $\operatorname{End}^0_A(E)$ respectively. The latter is the norm closure of the algebra of finite-rank operators, spanned by

$$ \Theta_{e,f} : g \mapsto e\,(f \mid g)_A, $$

for $e, f, g \in E$. The same notation is used for left C^*-modules, where the finite-rank operators now act on the right: $g\,\Theta_{e,f} := {}_A(g \mid e)\, f$.

► In [14], following [15], a *Morita equivalence bimodule* E between two C^*-algebras B and A is introduced as a bimodule $E = {}_B E_A$ which is both a full right C^*-module over A and a full left C^*-module over B, such that each algebra acts by adjointable operators on the module for the other, and both pairings satisfy a compatibility relation: for all $a \in A$, $b \in B$ and $e, f, g \in E$,

$$ {}_B(e\,a \mid f) = {}_B(e \mid f\,a^*), \qquad (b\,e \mid f)_A = (e \mid b^*\,f)_A, \qquad {}_B(e \mid f)\,g = e\,(f \mid g)_A. \tag{2.1} $$

If we wish instead to use bimodules relating dense subalgebras of C^*-algebras, the adjointability cannot be taken for granted, but it can be replaced by the following norm-continuity conditions [15, Defn. 6.10].

Definition 2.2. Let $\mathcal{A}$, $\mathcal{B}$ be dense subalgebras of C^*-algebras. A *pre-Morita equivalence bimodule* $\mathcal{E}$ between $\mathcal{B}$ and $\mathcal{A}$ is a $\mathcal{B}$-$\mathcal{A}$-bimodule that is both a full right hermitian $\mathcal{A}$-module and a full left hermitian $\mathcal{B}$-module, such that for all $a \in \mathcal{A}$, $b \in \mathcal{B}$ and $e, f, g \in \mathcal{E}$, the following relations hold:

$$ {}_{\mathcal{B}}(e\,a \mid e\,a) \leqslant \|a\|^2\, {}_{\mathcal{B}}(e \mid e), \quad (b\,e \mid b\,e)_{\mathcal{A}} \leqslant \|b\|^2\,(e \mid e)_{\mathcal{A}}, \quad {}_{\mathcal{B}}(e \mid f)\,g = e\,(f \mid g)_{\mathcal{A}}. \tag{2.2} $$

If $\mathcal{E}$ is a pre-Morita equivalence bimodule between $\mathcal{B}$ and $\mathcal{A}$, then $\mathcal{E}^\sharp$ is a pre-Morita equivalence bimodule between $\mathcal{A}$ and $\mathcal{B}$.

For Morita equivalence bimodules between C^*-algebras, the two conditions (2.1) and (2.2) are equivalent: see [14, Lemma 3.7].

Next we show that, starting from a pre-Morita equivalence bimodule $\mathcal{E}$ between *unital* algebras $\mathcal{B}$ and $\mathcal{A}$, that $\mathcal{E}$ is finitely generated and projective both as an $\mathcal{A}$-module and as a $\mathcal{B}$-module. In the case of a right C^*-module E (not necessarily full) over a unital C^*-algebra A, it is well known – see, for instance, [16, Lemma 6.5] or [13, Prop. 3.9] – that E is a finitely generated projective A-module if and only if 1_E is an A-compact endomorphism of E.

In the case that E is indeed a finitely generated projective right A-module, we can find elements $x_1, \dots, x_m, y_1, \dots, y_m \in E$ such that

$$1_E = \sum_{i=1}^{m} \Theta_{x_i, y_i}.$$

Then there is an idempotent $q \in M_m(A)$ and an isomorphism $E \to qA^m$ given by

$$e \mapsto \big[(y_i \mid e)_A\big]_i \in A^m, \qquad 1_E \mapsto q := \big[(y_i \mid x_j)_A\big]_{i,j} \in M_m(A). \tag{2.3}$$

These formulas also apply when A and E are replaced by a dense subalgebra $\mathcal{A} \subset A$ and a finitely generated projective hermitian right $\mathcal{A}$-module, $\mathcal{E}$. In the C^*-case, we can go a little further, see [13, Prop. 3.9], and assume that $y_i = x_i$ for each i, so that $1_E = \sum_{i=1}^m \Theta_{x_i, x_i}$ and $q = q^*$ in $M_m(A)$. This refinement is also available for dense subalgebras $\mathcal{A} \subset A$ that allow enough functional calculus to take positive square roots of positive elements.

The *holomorphic* functional calculus may also be invoked to refine the discussion. Recall that a dense subalgebra $\mathcal{A}$ of a C^*-algebra A is called a *pre-C^*-algebra* if for all $a \in \mathcal{A}$ and all functions f defined and holomorphic in a neighbourhood of the spectrum of a, we have $f(a) \in \mathcal{A}$. If $\mathcal{A}$ is a Fréchet pre-C^*-algebra, so also is $M_m(\mathcal{A})$ [17].

Lemma 2.3. *Suppose that $\mathcal{A}$ is a Fréchet pre-C^*-algebra with C^*-completion A and that E is a right C^*-module over A. Suppose that 1_E is an A-compact endomorphism of E. Then there exists $m \in \mathbb{N}$ and a projector $q \in M_m(\mathcal{A})$ such that $\mathcal{E} := q\mathcal{A}^m$ has E as its C^*-module completion.*

Proof. The hypothesis $1_E \in \mathrm{End}_A^0(E)$ implies that there is an A-module isomorphism $E \simeq \tilde{q}A^m$ for some m and some $\tilde{q} \in M_m(A)$. Since $\mathcal{A}$ is a Fréchet pre-C^*-algebra, so also is $M_m(\mathcal{A})$; thus $\tilde{q}$ can be norm-continuously homotopied to a projector $q \in M_m(\mathcal{A})$, [18, pp. 21–23]. In consequence, $q = u\tilde{q}u^*$ for some unitary $u \in M_m(A)$.

Thus without loss of generality, $E \simeq qA^m$ where the projector q can be taken in $M_m(\mathcal{A})$. However, by regarding $M_m(\mathcal{A})$ as the finite-rank endomorphisms of $\mathcal{A}^m$, we can find column vectors $w_1, \dots, w_m, z_1, \dots, z_m \in \mathcal{A}^m$ such that $q = \Theta_{w_1, z_1} + \cdots + \Theta_{w_m, z_m}$. Since, moreover,

$$q = q^2 = \sum_{i=1}^{m} \Theta_{qw_i, z_i} = \sum_{i=1}^{m} \Theta_{w_i, qz_i},$$

we can choose $w_i, z_i \in q\mathcal{A}^m$. Thus $\mathcal{E} := q\mathcal{A}^m$ is a finitely generated projective right hermitian $\mathcal{A}$-module, and by [14, Lemma 2.16] $\mathcal{E}$ may be completed in norm to a right C^*-module $\overline{\mathcal{E}}$ over A. It is now routine to show that $E \simeq \overline{\mathcal{E}}$ as right A-modules. □

Lemma 2.4. *Let $E = {}_BE_A$ be a Morita equivalence bimodule between the unital C^*-algebras B and A. Suppose moreover that $\mathcal{B} \subset B$ and $\mathcal{A} \subset A$ are unital Fréchet pre-C^*-algebras. Then* *there are $n, m \in \mathbb{N}$ and projectors $p \in M_n(\mathcal{B})$ and $q \in M_m(\mathcal{A})$ such that the left hermitian $\mathcal{B}$-module $\mathcal{E}_1 := \mathcal{B}^n p$ and the right hermitian $\mathcal{A}$-module $\mathcal{E}_2 := q\mathcal{A}^m$ both have C^*-module completions isomorphic to E.*

Proof. Since E is a Morita equivalence bimodule between unital C^*-algebras, there are isomorphisms $\operatorname{End}^0_A(E) \simeq B$ and $\operatorname{End}^0_B(E) \simeq A$, whence the identity map 1_E is a compact endomorphism for both module structures.

Using Lemma 2.3, we can write ${}_BE \simeq B^n p$ and $E_A \simeq qA^m$ for some $n, m \in \mathbb{N}$, $p \in M_n(\mathcal{B})$ and $q \in M_m(\mathcal{A})$. We may identify $\mathcal{E}_1 = \mathcal{B}^n p$ with a $\mathcal{B}$-submodule of E and $\mathcal{E}_2 = q\mathcal{A}^m$ with an $\mathcal{A}$-submodule of E, under these isomorphisms. Since both module structures induce the same norm on E, given by

$$\|e\|^2 := {}_{\mathcal{B}}(e \mid e) = (e \mid e)_{\mathcal{A}}\,, \tag{2.4}$$

by [14, Prop. 3.11], both of these submodules are norm-dense in E; thus the completions of $\mathcal{B}^n p$ and $q\mathcal{A}^m$ are each isomorphic to E. □

Proposition 2.5. *Let $\mathcal{E}$ be a pre-Morita equivalence bimodule between the unital pre-C^*-algebras $\mathcal{B}$ and $\mathcal{A}$. Then there are $n, m \in \mathbb{N}$ and projectors $p \in M_n(\mathcal{B})$ and $q \in M_m(\mathcal{A})$ that define algebra isomorphisms $\mathcal{A} \simeq pM_n(\mathcal{B})p$ and $\mathcal{B} \simeq qM_m(\mathcal{A})q$; and module isomorphisms $\mathcal{E}_{\mathcal{A}} \simeq q\mathcal{A}^m$ and ${}_{\mathcal{B}}\mathcal{E} \simeq \mathcal{B}^n p$.*

Proof. The existence of a pre-Morita equivalence bimodule entails that the pre-C^*-algebras $\mathcal{B}$ and $\mathcal{A}$ have well-defined completions to C^*-algebras, B and A respectively. (Equivalently: there is a unique continuous C^*-norm on $\mathcal{A}$ which defines A by completion in this norm; and likewise for $\mathcal{B}$.) The C^*-module completion E of $\mathcal{E}$ in the norm (2.4) is a Morita equivalence bimodule between the C^*-algebras B and A: this follows from the continuity conditions (2.2) on the inner products [14, Prop. 3.12]. Thus E is finitely generated and projective both as an A-module and as a B-module, by Lemma 2.4. Moreover we can choose the projectors p, q describing these module structures to lie over $\mathcal{B}$ and $\mathcal{A}$ respectively.

The vector space $\mathcal{I} = {}_{\mathcal{B}}(\mathcal{E} \mid \mathcal{E})$ is a dense ideal in $\mathcal{B}$, since $\mathcal{E}$ is $\mathcal{B}$-full by hypothesis, and it is also contained in ${}_B(E \mid E)$, which is thereby a dense ideal of B. (Recall that $\mathcal{B} \hookrightarrow B$ is a continuous dense inclusion.) Therefore, ${}_B(E \mid E) = B$ since the unital C^*-algebra B cannot have a proper dense ideal. But the pre-C^*-subalgebra $\mathcal{B}$ of B also has this "good" property, since $\{\, b \in \mathcal{B} : \|1 - b\|_B < 1 \,\}$ is an open neighbourhood of 1 in $\mathcal{B}$ from continuity of the inclusion $\mathcal{B} \hookrightarrow B$, and therefore the dense ideal $\mathcal{I}$ of $\mathcal{B}$ cannot be proper either.

Therefore ${}_{\mathcal{B}}(\mathcal{E} \mid \mathcal{E}) = \mathcal{B}$; and by the same token, $(\mathcal{E} \mid \mathcal{E})_{\mathcal{A}} = \mathcal{A}$. Thus we can write

$$1 = \sum_{i=1}^{m} {}_{\mathcal{B}}(x_i \mid y_i) \in \mathcal{B}, \qquad 1 = \sum_{k=1}^{n} (w_k \mid z_k)_{\mathcal{A}} \in \mathcal{A},$$

for some $x_i, y_i, w_k, z_k \in \mathcal{E}$. Therefore, the maps $e \mapsto \big[{}_{\mathcal{B}}(e \mid w_k)\big]_k$ and $e \mapsto \big[(y_i \mid e)_A\big]_i$ described in (2.3) yield isomorphisms $\mathcal{E} \simeq \mathcal{B}^n p$ of left $\mathcal{B}$-modules and $\mathcal{E} \simeq q\mathcal{A}^m$ of right $\mathcal{A}$-modules where

$$p := \big[{}_{\mathcal{B}}(z_l \mid w_k)\big]_{l,k} \in M_n(\mathcal{B}), \qquad q := \big[(y_i \mid x_j)_{\mathcal{A}}\big]_{i,j} \in M_m(\mathcal{A}).$$

The $*$-algebra isomorphisms $\mathcal{A} \simeq pM_n(\mathcal{B})p$ and $\mathcal{B} \simeq qM_m(\mathcal{A})q$ are now routine. □

Lemma 2.6. *Let $\mathcal{A}$ be a subalgebra of a unital C^*-algebra A with $1 \in \mathcal{A}$ and let $\mathcal{E} = q\mathcal{A}^m$ a finitely generated projective right $\mathcal{A}$-module. Then every Hermitian pairing on $\mathcal{E}$ is of the form*

$$(e \mid f)_{\mathcal{A}} = (e \mid f)_r \equiv \textstyle\sum_{j,k} e_j^* r_{jk} f_k \tag{2.5}$$

where $e = (e_1, \dots, e_m)^T$ with each $e_j \in \mathcal{A}$ and $qe = e$; similarly for f; and $r = [r_{jk}] \in qM_m(\mathcal{A})q$ is positive.

Proof. Write $x_j = qu_j = \sum_k q_{kj} u_k \in q\mathcal{A}^m$, where $\{u_1, \dots, u_m\}$ is the standard basis of $\mathcal{A}^m$. These x_j generate $\mathcal{E}$ as a right $\mathcal{A}$-module: $e = \sum_j x_j e_j$ for any $e \in \mathcal{E}$.

If $(\cdot \mid \cdot)_{\mathcal{A}}$ is a hermitian pairing on $\mathcal{E}$, then $0 \leqslant (e \mid e)_{\mathcal{A}} = \sum_{j,k} e_j^* (x_j \mid x_k)_{\mathcal{A}} e_k$. Hence the matrix $r := [(x_j \mid x_k)_{\mathcal{A}}]_{jk} \in M_m(\mathcal{A})$ is positive in $M_m(A)$, by [13, Proposition 1.20]. Next

$$\begin{aligned}(qr)_{ik} &= \textstyle\sum_j q_{ij} (x_j \mid x_k)_{\mathcal{A}} = \sum_j (x_j q_{ji} \mid x_k)_{\mathcal{A}} = \sum_{j,l} (q_{lj} u_l q_{ji} \mid x_k)_{\mathcal{A}} \\ &= \textstyle\sum_{j,l} (q_{lj} q_{ji} u_l \mid x_k)_{\mathcal{A}} = \sum_l (q_{li} u_l \mid x_k)_{\mathcal{A}} = (x_i \mid x_k)_{\mathcal{A}} = r_{ik},\end{aligned}$$

and similarly $rq = r$. $\square$

Lemma 2.7. *Let $\mathcal{A}$ be a dense pre-C^*-subalgebra of the unital C^*-algebra A with $1 \in \mathcal{A}$. If $q \in M_m(\mathcal{A})$ is a projector and if $(\cdot \mid \cdot)_{\mathcal{A}}$ is an $\mathcal{A}$-valued hermitian pairing on $\mathcal{E} = q\mathcal{A}^m$ making $\mathcal{E}$ full, then it coincides with the pairing* (2.5) *for some positive invertible $r \in qM_m(\mathcal{A})q$.*

Proof. By [14, Lemma 2.16], the Hermitian form $(\cdot \mid \cdot)_{\mathcal{A}}$ on $\mathcal{E}$ has a canonical extension to an A-valued pairing on the completion $E = qA^m$, which is a full right A-module. By Lemma 2.6, the original inner product and this extension are both given by the formula (2.5), for some positive element $r \in qM_m(\mathcal{A})q$.

The compact endomorphisms of the full right A-module E are given by $\operatorname{End}_A^0(E) = qM_m(A)q$; this algebra is generated by the rank-one operators $\Theta^r_{e,f} \colon E \to E : g \mapsto e(f \mid g)_r$. In terms of the standard pairing on $E = qA^m$ given by $(e \mid f)_A := \sum_{j,k} e_j^* q_{jk} f_k$, it follows that $\Theta^r_{e,f}(g) = \Theta_{e,f}(rg)$. If r were not invertible in $qM_m(A)q$, the operators $\Theta^r_{e,f} = \Theta_{e,f}\, r$ and their adjoints would generate a proper two-sided ideal of this C^*-algebra, contradicting fullness of E.

We remark in passing that r is invertible in $qM_m(A)q$ if and only if $r + (1 - q)$ is invertible in $M_m(A)$, with inverse $r^{-1} + (1 - q)$, as is easily checked. Using the stability under the holomorphic functional calculus of $M_m(\mathcal{A})$, we find that r^{-1} also lies in $qM_m(\mathcal{A})q$. $\square$

Later on, we shall consider Hilbert spaces arising as completions of finitely generated projective modules.

Definition 2.8. Let $\mathcal{A}$ be a unital pre-C^*-algebra, $\mathcal{E}$ a right hermitian $\mathcal{A}$-module and $\psi \colon \mathcal{A} \to \mathbb{C}$ a faithful bounded positive linear functional. Let $L^2(\mathcal{E}, \psi)$ denote the Hilbert space completion of $\mathcal{E}$ with respect to the scalar product

$$\langle e \mid f \rangle := \psi\big((e \mid f)_{\mathcal{A}}\big).$$

Proposition 2.9. *Let $\mathcal{H}_\infty \subset \mathcal{H}$ be a dense subspace of the Hilbert space $\mathcal{H}$, and suppose that $\mathcal{H}_\infty$ is a finite projective right hermitian $\mathcal{A}$-module, $\mathcal{H}_\infty \simeq q\mathcal{A}^m$. Suppose moreover that $\psi \colon \mathcal{A} \to \mathbb{C}$ is a faithful bounded positive linear functional such that $\mathcal{H} = L^2(\mathcal{H}_\infty, \psi)$.*

Let $T \colon \mathcal{H}_\infty \to \mathcal{H}_\infty$ be right $\mathcal{A}$-linear. Then T extends to a bounded operator on $\mathcal{H}$.

Remark 2.10. Of course Proposition 2.9 applies equally well to left modules when the operator T is linear over the left action of $\mathcal{A}$.

Proof. The projective right $\mathcal{A}$-module $\mathcal{H}_\infty \simeq q\mathcal{A}^m$ has a finite set of generators $\xi_1, \dots, \xi_m$ such that any $\xi \in \mathcal{H}_\infty$ may be written as

$$\xi = \textstyle\sum_{j=1}^m \xi_j a_j = \sum_{j,r=1}^m \xi_j q_{jr} a_r, \quad \text{for some} \quad a_1, \dots, a_m \in \mathcal{A}.$$

The generators may thus be taken to satisfy the relations $\xi_r = \sum_{j=1}^m \xi_j q_{jr}$, for $r = 1, \dots, m$. If $\eta = \sum_{j=1}^m \xi_j b_j$ also, the $\mathcal{A}$-valued inner product of $\eta, \xi \in \mathcal{H}_\infty$ coming from this isomorphism is well defined by

$$(\eta \mid \xi)_{\mathcal{A}} := \sum_{j,r=1}^m b_j^* q_{jr} a_r.$$

The right $\mathcal{A}$-linearity of T gives $T\xi = \sum_{j=1}^m (T\xi_j) a_j$ and $T\xi_r = \sum_{j=1}^m (T\xi_j) q_{jr}$. Thus

$$(T\xi_j \mid T\xi_k)_{\mathcal{A}} = \sum_{r,s} q_{jr}\, (T\xi_r \mid T\xi_s)_{\mathcal{A}}\, q_{sk}.$$

The inner product $(T\xi \mid T\xi)_{\mathcal{A}}$ may be expanded as follows:

$$\begin{aligned}
(T\xi \mid T\xi)_{\mathcal{A}} &= \sum_{j,k} a_j^* (T\xi_j \mid T\xi_k)_{\mathcal{A}}\, a_k = \sum_{j,k,r,s} a_j^* q_{jr}\, (T\xi_r \mid T\xi_s)_{\mathcal{A}}\, q_{sk} a_k \\
&= \sum_{r,s} (\xi \mid \xi_r)_{\mathcal{A}} (T\xi_r \mid T\xi_s)_{\mathcal{A}} (\xi_s \mid \xi)_{\mathcal{A}} = \sum_{r,s} \big(T\xi_r\, (\xi_r \mid \xi)_{\mathcal{A}} \,\big|\, T\xi_s\, (\xi_s \mid \xi)_{\mathcal{A}}\big)_{\mathcal{A}} \\
&= \sum_{r,s} \big(\Theta_{T\xi_r, \xi_r} \xi \,\big|\, \Theta_{T\xi_s, \xi_s} \xi\big)_{\mathcal{A}} = \sum_{r,s} \big(\xi \,\big|\, \Theta_{\xi_r (T\xi_r \mid T\xi_s)_{\mathcal{A}}, \xi_s} \xi\big)_{\mathcal{A}} \\
&\leqslant \sum_{r,s} \big\|\Theta_{\xi_r (T\xi_r \mid T\xi_s)_{\mathcal{A}}, \xi_s}\big\|\, (\xi \mid \xi)_{\mathcal{A}}\,.
\end{aligned}$$

In the last line here the norm is the endomorphism norm (of the C^*-completion), which satisfies $\|\Theta_{\xi,\eta}\| = \|(\eta \mid \xi)_{\mathcal{A}}\|$ for $\xi, \eta \in \mathcal{H}_\infty$ by [14, Lemma 2.30].

Now we can estimate the operator norm of T; for $\xi \in \mathcal{H}_\infty$ we get the bound

$$\begin{aligned}
\langle T\xi \mid T\xi \rangle = \psi\big((T\xi \mid T\xi)_{\mathcal{A}}\big) &\leqslant \sum_{r,s=1}^m \big\|\big(\xi_s \,\big|\, \xi_r (T\xi_r \mid T\xi_s)_{\mathcal{A}}\big)_{\mathcal{A}}\big\|\, \psi\big((\xi \mid \xi)_{\mathcal{A}}\big) \\
&= \sum_{r,s=1}^m \big\|\big(\xi_s \,\big|\, \xi_r (T\xi_r \mid T\xi_s)_{\mathcal{A}}\big)_{\mathcal{A}}\big\|\, \langle \xi \mid \xi \rangle\,.
\end{aligned}$$

Therefore, T extends by continuity to a bounded operator on $\mathcal{H}$, with

$$\|T\|^2 \leqslant \sum_{r,s=1}^m \big\|(\xi_s \mid \xi_r)_{\mathcal{A}} (T\xi_r \mid T\xi_s)_{\mathcal{A}}\big\|. \qquad \square$$

3 Bimodule connections and Kasparov modules

In this section we shall use the Kasparov product to produce new spectral triples from old ones, in the presence of a Morita equivalence bimodule. This is essentially the noncommutative formulation of twisting an elliptic operator by a vector bundle. Spectral triples yield unbounded Kasparov modules, so we use the work of Kucerovsky [6] to implement this product. The main technical requirement is the construction of suitable bimodule connections to set up the unbounded Kasparov product.

3.1 Connections on bimodules and the first order condition

In this subsection the $*$-algebras $\mathcal{A}$ and $\mathcal{B}$ will always be unital *Fréchet pre-C^*-algebras* with a unique continuous C^*-norm; and $\mathcal{E}$ will denote a $\mathcal{B}$-$\mathcal{A}$-bimodule with a complete locally convex topology for which the module operations are continuous. Moreover, we shall assume that $\mathcal{E}$ carries both a left-linear inner product with values in $\mathcal{B}$ and a right-linear inner product with values in $\mathcal{A}$, such that the right action of $\mathcal{A}$ on $\mathcal{E}$ is $\mathcal{B}$-linear and adjointable with respect to ${}_{\mathcal{B}}(\cdot \mid \cdot)$; and vice versa.

We do not, at this stage, assume any fullness conditions, so that $\mathcal{E}$ need not be a pre-Morita equivalence bimodule between $\mathcal{B}$ and $\mathcal{A}$, although this is of course the main example.

Let $\operatorname{End}_{\mathcal{A}}(\mathcal{E})$ be the $*$-algebra of $\mathcal{A}$-linear adjointable maps from $\mathcal{E}$ to $\mathcal{E}$. Such maps need not, *a priori*, be bounded with respect to the C^*-module norm on $\mathcal{E}$ coming from $(\cdot \mid \cdot)_{\mathcal{A}}$; although we shall have occasion later, in the finitely generated and projective case, to obtain such bounds by applying Proposition 2.9. Similarly, let $\operatorname{End}_{\mathcal{B}}(\mathcal{E})$ be the $*$-algebra of $\mathcal{B}$-linear adjointable maps on $\mathcal{E}$. Our assumptions give us the inclusions:

$$\mathcal{A} \subseteq \operatorname{End}_{\mathcal{B}}(\mathcal{E}) \text{ acting on the right}, \qquad \mathcal{B} \subseteq \operatorname{End}_{\mathcal{A}}(\mathcal{E}) \text{ acting on the left}.$$

If $a \in \mathcal{A}$, we write a° for the corresponding right multiplication operator in $\operatorname{End}_{\mathcal{B}}(\mathcal{E})$, since the right action of $\mathcal{A}$ on $\mathcal{E}$ gives a left action of the opposite algebra $\mathcal{A}^\circ$.

We need to deal with tensor products over the Fréchet algebras $\mathcal{A}$ and $\mathcal{B}$. We recall that the completed projective tensor product (over $\mathbb{C}$) of $\mathcal{A}$ with itself, which we write simply as $\mathcal{A} \otimes \mathcal{A}$ rather than $\mathcal{A} \mathbin{\widehat{\otimes}} \mathcal{A}$, is a Fréchet space. Since the multiplication of a Fréchet algebra is jointly continuous, the kernel $\Omega^1\mathcal{A}$ of the corresponding linear map $m\colon \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ is closed and thus is a Fréchet space, as well as being an $\mathcal{A}$-$\mathcal{A}$-bimodule for which the left and right actions of $\mathcal{A}$ on $\Omega^1\mathcal{A}$ are continuous: see Section II.4 of [19].

Likewise, we may define *balanced* tensor products such as $\mathcal{E} \otimes_{\mathcal{A}} \Omega^1\mathcal{A}$, by quotienting the (completed) projective tensor product $\mathcal{E} \otimes \Omega^1\mathcal{A}$ by the closure of the linear span of the tensors $e\,a \otimes \omega - e \otimes a\,\omega$, where $e \in \mathcal{E}$, $a \in \mathcal{A}$ and $\omega \in \Omega^1\mathcal{A}$. Thus $\mathcal{E} \otimes_{\mathcal{A}} \Omega^1\mathcal{A}$ is again a complete locally convex space, and moreover is a topological $\mathcal{B}$-$\mathcal{A}$-bimodule, by [19, Prop. 5.15].

► Assume further that $\mathcal{E}$ is projective both as a right $\mathcal{A}$-module and as a left $\mathcal{B}$-module. This assumption enables us to choose connections [13,20]:

$$\nabla_{\mathcal{B}} : \mathcal{E} \to \Omega^1\mathcal{B} \otimes_{\mathcal{B}} \mathcal{E}, \qquad \nabla_{\mathcal{A}} : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega^1\mathcal{A},$$

which are $\mathbb{C}$-linear maps such that, for all $a \in \mathcal{A}$, $b \in \mathcal{B}$, $e \in \mathcal{E}$,

$$\nabla_{\mathcal{B}}(be) = db \otimes e + b\,\nabla_{\mathcal{B}}(e), \qquad \nabla_{\mathcal{A}}(ea) = e \otimes da + \nabla_{\mathcal{A}}(e)\,a.$$

The graded differential algebra $\Omega^\bullet\mathcal{A}$ is densely generated by elements $a \in \mathcal{A}$, $da \in \Omega^1\mathcal{A}$ subject to the preexisting algebra relations of $\mathcal{A}$, the derivation rule $d(ab) = a\,db + da\,b$, and the relations

$$d(a_0\,da_1 \cdots da_k) = da_0\,da_1 \cdots da_k, \qquad d(da_1\,da_2 \cdots da_k) = 0.$$

Note that $\Omega^0\mathcal{A} = \mathcal{A}$, so $\mathcal{A}$ is to be regarded as a subalgebra of $\Omega^\bullet\mathcal{A}$. Also, since $\mathcal{A}$ is a $*$-algebra, $\Omega^\bullet\mathcal{A}$ becomes a $*$-algebra by adding the rule $(da)^* = -d(a^*)$.

The connection $\nabla_\mathcal{A}$ extends to an operator on the module $\mathcal{E} \otimes_\mathcal{A} \Omega^\bullet\mathcal{A}$, using a graded Leibniz rule, and similarly for $\nabla_\mathcal{B}$ on the module $\Omega^\bullet\mathcal{B} \otimes_\mathcal{B} \mathcal{E}$.

While $\nabla_\mathcal{A}$ does not intertwine the left $\mathcal{B}$-module structures of $\mathcal{E}$ and $\Omega^1\mathcal{B} \otimes_\mathcal{B} \mathcal{E}$ (and similarly for $\nabla_\mathcal{B}$), the following linearity relations do hold.

Lemma 3.1. *With the notation as above, all $a \in \mathcal{A}$ and $b \in \mathcal{B}$ satisfy*

$$[\nabla_\mathcal{B}, b] \in \mathrm{Hom}_\mathcal{A}(\mathcal{E}, \Omega^1\mathcal{B} \otimes_\mathcal{B} \mathcal{E}), \qquad [\nabla_\mathcal{A}, a^\circ] \in \mathrm{Hom}_\mathcal{B}(\mathcal{E}, \mathcal{E} \otimes_\mathcal{A} \Omega^1\mathcal{A}).$$

Proof. This follows immediately from the definition of connection, since if $e \in \mathcal{E}$ then

$$\nabla_\mathcal{B}(be) - b\,\nabla_\mathcal{B}(e) = db \otimes e.$$

Since $\Omega^1\mathcal{B} \otimes_\mathcal{B} \mathcal{E}$ and $\mathcal{E}$ carry the same action of $\mathcal{A}$ on the right of $\mathcal{E}$, the first statement is proved. If $a \in \mathcal{A}$, an identical argument shows that $[\nabla_\mathcal{A}, a^\circ]$ respects the left actions of $\mathcal{B}$. $\square$

Suppose now that we are given a pair of $*$-homomorphisms:

$$c_\mathcal{B} : \Omega^\bullet\mathcal{B} \to \mathrm{End}_\mathcal{A}(\mathcal{E}), \qquad c_\mathcal{A} : (\Omega^\bullet\mathcal{A})^\circ \to \mathrm{End}_\mathcal{B}(\mathcal{E}).$$

We suppose, for compatibility with the tensor products, that $c_\mathcal{B}\big|_{\Omega^0\mathcal{B}}$ agrees with the left action of $\mathcal{B}$ on $\mathcal{E}$, and similarly for $c_\mathcal{A}$. With this assumption, we also obtain adjointable module maps

$$\gamma_\mathcal{B} : \Omega^\bullet\mathcal{B} \otimes_\mathcal{B} \mathcal{E} \to \mathcal{E}, \qquad \gamma_\mathcal{A} : \mathcal{E} \otimes_\mathcal{A} \Omega^\bullet\mathcal{A} \to \mathcal{E}, \tag{3.1}$$

given by $\gamma_\mathcal{B}(\beta \otimes e) := c_\mathcal{B}(\beta)(e)$ and $\gamma_\mathcal{A}(e \otimes \alpha) := c_\mathcal{A}(\alpha^\circ)(e)$.

Example 3.2. The main classical example we have in mind is a Clifford module. Let M be a closed C^∞ manifold with Riemannian metric g, and $E \to M$ a smooth complex vector bundle with the additional property that each fibre E_x, $x \in M$, is a module for $\mathbb{C}\ell_x$, the complex Clifford algebra for T_x^*M with inner product given by g_x^{-1}. Denoting the exterior derivative by $\underline{\mathrm{d}}$ and the Clifford action at $x \in M$ by c_x, we can define an algebra homomorphism

$$c_M : \Omega^\bullet C^\infty(M) \to \mathrm{End}_{C^\infty(M)}\big(C^\infty(M, E)\big)$$

by

$$c_M(f_0\,df_1 \,\cdots df_k)s : x \longmapsto f_0(x)c_x(\underline{\mathrm{d}}f_1)\dots c_x(\underline{\mathrm{d}}f_k)s(x), \quad f_j \in C^\infty(M),\ s \in C^\infty(M, E).$$

It is straightforward to check that $\mathcal{A} = \mathcal{B} = C^\infty(M)$ is a Fréchet pre-C^*-algebra and that $\mathcal{E} = C^\infty(M, E)$ is a $\mathbb{Z}_2$-graded hermitian bimodule over this algebra with continuous actions; and that the restriction of the left Clifford action to functions is just the multiplication of sections by smooth functions.

Returning to the general case, we note the following preliminary result.

Lemma 3.3. *With the notation as above, define two linear operators on $\mathcal{E}$ by $\mathcal{D}_{\mathcal{B}} := \gamma_{\mathcal{B}} \circ \nabla_{\mathcal{B}}$ and $\mathcal{D}_{\mathcal{A}} := \gamma_{\mathcal{A}} \circ \nabla_{\mathcal{A}}$. Then $[\mathcal{D}_{\mathcal{B}}, b] \in \mathrm{End}_{\mathcal{A}}(\mathcal{E})$ with $[\mathcal{D}_{\mathcal{B}}, b]^* = -[\mathcal{D}_{\mathcal{B}}, b^*]$ for all $b \in \mathcal{B}$ and $[\mathcal{D}_{\mathcal{A}}, a^\circ] \in \mathrm{End}_{\mathcal{B}}(\mathcal{E})$ with $[\mathcal{D}_{\mathcal{A}}, a^\circ]^* = -[\mathcal{D}_{\mathcal{A}}, (a^\circ)^*]$ for all $a \in \mathcal{A}$.*

Proof. The right $\mathcal{A}$-linearity of $[\mathcal{D}_{\mathcal{B}}, b]$ is a straightforward check:

$$\begin{aligned}
[\mathcal{D}_{\mathcal{B}}, b](ea) - ([\mathcal{D}_{\mathcal{B}}, b]e)a &= \gamma_{\mathcal{B}}\nabla_{\mathcal{B}}(bea) - b\,\gamma_{\mathcal{B}}\nabla_{\mathcal{B}}(ea) - \gamma_{\mathcal{B}}\nabla_{\mathcal{B}}(be)a + b(\gamma_{\mathcal{B}}\nabla_{\mathcal{B}}(e))a \\
&= \gamma_{\mathcal{B}}\big(\nabla_{\mathcal{B}}(bea) - b\,\nabla_{\mathcal{B}}(ea)\big) - \gamma_{\mathcal{B}}\big(\nabla_{\mathcal{B}}(be) + b\,\nabla_{\mathcal{B}}(e)\big)a \\
&= \gamma_{\mathcal{B}}(db \otimes ea) - \gamma_{\mathcal{B}}(db \otimes e)a \\
&= \gamma_{\mathcal{B}}((db \otimes e)a) - \gamma_{\mathcal{B}}(db \otimes e)a = 0,
\end{aligned}$$

since $\gamma_{\mathcal{B}}$ is a right $\mathcal{A}$-module map.

For the adjointability, the previous calculation, with $a = 1$, gives

$$[\mathcal{D}_{\mathcal{B}}, b](e) = \gamma_{\mathcal{B}}(db \otimes e) = c_{\mathcal{B}}(db)(e),$$

and by definition, $c_{\mathcal{B}}(db) \in \mathrm{End}_{\mathcal{A}}(\mathcal{E})$ is assumed adjointable. Thus also, $[\mathcal{D}_{\mathcal{B}}, b]^* = c_{\mathcal{B}}(db)^* = -c_{\mathcal{B}}(d(b^*)) = -[\mathcal{D}_{\mathcal{B}}, b^*]$ since $c_{\mathcal{B}}$ is a $*$-homomorphism. The analysis of $[\mathcal{D}_{\mathcal{A}}, a^\circ]$ follows the same pattern. ◻

Example 3.4. In the context of Example 3.2, Lemma 3.3 defines a Dirac-type operator $\mathcal{D}$ on the smooth sections $C^\infty(M, E)$ of the Clifford bundle; that is to say, a first-order differential operator satisfying $[\mathcal{D}, f] = c_M(\underline{\mathrm{d}}f)$ for all $f \in C^\infty(M)$. Such Dirac-type operators typically are of the form $\mathcal{D} = c_M \circ \nabla^E$ where $\nabla^E\colon C^\infty(M, E) \to C^\infty(M, T^*M \otimes E)$ is a Clifford connection: see [21, Sect. 3.3].

We have used the left action of $C^\infty(M)$ on $C^\infty(M, E)$ to define the Dirac operator. In this example we could also use the right action to define another Dirac operator. The relation between these two possible definitions plays a prominent role in [12], where supersymmetry is used to discuss how one can model additional geometric structures (oriented, spin, complex, Kähler, . . .).

Remark 3.5. Lemma 3.3 gives an algebraic version of the first order condition for spectral triples (see below) in a very general setting. We can make some interesting deductions about the operators $\gamma \circ \nabla$ that can be defined in this way. The lemma gives the commutator equations

$$[[\mathcal{D}_{\mathcal{A}}, a^\circ], b] = 0, \qquad [[\mathcal{D}_{\mathcal{B}}, b], a^\circ] = 0,$$

which are respectively equivalent to

$$[[\mathcal{D}_{\mathcal{A}}, b], a^\circ] = 0, \qquad [[\mathcal{D}_{\mathcal{B}}, a^\circ], b] = 0.$$

This observation allows us to say a little more about commutators with the connection itself as well, and strengthens the linearity properties of commutators with $\mathcal{D}_{\mathcal{A}}$ and $\mathcal{D}_{\mathcal{B}}$.

Lemma 3.6. *For all $a \in \mathcal{A}$, the map $[\nabla_{\mathcal{B}}, a^\circ] : \mathcal{E} \to \Omega^1\mathcal{B} \otimes_{\mathcal{B}} \mathcal{E}$ is left $\mathcal{B}$-linear. Similarly, $[\nabla_{\mathcal{A}}, b] : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega^1\mathcal{A}$ is right $\mathcal{A}$-linear for all $b \in \mathcal{B}$.*

Proof. For $a \in \mathcal{A}$, $b \in \mathcal{B}$, $e \in \mathcal{E}$ we find that

$$\begin{aligned} [\,[\nabla_{\mathcal{B}}, a^\circ], b]\, e &= [\nabla_{\mathcal{B}}, a^\circ]\, be - b\,[\nabla_{\mathcal{B}}, a^\circ]\, e = \nabla_{\mathcal{B}}(bea) - \nabla_{\mathcal{B}}(be)a - b\,\nabla_{\mathcal{B}}(ea) + b\,\nabla_{\mathcal{B}}(e)a \\ &= [\nabla_{\mathcal{B}}, b](ea) - ([\nabla_{\mathcal{B}}, b]e)a = [\,[\nabla_{\mathcal{B}}, b], a^\circ]\, e = 0, \end{aligned}$$

since we know from Lemma 3.1 that $[\nabla_{\mathcal{B}}, b]$ is right $\mathcal{A}$-linear. □

Corollary 3.7. *For any scalars λ_1, λ_2, and all $a \in \mathcal{A}$, $[\lambda_1 \mathcal{D}_{\mathcal{A}} + \lambda_2 \mathcal{D}_{\mathcal{B}}, a^\circ] \in \mathrm{End}_{\mathcal{B}}(\mathcal{E})$; and if $b \in B$, then $[\lambda_1 \mathcal{D}_{\mathcal{A}} + \lambda_2 \mathcal{D}_{\mathcal{B}}, b] \in \mathrm{End}_{\mathcal{A}}(\mathcal{E})$.*

Theorem 3.8. *Suppose that $\mathcal{D}\colon \mathcal{E} \to \mathcal{E}$ satisfies $[\mathcal{D}, a^\circ] \in \mathrm{End}_{\mathcal{B}}(\mathcal{E})$ and $[\mathcal{D}, b] \in \mathrm{End}_{\mathcal{A}}(\mathcal{E})$, for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$. For given connections $\nabla_{\mathcal{B}}$ and $\nabla_{\mathcal{A}}$ on $\mathcal{E}$, there exist module maps $\gamma_{\mathcal{B}}$, $\gamma_{\mathcal{A}}$ as in* (3.1) *and endomorphisms $T \in \mathrm{End}_{\mathcal{B}}(\mathcal{E})$ and $S \in \mathrm{End}_{\mathcal{A}}(\mathcal{E})$ such that*

$$\mathcal{D} = \gamma_{\mathcal{A}} \circ \nabla_{\mathcal{A}} + S = \gamma_{\mathcal{B}} \circ \nabla_{\mathcal{B}} + T.$$

Proof. Given such a $\mathcal{D}$ we define $\gamma_{\mathcal{B}}\colon \Omega^1\mathcal{B} \otimes_{\mathcal{B}} \mathcal{E} \to \mathcal{E}$ by $\gamma_{\mathcal{B}}(db \otimes e) := [\mathcal{D}, b]\, e$, and likewise define $\gamma_{\mathcal{A}}\colon \mathcal{E} \otimes_{\mathcal{A}} \Omega^1\mathcal{A} \to \mathcal{E}$ by $\gamma_{\mathcal{A}}(e \otimes da) := [\mathcal{D}, a^\circ]\, e$.

The associated maps $c_{\mathcal{B}}\colon b_0\, db_1 \mapsto b_0\,[\mathcal{D}, b_1]$ and $c_{\mathcal{A}}\colon (a_0\, da_1)^\circ \mapsto [\mathcal{D}, a_1^\circ]\, a_0^\circ$ are defined on $\Omega^1\mathcal{B}$ and $(\Omega^1\mathcal{A})^\circ$ respectively. These extend to algebra homomorphisms $\Omega^\bullet\mathcal{B} \to \mathrm{End}_{\mathcal{A}}(\mathcal{E})$ and $(\Omega^\bullet\mathcal{A})^\circ \to \mathrm{End}_{\mathcal{B}}(\mathcal{E})$, respectively. We may then check that

$$T := \mathcal{D} - \gamma_{\mathcal{B}} \circ \nabla_{\mathcal{B}} \in \mathrm{End}_{\mathcal{B}}(\mathcal{E}), \qquad S := \mathcal{D} - \gamma_{\mathcal{A}} \circ \nabla_{\mathcal{A}} \in \mathrm{End}_{\mathcal{A}}(\mathcal{E}).$$ □

To deal eventually with general KK-classes, we need to take into account $\mathbb{Z}_2$-graded algebras. So let $\mathcal{E}$ be a $\mathcal{B}$-$\mathcal{A}$-bimodule as above, but now suppose that $\mathcal{B}$ is a $\mathbb{Z}_2$-graded algebra, and that $\mathcal{E}$ is $\mathbb{Z}_2$-graded by ε such that $\varepsilon b_\pm \varepsilon = \pm b_\pm$ where b_+ and b_- are the even and odd components of $b \in \mathcal{B}$. We assume that $\mathcal{A}$ commutes with the grading ε. When we assume that $\mathcal{E} \simeq \mathcal{B}^n p$ is projective over $\mathcal{B}$, we shall always take $p = p^2$ in the even subalgebra of $M_n(\mathcal{B})$. Denoting the graded commutator by $[\cdot,\cdot]_\pm$ we obtain the following variant of Theorem 3.8.

Lemma 3.9. *If $\mathcal{D}\colon \mathcal{E} \to \mathcal{E}$ satisfies the graded first-order condition $[[\mathcal{D}, b]_\pm, a^\circ] = 0$, for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$, then for a given connection $\nabla_{\mathcal{B}}$ on $\mathcal{E}$, there exists a module map $\gamma_{\mathcal{B}}$ such that $\mathcal{D} = \varepsilon\gamma_{\mathcal{B}} \circ \nabla_{\mathcal{B}} + \varepsilon T$ where $T\colon \mathcal{E} \to \mathcal{E}$ is left $\mathcal{B}$-linear.*

Proof. The proof of Theorem 3.8 applies, with only the following differences. We define the map $\gamma_{\mathcal{B}}\colon \Omega^1\mathcal{B} \otimes_{\mathcal{B}} \mathcal{E} \to \mathcal{E}$ by $\gamma_{\mathcal{B}}(db \otimes e) := [\varepsilon\mathcal{D}, b]\, e$, using here the ordinary commutator (to ensure that the Leibniz rule is represented correctly). Also, $[\varepsilon\mathcal{D}, b_\pm] = \varepsilon[\mathcal{D}, b_\pm]_\mp$. Then, choosing a connection $\nabla_{\mathcal{B}}$, we find that

$$T := \varepsilon\, \mathcal{D} - \gamma_{\mathcal{B}} \circ \nabla_{\mathcal{B}}$$

is $\mathcal{B}$-linear. Hence $\mathcal{D} = \varepsilon\, \gamma_{\mathcal{B}} \circ \nabla_{\mathcal{B}} + \varepsilon\, T$. □

Recall that a connection ∇ on a right $\mathcal{A}$-module $\mathcal{E}$ is said to be *compatible with the $\mathcal{A}$-valued inner product* $(\cdot \mid \cdot)_{\mathcal{A}}$ if

$$d\big((e \mid f)_{\mathcal{A}}\big) = (e \mid \nabla f)_{\mathcal{A}} - (\nabla e \mid f)_{\mathcal{A}} \quad \text{for all} \quad e, f \in \mathcal{E},$$

where on the right hand side the hermitian pairings are extended to take values in $\Omega^1\mathcal{A}$, as follows: if $\nabla f = \sum_i g_i \otimes \alpha_i$ with $g_i \in \mathcal{E}$ and $\alpha_i \in \Omega^1\mathcal{A}$, then

$$(e \mid \nabla f)_{\mathcal{A}} := \textstyle\sum_i (e \mid g_i)_{\mathcal{A}}\, \alpha_i, \qquad (\nabla f \mid e)_{\mathcal{A}} := \sum_i \alpha_i^* \,(g_i \mid e)_{\mathcal{A}}\,.$$

For a hermitian left $\mathcal{B}$-module, compatibility of a connection ∇ is expressed, *mutatis mutandis*, by $d\big({}_{\mathcal{B}}(e \mid f)\big) = -{}_{\mathcal{B}}(e \mid \nabla f) + {}_{\mathcal{B}}(\nabla e \mid f)$.

A compatible connection on a finitely generated projective right $\mathcal{A}$-module $\mathcal{E} = q\mathcal{A}^m$ is of the form $\nabla = q \circ (d \otimes 1_m) + A$, where $A \in \operatorname{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \Omega^1\mathcal{A})$ is self-adjoint, in the sense that $(Ae \mid f)_{\mathcal{A}} = (e \mid Af)_{\mathcal{A}}$ in $\Omega^1\mathcal{A}$, for all $e, f \in \mathcal{E}$ [20, Prop. III.3.6].

3.2 Spectral triples

In this section we recall the definition of spectral triples and those basic features and additional properties we need to discuss the Kasparov product.

Definition 3.10. A *spectral triple* $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ consists of a unital[2] $*$-algebra $\mathcal{A}$, faithfully represented by bounded operators on a Hilbert space $\mathcal{H}$ (we write simply a for the operator representing an element $a \in \mathcal{A}$); together with a selfadjoint operator $\mathcal{D}$ on $\mathcal{H}$, with dense domain $\operatorname{Dom}\mathcal{D}$, such that $\langle\mathcal{D}\rangle^{-1} \equiv (1 + \mathcal{D}^2)^{-1/2}$ is a compact operator and, for each $a \in \mathcal{A}$, $a(\operatorname{Dom}\mathcal{D}) \subseteq \operatorname{Dom}\mathcal{D}$ and the commutator $[\mathcal{D}, a]$ extends to a bounded operator on $\mathcal{H}$.

The spectral triple is said to be *even* if there is a selfadjoint unitary operator $\Gamma = \Gamma^*$ on $\mathcal{H}$ (so that $\Gamma^2 = 1$ and thus Γ determines a $\mathbb{Z}_2$-grading on $\mathcal{H}$), for which $[\Gamma, a] = 0$ for all $a \in \mathcal{A}$ and $\Gamma\mathcal{D} + \mathcal{D}\Gamma = 0$. (Note the consequence for even spectral triples that $[\mathcal{D}, a] \in \mathcal{A}$ only if $[\mathcal{D}, a] = 0$.) If no such grading is available, the spectral triple is called *odd*.

Remark 3.11. Since $\mathcal{A}$ is faithfully represented on $\mathcal{H}$, we regard $\mathcal{A}$ as a $*$-subalgebra of $\mathcal{B}(\mathcal{H})$; its norm closure A is a C^*-algebra.

Remark 3.12. We can talk about *even* spectral triples for $\mathbb{Z}_2$-graded algebras simply by interpreting all commutators $[\mathcal{D}, a]$, $[\Gamma, a]$ as graded commutators. While this is also possible for odd spectral triples, it is not appropriate from a KK-point of view.

Example 3.13. The Dirac-type operator of a Clifford bundle $E \to M$ on a closed C^∞ manifold, alluded to in Example 3.4, gives rise to a spectral triple $\big(C^\infty(M), L^2(M, E), \mathcal{D} = c_M \circ \nabla^E\big)$ over the algebra $C^\infty(M)$.

Definition 3.14. The operator $\mathcal{D}$ gives rise to two (commuting) derivations of operators on $\mathcal{H}$; we shall denote them by

$$\underline{\mathrm{d}}T := [\mathcal{D}, T], \qquad \delta T := [|\mathcal{D}|, T], \quad \text{for} \quad T \in \mathcal{B}(\mathcal{H}).$$

Note that $\mathcal{A}$ lies within $\operatorname{Dom}\underline{\mathrm{d}} := \{\, T \in \mathcal{B}(\mathcal{H}) : T(\operatorname{Dom}\mathcal{D}) \subseteq \operatorname{Dom}\mathcal{D};\ [\mathcal{D}, T] \in \mathcal{B}(\mathcal{H}) \,\}$.

A spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is called QC^∞, in the terminology of [22], if $\mathcal{A} + \underline{\mathrm{d}}\mathcal{A} \subseteq \operatorname{Dom}^\infty \delta$. (The terms *regular* [1, 13] and *smooth* [23] are synonymous with QC^∞.)

[2] Spectral triples can also be defined over nonunital algebras; but those are not needed for the present purpose.

Remark 3.15. One may replace the derivation $\delta = [|\mathcal{D}|, \cdot]$ by $\tilde{\delta} := [\langle\mathcal{D}\rangle, \cdot]$ in the definition of a QC^∞ spectral triple, since $\mathrm{Dom}^\infty \tilde{\delta} = \mathrm{Dom}^\infty \delta$, as is easily checked. This is often useful to sidestep issues that arise when $\ker \mathcal{D} \neq 0$.

Definition 3.16. If $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a QC^∞ spectral triple, one can gift $\mathcal{A}$ with a locally convex topology, finer than the norm topology of A, defined by the family of seminorms

$$q_m(a) := \|\delta^m a\| \quad \text{and} \quad q'_m(a) := \|\delta^m([\mathcal{D}, a])\|, \quad m = 0, 1, 2, \dots \tag{3.2}$$

for which the involution $a \mapsto a^*$ is continuous. (Note that q_0 is just the operator norm of A.) Or one can replace the seminorms $\{\, q_m : m \in \mathbb{N} \,\}$ by the equivalent family of seminorms [3]:

$$p_m(a) := \|\rho_m(a)\|, \quad \text{where} \quad \rho_m(a) := \begin{pmatrix} a & \delta(a) & \cdots & \delta^m(a) \\ 0 & a & \ddots & \vdots \\ \vdots & \ddots & a & \delta(a) \\ 0 & \cdots & 0 & a \end{pmatrix}.$$

The seminorms p_m are submultiplicative: $p_m(ab) \leqslant p_m(a)\, p_m(b)$, since ρ_m is a representation of $\mathcal{A}$. If $\mathcal{A}$ is *complete* in this topology, then $\mathcal{A}$ is a *Fréchet algebra*[3] for which the seminorms q'_m are continuous, by [3, Prop. 2.2].

Alternatively, if $\mathcal{A}$ is not complete in the topology given by the seminorms (3.2), one can replace $\mathcal{A}$ by its completion $\mathcal{A}_\delta$. Assuming that $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a QC^∞ spectral triple, it follows from [23, Lemma 16] that $(\mathcal{A}_\delta, \mathcal{H}, \mathcal{D})$ is also a QC^∞ spectral triple, and moreover that $\mathcal{A}_\delta$ is a pre-C^*-algebra.

Thus, whenever we are given a QC^∞ spectral triple, we may and shall always assume, by completing its algebra $\mathcal{A}$ if necessary, that $\mathcal{A}$ is a *Fréchet pre-C^*-algebra*.

If $T \in \mathrm{Dom}\,\delta^m$ and $\xi \in \mathrm{Dom}\,|\mathcal{D}|^m$, then $T\xi \in \mathrm{Dom}\,|\mathcal{D}|^m$ and the equality

$$|\mathcal{D}|^m T\xi = \sum_{k=0}^{m} \binom{m}{k} \delta^k(T)\, |\mathcal{D}|^{m-k} \xi \tag{3.3}$$

holds (by induction on m). There is a similar formula with δ and $|\mathcal{D}|$ replaced by $\tilde{\delta}$ and $\langle\mathcal{D}\rangle$, if desired. Thus, if $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is QC^∞, then the subspace

$$\mathcal{H}_\infty := \mathrm{Dom}^\infty \mathcal{D} = \mathrm{Dom}^\infty |\mathcal{D}| = \mathrm{Dom}^\infty \langle\mathcal{D}\rangle$$

is mapped to itself by any $a \in \mathcal{A}$.

Definition 3.17. Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a spectral triple and $\mathcal{I} \subset \mathcal{K}(\mathcal{H})$ a (two-sided) symmetric ideal of compact operators. We say that $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is $\mathcal{I}$-*summable* if $\langle\mathcal{D}\rangle^{-1} \in \mathcal{I}$.

Let $\mathcal{L}^s = \mathcal{L}^s(\mathcal{H})$, for $s \geqslant 1$, be the Schatten ideal of operators T for which $|T|^s$ is trace-class. If the spectral triple is $\mathcal{L}^s$ summable for all $s > p$ (with $p \geqslant 1$), then $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is *finitely summable*, and the infimum of such p is called its *spectral dimension*. This holds true in the important special cases where we can take $\mathcal{I} = \mathcal{L}^{p,\infty}$ or the larger ideal $\mathcal{Z}_p$ studied in [24] and [25]. (A positive operator A lies in $\mathcal{Z}_p$ if and only if $A^p \in \mathcal{Z}_1 = \mathcal{L}^{1,\infty}$, the Dixmier ideal.)

[3] A Fréchet algebra is defined to be a complete locally convex algebra whose topology is defined by a countable family of *submultiplicative* seminorms. Note that the seminorm $p'_m(a) := p_m([\mathcal{D}, a])$ is not submultiplicative, but the sum $p_m + p'_m$ will be.

The next interesting property of spectral triples, namely, the first order condition, only makes sense for spectral triples defined over tensor products of algebras.

Definition 3.18. The notation $(\mathcal{A} \otimes \mathcal{B}, \mathcal{H}, \mathcal{D})$ for a spectral triple means that two algebras $\mathcal{A}$ and $\mathcal{B}$ are faithfully represented on $\mathcal{H}$ by *commuting* bounded operators, so that the tensor product $\mathcal{A} \otimes \mathcal{B}$ acts on $\mathcal{H}$ and elements of $\mathcal{A} \otimes \mathcal{B}$ have bounded commutators with $\mathcal{D}$.

We say that the spectral triple $(\mathcal{A} \otimes \mathcal{B}, \mathcal{H}, \mathcal{D})$ satisfies the *first order condition* if $[[\mathcal{D}, a], b] = 0$ for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$.

Definition 3.19. If $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is an even spectral triple, we shall use the notation $\mathcal{C} \equiv \mathcal{C}_{\mathcal{D}}(\mathcal{A})$ for the $\mathbb{Z}_2$-graded subalgebra of $\mathcal{B}(\mathcal{H})$ generated by $\mathcal{A}$ (of even degree) and $\{ [\mathcal{D}, a] : a \in \mathcal{A} \}$ (of odd degree). There is an algebra homomorphism $\pi_{\mathcal{D}} \colon \Omega^\bullet \mathcal{A} \to \mathcal{C}_{\mathcal{D}}(\mathcal{A})$ given by

$$\pi_{\mathcal{D}}(a_0\, da_1 \cdots da_k) := a_0\, [\mathcal{D}, a_1] \cdots [\mathcal{D}, a_k]. \tag{3.4}$$

If $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is an odd spectral triple, we can consider the even spectral triple $(\mathcal{A}, \mathcal{H} \oplus \mathcal{H}, \mathcal{D}')$ where $\mathcal{A}$ acts diagonally, $\mathcal{D}' = \begin{pmatrix} \mathcal{D} & 0 \\ 0 & -\mathcal{D} \end{pmatrix}$, and the grading is given by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then we obtain a $\mathbb{Z}_2$-grading on $\mathcal{C}_{\mathcal{D}'}(\mathcal{A})$.

Recall that $\Omega^\bullet \mathcal{A}$ becomes an involutive algebra by setting $(da)^* := -d(a^*)$; then $\pi_{\mathcal{D}}$ becomes a $*$-representation of the differential forms, and so also the Hochschild chains [26], of $\mathcal{A}$ by operators on $\mathcal{H}$.

Proposition 3.20. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \Gamma)$ be an even spectral triple and suppose that $\mathcal{H}_\infty$ is a finite projective left $\mathcal{A}$-module and that $\mathcal{H} = L^2(\mathcal{H}_\infty, \psi)$. Then with*

$$\mathcal{B} = \{ T \in \mathcal{B}(\mathcal{H}) : T(\mathcal{H}_\infty) \subseteq \mathcal{H}_\infty,\ [T, \Gamma] = 0,\ [T, w] = 0 \textit{ for } w \in \mathcal{C}_{\mathcal{D}}(\mathcal{A}) \}, \tag{3.5}$$

we find that $(\mathcal{A} \otimes \mathcal{B}, \mathcal{H}, \mathcal{D}, \Gamma)$ is an even spectral triple satisfying the first order condition.

Proof. All we need to show is the boundedness of $[\mathcal{D}, T]$ for $T \in \mathcal{B}$. However for $a \in \mathcal{A}$ we get

$$[[\mathcal{D}, T], a] = -[T, [\mathcal{D}, a]] = 0,$$

because T commutes with $\mathcal{C}_{\mathcal{D}}(\mathcal{A})$ by assumption. Hence $[\mathcal{D}, T]$ is left $\mathcal{A}$-linear and maps $\mathcal{H}_\infty$ to itself, so Proposition 2.9 implies that $[\mathcal{D}, T]$ is bounded. The first order condition is obvious. □

Example 3.21. In the context of Examples 3.4 and 3.13, Proposition 3.20 shows that the data $\left(C^\infty(M) \otimes C^\infty(M), L^2(M, E), \mathcal{D} = c_M \circ \nabla^E\right)$ form a spectral triple. This follows since both the left and right actions of $C^\infty(M)$ commute with the action of the Clifford algebra, and moreover the smooth sections of E are finite projective over $C^\infty(M)$ and form the smooth domain of $\mathcal{D}$; see [23].

Remark 3.22. If $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \Gamma)$ is QC^∞ then $(\mathcal{A} \otimes \mathcal{B}, \mathcal{H}, \mathcal{D}, \Gamma)$ is QC^∞ for the action of $\mathcal{A}$, but not necessarily for the action of $\mathcal{B}$. Thus we shall say "QC^∞ for $\mathcal{A}$" in such cases, when considering spectral triples defined over a tensor product of algebras.

Later we shall also want some information about the $\mathcal{A}$-module structure of $\mathcal{C}_{\mathcal{D}}(\mathcal{A})$. It is immediate that $\mathcal{C}_{\mathcal{D}}(\mathcal{A})$ is an $\mathcal{A}$-bimodule. Regarding $\mathcal{A}$-valued inner products on $\mathcal{C}_{\mathcal{D}}(\mathcal{A})$, the following result is helpful. Recall that an *operator-valued weight* is a positive linear map $\Psi \colon \mathcal{C} \to \mathcal{A}$ from a $*$-algebra $\mathcal{C}$ onto a $*$-subalgebra $\mathcal{A}$ that satisfies $\Psi(awb) = a\,\Psi(w)\,b$ for $w \in \mathcal{C}$ and $a, b \in \mathcal{A}$; thus it behaves like a conditional expectation except that it need not be unit-preserving [27, Appendix A], nor need it extend to the C^*-completion of $\mathcal{C}$ as a bounded map.

Lemma 3.23. *Let $\mathcal{C}$ be a unital $*$-algebra and let $\mathcal{A}$ be a unital $*$-subalgebra with the same unit* 1 *(i.e., the inclusion $\mathcal{A} \hookrightarrow \mathcal{C}$ is unit-preserving). The existence of a left $\mathcal{A}$-valued inner product ${}_{\mathcal{A}}(\cdot|\cdot)$ on $\mathcal{C}$ such that right multiplication of $\mathcal{C}$ on itself defines an adjointable action is equivalent to the existence of a faithful operator-valued weight $\Psi\colon \mathcal{C} \to \mathcal{A}$.*

Proof. Suppose first that ${}_{\mathcal{A}}(\cdot \mid \cdot)$ is an $\mathcal{A}$-valued left inner product on $\mathcal{C}$. Define

$$\Psi : \mathcal{C} \to \mathcal{A} \quad \text{by} \quad \Psi(w) := {}_{\mathcal{A}}(w1 \mid 1) = {}_{\mathcal{A}}(w \mid 1).$$

Then, assuming right multiplication to be adjointable, we get, for $a, b \in \mathcal{A}$ and $w \in \mathcal{C}$,

$$\begin{aligned}\Psi(awb) &= {}_{\mathcal{A}}(awb \mid 1) = a\, {}_{\mathcal{A}}(wb \mid 1) \\ &= a\, {}_{\mathcal{A}}(w \mid b^*) = a\, {}_{\mathcal{A}}(w \mid 1)\, b = a\, \Psi(w)\, b.\end{aligned}$$

Similarly, positivity of Ψ follows from

$$\Psi(w^*w) = {}_{\mathcal{A}}(w^*w \mid 1) = {}_{\mathcal{A}}(w^* \mid w^*) \geqslant 0,$$

with equality if and only if $w^* = 0$, if and only if $w = 0$.

Conversely, suppose that a faithful operator valued weight $\Psi : \mathcal{C} \to \mathcal{A}$ is given. Define

$${}_{\mathcal{A}}(u \mid v) := \Psi(uv^*), \quad \text{for} \quad u, v \in \mathcal{C}.$$

One verifies easily that this defines a positive definite left $\mathcal{A}$-module hermitian form on $\mathcal{C}$, for the left action of $\mathcal{A}$ coming from the inclusion $\mathcal{A} \subset \mathcal{C}$. The adjointability of right multiplication by $\mathcal{C}$ is clear. □

Remark 3.24. If ${}_{\mathcal{A}}(1|1) = 1 \in \mathcal{A}$ then the corresponding operator-valued weight Ψ is an expectation, and conversely.

3.3 Kasparov products using bimodule connections

In order to make the passage from spin^c to Riemannian manifolds, we shall employ unbounded Kasparov products as described in [6].

In this subsection we assume that $(\mathcal{A} \otimes \mathcal{B}^\circ, \mathcal{H}, \mathcal{D}, \varepsilon)$ is an even spectral triple (in the $\mathbb{Z}_2$-graded sense for the algebra $\mathcal{B}$) satisfying smoothness and first order conditions. We also require that $\mathcal{H}_\infty$ be finitely generated and projective as a right module over $\mathcal{B}$.

From subsection 3.1, this means that we can represent $\mathcal{D}$ using a $\mathcal{B}$-compatible connection, so that $\mathcal{D} = \varepsilon(\gamma \circ \nabla^{\mathcal{H}}_{\mathcal{B}}) + \varepsilon T$ where T is $\mathcal{B}$-linear, $\nabla^{\mathcal{H}}_{\mathcal{B}} : \mathcal{H}_\infty \to \mathcal{H}_\infty \otimes_{\mathcal{B}} \Omega^1\mathcal{B}$, and

$$\gamma : \mathcal{H}_\infty \otimes_{\mathcal{B}} \Omega^1\mathcal{B} \to \mathcal{H}_\infty \quad \text{is given by} \quad \gamma(\xi \otimes b_0\, db_1) = [\varepsilon\mathcal{D}, b_1^\circ](\xi\, b_0). \tag{3.6}$$

(The grading ε is required only when $\mathcal{B}$ is $\mathbb{Z}_2$-graded and we employ a graded first order condition. Otherwise, just put $\varepsilon = 1$.) Note that γ is right $\mathcal{B}$-linear since

$$\begin{aligned}\gamma(\xi \otimes b_0\, db_1\, b_2) &= \gamma(\xi \otimes b_0\, d(b_1 b_2)) - \gamma(\xi \otimes b_0 b_1\, db_2) \\ &= [\varepsilon\mathcal{D}, b_2^\circ b_1^\circ](\xi\, b_0) - [\varepsilon\mathcal{D}, b_2^\circ](\xi\, b_0 b_1) = b_2^\circ\, [\varepsilon\mathcal{D}, b_1^\circ](\xi\, b_0) \\ &= [\varepsilon\mathcal{D}, b_1^\circ](\xi\, b_0) b_2 = \gamma(\xi \otimes b_0\, db_1) b_2.\end{aligned}$$

Assume now that we are also given a $\mathcal{B}$-$\mathcal{C}$*-bimodule* $\mathcal{E}$ which is finitely generated and projective as a left $\mathcal{B}$-module, so $\mathcal{E} \simeq \mathcal{B}^n q$ where $q \in M_n(\mathcal{B})$ is a projector. We take the $\mathcal{B}$-valued inner product ${}_{\mathcal{B}}(\cdot \mid \cdot)$ on $\mathcal{E}$ given by this identification. Choose an (arbitrary but fixed) connection $\nabla^{\mathcal{E}}_{\mathcal{B}} \colon \mathcal{E} \to \Omega^1\mathcal{B} \otimes_{\mathcal{B}} \mathcal{E}$ compatible with this inner product. We let ε' be a $\mathbb{Z}_2$-grading of $\mathcal{E}$ such that

$$\varepsilon' b_\pm \varepsilon' = \pm\, b_\pm \text{ for } b \in \mathcal{B}, \qquad c\varepsilon' = \varepsilon' c \text{ for } c \in \mathcal{C}.$$

We marry the connections $\nabla^{\mathcal{H}}_{\mathcal{B}}$ and $\nabla^{\mathcal{E}}_{\mathcal{B}}$ in the usual way, by defining a linear map $\nabla_{\mathcal{B}}$ on the balanced tensor product $\mathcal{H}_\infty \otimes_{\mathcal{B}} \mathcal{E}$ by

$$\nabla_{\mathcal{B}} : \mathcal{H}_\infty \otimes_{\mathcal{B}} \mathcal{E} \to \mathcal{H}_\infty \otimes_{\mathcal{B}} \Omega^1\mathcal{B} \otimes_{\mathcal{B}} \mathcal{E}, \qquad \nabla_{\mathcal{B}}(\xi \otimes e) := \nabla^{\mathcal{H}}_{\mathcal{B}}(\xi) \otimes e + \xi \otimes \nabla^{\mathcal{E}}_{\mathcal{B}}(e).$$

To see that $\nabla_{\mathcal{B}}$ is indeed well defined, we remark that

$$\begin{aligned}\nabla_{\mathcal{B}}(\xi b \otimes e) &= \nabla^{\mathcal{H}}_{\mathcal{B}}(\xi b) \otimes e + \xi b \otimes \nabla^{\mathcal{E}}_{\mathcal{B}}(e) = \nabla^{\mathcal{H}}_{\mathcal{B}}(\xi) b \otimes e + \xi \otimes db \otimes e + \xi \otimes b\, \nabla^{\mathcal{E}}_{\mathcal{B}}(e) \\ &= \nabla^{\mathcal{H}}_{\mathcal{B}}(\xi) \otimes be + \xi \otimes \nabla^{\mathcal{E}}_{\mathcal{B}}(be) = \nabla_{\mathcal{B}}(\xi \otimes be).\end{aligned}$$

Similarly the operator $\varepsilon \otimes \varepsilon'$ is well-defined on the tensor product.

In consequence, the linear operator $\widehat{\mathcal{D}} \colon \mathcal{H}_\infty \otimes_{\mathcal{B}} \mathcal{E} \to \mathcal{H}_\infty \otimes_{\mathcal{B}} \mathcal{E}$ given by

$$\widehat{\mathcal{D}} := (\varepsilon \otimes \varepsilon')(\gamma \otimes 1_{\mathcal{E}}) \circ \nabla_{\mathcal{B}} + \varepsilon T \otimes \varepsilon'$$

is also well defined.

► To examine $\widehat{\mathcal{D}}$ more closely, it helps to work in a framework where the isomorphism $\mathcal{E} \simeq \mathcal{B}^n q$ is explicit. If $e = (b_1, \dots, b_n) = (b_1, \dots, b_n)q \in \mathcal{E}$, we write

$$\phi : \mathcal{H}_\infty \otimes_{\mathcal{B}} \mathcal{E} \to (\mathcal{H}_\infty \otimes \mathbb{C}^n)q, \quad \phi(\xi \otimes e) := (\xi b_1, \dots, \xi b_n)q = (\xi b_1, \dots, \xi b_n),$$

and likewise

$$\begin{gathered}\hat{\phi} : \mathcal{H}_\infty \otimes_{\mathcal{B}} \Omega^1\mathcal{B} \otimes_{\mathcal{B}} \mathcal{E} \to \mathcal{H}_\infty \otimes_{\mathcal{B}} (\Omega^1\mathcal{B})^n q, \\ \hat{\phi}(\xi \otimes \omega \otimes e) := (\xi \otimes \omega b_1, \dots, \xi \otimes \omega b_n)q = (\xi \otimes \omega b_1, \dots, \xi \otimes \omega b_n).\end{gathered}$$

Lemma 3.25. *The linear map ϕ is an $\mathcal{A}$-$\mathcal{C}$-bimodule isomorphism satisfying*

(a) $\phi \circ (\gamma \otimes 1_{\mathcal{E}}) = (\gamma \otimes 1_n) \circ \hat{\phi}$*; and*

(b) $\phi \circ \widehat{\mathcal{D}} \circ \phi^{-1} = q^\circ(\mathcal{D} \otimes \varepsilon' 1_n)q^\circ + \widehat{A}$*, where $\widehat{A}$ is a bounded and selfadjoint operator on* $(\mathcal{H} \otimes \mathbb{C}^n)q = \mathcal{H}^n q$*.*

Moreover, $\widehat{\mathcal{D}}$ is a selfadjoint operator on the Hilbert space $\mathcal{H} \otimes_{\mathcal{B}} \mathcal{E}$.

Remark 3.26. Since $1 \in \mathcal{B}$ and $q \in M_n(\mathcal{B})$ are even elements, we can define a $\mathbb{Z}_2$-grading on $\mathcal{B}^n q$ by setting $(b_1, \dots, b_n)_+ := (b_{1+}, \dots, b_{n+})$. If, abusing notation, we denote this grading by ε', then $\phi \circ (\varepsilon \otimes \varepsilon') = (\varepsilon \otimes \varepsilon') \circ \phi$. Similarly, if $T \colon \mathcal{H}_\infty \to \mathcal{H}_\infty$ is $\mathcal{B}$-linear, $\phi \circ (\varepsilon T \otimes \varepsilon') = (\varepsilon T \otimes \varepsilon') \circ \phi$.

Proof. The left $\mathcal{A}$-linearity of ϕ is obvious; its right $\mathcal{C}$-linearity comes from that of the identification $\mathcal{E} \simeq \mathcal{B}^n q$.

Ad (a): Using the right $\mathcal{B}$-linearity of γ, it suffices to evaluate

$$\begin{aligned}\phi \circ (\gamma \otimes 1_{\mathcal{E}})(\xi \otimes \omega \otimes e) &= \phi(\gamma(\xi \otimes \omega) \otimes e) = (\gamma(\xi \otimes \omega)b_1, \dots, \gamma(\xi \otimes \omega)b_n) \\ &= (\gamma(\xi \otimes \omega b_1), \dots, \gamma(\xi \otimes \omega b_n)) = (\gamma \otimes 1_n)(\xi \otimes \omega b_1, \dots, \xi \otimes \omega b_n) \\ &= (\gamma \otimes 1_n) \circ \hat{\phi}(\xi \otimes \omega \otimes e).\end{aligned}$$

Ad (b): We write $\nabla^{\mathcal{E}}_{\mathcal{B}} = q^\circ(d \otimes 1_n) + A$ with $A \in \mathrm{Hom}_{\mathcal{B}}(\mathcal{E}, \Omega^1\mathcal{B} \otimes_{\mathcal{B}} \mathcal{E})$, and let u_k denote the standard unit vectors in $\mathbb{C}^n$. Then we find that

$$\begin{aligned} q^\circ(\gamma \otimes 1_n)(\nabla^{\mathcal{H}}_{\mathcal{B}} \otimes 1_n)q^\circ\big(\phi(\xi \otimes e)\big) &= q^\circ(\gamma \otimes 1_n)\big((\nabla^{\mathcal{H}}_{\mathcal{B}} \otimes 1_n)(\xi b_1, \dots, \xi b_n)\big) \\ &= q^\circ(\gamma \otimes 1_n)\big(\nabla^{\mathcal{H}}_{\mathcal{B}}(\xi)b_1, \dots, \nabla^{\mathcal{H}}_{\mathcal{B}}(\xi)b_n\big) + q^\circ(\gamma \otimes 1_n)(\xi \otimes db_1, \dots, \xi \otimes db_n) \\ &= \big(\gamma(\nabla^{\mathcal{H}}_{\mathcal{B}}(\xi))b_1, \dots, \gamma(\nabla^{\mathcal{H}}_{\mathcal{B}}(\xi))b_n\big)q + \big(\gamma(\xi \otimes db_1), \dots, \gamma(\xi \otimes db_n)\big)q \\ &= (\gamma \otimes 1_n)\big(\hat{\phi}(\nabla^{\mathcal{H}}_{\mathcal{B}}(\xi) \otimes e)\big) + \textstyle\sum_{j,k=1}^n (\gamma \otimes 1_n)\big(\hat{\phi}(\xi \otimes db_j \otimes u_k q_{kj})\big) \\ &= \phi(\gamma \otimes 1_{\mathcal{E}})\big(\nabla^{\mathcal{H}}_{\mathcal{B}}(\xi) \otimes e + \textstyle\sum_{j,k=1}^n \xi \otimes db_j \otimes u_k q_{kj}\big) \\ &= \phi(\gamma \otimes 1_{\mathcal{E}})\big(\nabla^{\mathcal{H}}_{\mathcal{B}}(\xi) \otimes e + \xi \otimes \nabla^{\mathcal{E}}_{\mathcal{B}}(e) - \xi \otimes A(e)\big) \\ &= \phi(\gamma \otimes 1_{\mathcal{E}})\big(\nabla_{\mathcal{B}}(\xi \otimes e) - \xi \otimes A(e)\big).\end{aligned}$$

Consequently, multiplying by $\varepsilon \otimes \varepsilon'$ we find

$$q^\circ(\mathcal{D} \otimes \varepsilon' 1_n)q^\circ(\phi(\xi \otimes e)) = \phi(\widehat{\mathcal{D}}(\xi \otimes e)) - \phi\big((\varepsilon \otimes \varepsilon')(\gamma \otimes 1_{\mathcal{E}})(1_{\mathcal{H}_\infty} \otimes A)(\xi \otimes e)\big).$$

On setting $\widehat{A} := \phi \circ (\varepsilon \otimes \varepsilon') \circ (\gamma \otimes 1_{\mathcal{E}}) \circ (1_{\mathcal{H}_\infty} \otimes A) \circ \phi^{-1}$, we find that

$$\phi \circ \widehat{\mathcal{D}} \circ \phi^{-1} = q^\circ(\mathcal{D} \otimes \varepsilon' 1_n)q^\circ + \widehat{A}$$

as operators on $(\mathcal{H}_\infty \otimes \mathbb{C}^n)q = \mathcal{H}^n_\infty q$.

We may now write $A = \sum_{i,j=1}^n \omega_{ij} \otimes e_{ij}$ where the e_{ij} are matrix units and $\omega_{ij} \in \Omega^1\mathcal{B}$. Mindful of (3.6), for $\rho = b_0\, db_1 \in \Omega^1\mathcal{B}$ we write $c(\rho^\circ) := [\varepsilon\mathcal{D}, b_1^\circ]\, b_0^\circ$ for the right $\mathcal{B}$-linear operator on $\mathcal{H}_\infty$ corresponding to ρ. Using Theorem 3.8, the selfadjointness of $\mathcal{D}$ and the $\mathcal{B}$-compatibility of the connection $\nabla^{\mathcal{H}}_{\mathcal{B}}$ imply that $c(\omega^\circ_{ij})^* = c(\omega^\circ_{ji})$ for each i, j. Then

$$\widehat{A}\biggl(\sum_{j=1}^n \xi b_j\, u_j\biggr) = \sum_{i,j=1}^n c(\omega^\circ_{ij})(\xi) b_i\, u_j.$$

The scalar product on $\mathcal{H}^n_\infty q$ is given by

$$\biggl\langle \sum_j \xi b_j\, u_j \biggm| \sum_k \eta b'_k\, u_k \biggr\rangle = \sum_j \langle \xi b_j \mid \eta b'_j \rangle = \sum_{j,k} \langle \xi b_j q_{jk} \mid \eta b'_k \rangle\,. \tag{3.7}$$

With these formulas, it is straightforward to check that $\widehat{A}$ is a symmetric operator on the Hilbert space $\mathcal{H}^n q$ (the completion of $\mathcal{H}^n_\infty q$ for this scalar product). Now, ignoring $\varepsilon \otimes \varepsilon'$, $\widehat{A}$ is just multiplication by the matrix with bounded entries $c(\omega^\circ_{ij})$, which is manifestly bounded; so $\widehat{A}$ is a bounded selfadjoint operator on $\mathcal{H}^n q$.

The Hilbert space $\mathcal{H} \otimes_{\mathcal{B}} \mathcal{E}$ is (by definition) the completion of $\mathcal{H}_\infty \otimes_{\mathcal{B}} \mathcal{E}$ in the corresponding scalar product, so that ϕ extends to a unitary isomorphism from $\mathcal{H} \otimes_{\mathcal{B}} \mathcal{E}$ to $\mathcal{H}^n q$. To show that $\widehat{\mathcal{D}}$ is a selfadjoint operator on $\mathcal{H} \otimes_{\mathcal{B}} \mathcal{E}$, it is thus enough to show that $q^\circ(\mathcal{D} \otimes \varepsilon' 1_n)q^\circ$ is a selfadjoint operator on $\mathcal{H}^n q$, since $\widehat{A}$ delivers a bounded selfadjoint perturbation of it.

Write $\mathcal{D}_n \equiv \mathcal{D} \otimes \varepsilon' 1_n$. Observe that $q^\circ \mathcal{D}_n q^\circ$ is symmetric on the dense domain

$$\operatorname{Dom}(q^\circ \mathcal{D}_n q^\circ) = \{\, \xi \in \operatorname{Dom} \mathcal{D}_n \subset \mathcal{H}^n : \xi q = \xi \,\}.$$

The domain of the adjoint $\operatorname{Dom}((q^\circ \mathcal{D}_n q^\circ)^*)$ on $\mathcal{H}^n q$ consists of all $\xi \in \mathcal{H}^n q$ such that for all $\eta \in \operatorname{Dom}(q^\circ \mathcal{D}_n q^\circ)$ there is some $\zeta \in \mathcal{H}^n q$ for which $\langle q^\circ \mathcal{D}_n q^\circ \eta \mid \xi \rangle = \langle \eta \mid \zeta \rangle$. However, we see that

$$\langle q^\circ \mathcal{D}_n q^\circ \eta \mid \xi \rangle = \langle \mathcal{D}_n q^\circ \eta \mid q^\circ \xi \rangle = \langle \mathcal{D}_n \eta \mid \xi \rangle \,.$$

Therefore

$$\operatorname{Dom}((q^\circ \mathcal{D}_n q^\circ)^*) = \{\, \xi \in \operatorname{Dom} \mathcal{D}_n^* \subset \mathcal{H}^n : \xi q = \xi \,\} = \operatorname{Dom}(q^\circ \mathcal{D}_n q^\circ)$$

since $\mathcal{D}_n$ is selfadjoint on $\mathcal{H}^n$. Thus $q^\circ \mathcal{D}_n q^\circ$ is selfadjoint. □

We now come to an important point: the passage from $\mathcal{D}$ to $\widehat{\mathcal{D}}$ does not change the order of summability of the corresponding spectral triples.

Proposition 3.27. *If $(\mathcal{A} \otimes \mathcal{B}^\circ, \mathcal{H}, \mathcal{D})$ is $\mathcal{I}$-summable, then so also is $(\mathcal{A}, \mathcal{H} \otimes_{\mathcal{B}} \mathcal{E}, \widehat{\mathcal{D}})$, for either of the ideals $\mathcal{I} = \mathcal{L}^s$, $s \geqslant 1$; or $\mathcal{I} = \mathcal{Z}_p$, $p \geqslant 1$.*

Proof. First consider the summability of the spectral triple $(\mathcal{A}, \mathcal{H}^n q, q^\circ \mathcal{D}_n q^\circ)$. Note that

$$\begin{aligned}
(q^\circ \mathcal{D}_n q^\circ)^2 &= q^\circ \mathcal{D}_n q^\circ \mathcal{D}_n q^\circ = q^\circ [\mathcal{D}_n, q^\circ] \mathcal{D}_n q^\circ + q^\circ \mathcal{D}_n^2 q^\circ \\
&= q^\circ [\mathcal{D}_n, q^\circ][\mathcal{D}_n, q^\circ] + q^\circ [\mathcal{D}_n, q^\circ] q^\circ \mathcal{D}_n + q^\circ \mathcal{D}_n^2 q^\circ \\
&= q^\circ [\mathcal{D}_n, q^\circ]\, [\mathcal{D}_n, q^\circ] + q^\circ \mathcal{D}_n^2 q^\circ,
\end{aligned}$$

because $p\,\delta(p)p = 0$ for any projector p and any derivation δ with $p \in \operatorname{Dom}\delta$.

Since q° acts as the identity operator on $\mathcal{H}^n q$, we find that

$$\langle q^\circ \mathcal{D}_n q^\circ \rangle^{-1} = (q^\circ + (q^\circ \mathcal{D}_n q^\circ)^2)^{-1/2} = q^\circ \big(1 + \mathcal{D}_n^2 + q^\circ [\mathcal{D}_n, q^\circ][\mathcal{D}_n, q^\circ]\big)^{-1/2} q^\circ. \tag{3.8}$$

So suppose that $\langle \mathcal{D} \rangle^{-1} = (1 + \mathcal{D}^2)^{-1/2}$ is contained in the symmetric ideal $\mathcal{I} = \mathcal{I}(\mathcal{H})$ where $\mathcal{I} = \mathcal{L}^s$ or $\mathcal{Z}_p$, say. Then $\langle \mathcal{D}_n \rangle^{-1} = (1 + \mathcal{D}_n^2)^{-1/2}$ lies in $\mathcal{I}(\mathcal{H}^n)$, and so $q^\circ(1 + \mathcal{D}_n^2)^{-1/2} q^\circ$ lies in $\mathcal{I}(\mathcal{H}^n q)$.

From (3.8) it follows that

$$\begin{aligned}
\langle q^\circ \mathcal{D}_n q^\circ \rangle^{-2} &= (q^\circ + (q^\circ \mathcal{D}_n q^\circ)^2)^{-1} = \big(q^\circ (1 + \mathcal{D}_n^2) q^\circ + q^\circ [\mathcal{D}_n, q^\circ][\mathcal{D}_n, q^\circ]\big)^{-1} \\
&= q^\circ (1 + \mathcal{D}_n^2)^{-1} q^\circ - q^\circ (1 + \mathcal{D}_n^2)^{-1} q^\circ [\mathcal{D}_n, q^\circ]\, [\mathcal{D}_n, q^\circ]\, \big(q^\circ + (q^\circ \mathcal{D}_n q^\circ)^2\big)^{-1}.
\end{aligned}$$

This shows that if $q^\circ (1 + \mathcal{D}_n^2)^{-1} q^\circ \in \mathcal{I}^2$ then $(q^\circ + (q^\circ \mathcal{D}_n q^\circ)^2)^{-1} \in \mathcal{I}^2$, too. For $\mathcal{I} = \mathcal{L}^s$ or $\mathcal{I} = \mathcal{Z}_p$, this then implies that $\langle q^\circ \mathcal{D}_n q^\circ \rangle^{-1} \in \mathcal{I}$, as desired.

To finish, in view of Lemma 3.25, we only need to show that if A is a bounded selfadjoint operator on $\mathcal{H}$ and $(1+\mathcal{D}^2)^{-1/2} \in \mathcal{J}(\mathcal{H})$, then $(1+(\mathcal{D}+A)^2)^{-1/2} \in \mathcal{J}(\mathcal{H})$, too. Observe that

$$(i+\mathcal{D})^{-1} = (1+\mathcal{D}^2)^{-1/2}(1+\mathcal{D}^2)^{1/2}(i+\mathcal{D})^{-1},$$

and since $(1+\mathcal{D}^2)^{1/2}(i+\mathcal{D})^{-1}$ is unitary, we can start from $(i+\mathcal{D})^{-1} \in \mathcal{J}(\mathcal{H})$. Using the identity

$$(i+\mathcal{D}+A)^{-1} = (i+\mathcal{D})^{-1} - (i+\mathcal{D}+A)^{-1}A(i+\mathcal{D})^{-1}, \tag{3.9}$$

we conclude that $(i+\mathcal{D}+A)^{-1} \in \mathcal{J}(\mathcal{H})$ and thus $(1+(\mathcal{D}+A)^2)^{-1/2} \in \mathcal{J}(\mathcal{H})$. □

Having now constructed $\widehat{\mathcal{D}}$, one could expect that since $\mathcal{A}$ and $\mathcal{C}$ have commuting actions on $\mathcal{H} \otimes_{\mathcal{B}} \mathcal{E}$, one could produce a spectral triple over the algebra $\mathcal{A} \otimes \mathcal{C}^\circ$. In order to obtain it, we need finite projectivity under the left action of $\mathcal{A}$.

Theorem 3.28. *Let the even spectral triple $(\mathcal{A} \otimes \mathcal{B}^\circ, \mathcal{H}, \mathcal{D})$ be QC^∞ for $\mathcal{A}$, $\mathbb{Z}_2$-graded for the $\mathbb{Z}_2$-graded algebra $\mathcal{B}^\circ$, and $\mathcal{J}$-summable; and let $\mathcal{E}$ be a $\mathcal{B}$-$\mathcal{C}$-bimodule, finitely generated and projective over $\mathcal{B}$, and $\mathbb{Z}_2$-graded for $\mathcal{B}$. Then the associated spectral triple $(\mathcal{A}, \mathcal{H} \otimes_{\mathcal{B}} \mathcal{E}, \widehat{\mathcal{D}})$ is QC^∞ and $\mathcal{J}$-summable, and has the same spectral dimension.*

Moreover, if $\mathcal{H}_\infty = \mathrm{Dom}^\infty \mathcal{D}$ is finitely generated and projective as a left $\mathcal{A}$-module and if $\mathcal{H} = L^2(\mathcal{H}_\infty, \psi)$ for some positive linear functional ψ on $\mathcal{A}$, then $(\mathcal{A} \otimes \mathcal{C}^\circ, \mathcal{H} \otimes_{\mathcal{B}} \mathcal{E}, \widehat{\mathcal{D}})$ is a spectral triple satisfying the first order condition, and it is QC^∞ for the action of $\mathcal{A}$.

Proof. The left actions of $\mathcal{A}$ on $\mathcal{H}$ and $\mathcal{H} \otimes_{\mathcal{B}} \mathcal{E}$ satisfy

$$[\widehat{\mathcal{D}}, a] = (\varepsilon \otimes \varepsilon')\,(\gamma \otimes 1_{\mathcal{E}})\,[\nabla_{\mathcal{B}}, a] = \varepsilon\gamma\,[\nabla^{\mathcal{H}}_{\mathcal{B}}, a] \otimes \varepsilon' = [\mathcal{D}, a] \otimes \varepsilon', \tag{3.10}$$

since γ is left $\mathcal{A}$-linear: recall that we have assumed that the spectral triple $(\mathcal{A} \otimes \mathcal{B}^\circ, \mathcal{H}, \mathcal{D})$ satisfies the (graded) first order condition. Thus the commutators $[\widehat{\mathcal{D}}, a]$ are bounded and generate a representation of $\mathcal{C}_{\mathcal{D}}(\mathcal{A})$ on $\mathcal{H} \otimes_{\mathcal{B}} \mathcal{E}$. The right action of $\mathcal{C}$ on $\mathcal{E}$ extends in the obvious way to a representation of $\mathcal{C}^\circ$ on $\mathcal{H} \otimes_{\mathcal{B}} \mathcal{E}$ commuting with this left action of $\mathcal{C}_{\mathcal{D}}(\mathcal{A})$.

We can identify $\mathcal{H} \otimes_{\mathcal{B}} \mathcal{E}$ with $\mathcal{H}^n q$ as before – omitting the explicit isomorphism ϕ from the notation – so that $\widehat{\mathcal{D}}^2$ is a bounded perturbation of $q^\circ(\mathcal{D}^2 \otimes 1_n)q^\circ$. Since q° commutes with the left action of $\mathcal{A}$, it follows that a and $[\widehat{\mathcal{D}}, a]$ lie in $\mathrm{Dom}^\infty\big[|\widehat{\mathcal{D}}|, \cdot\big]$ so that $(\mathcal{A}, \mathcal{H} \otimes_{\mathcal{B}} \mathcal{E}, \widehat{\mathcal{D}})$ is indeed QC^∞.

For example, a short calculation shows that

$$\big(q^\circ + q^\circ(\mathcal{D}^2 \otimes 1_n)q^\circ\big)^{-1/2}\big[q^\circ(\mathcal{D}^2 \otimes 1_n)q^\circ, [\mathcal{D}, a] \otimes \varepsilon'\big] = q^\circ\big((1+\mathcal{D}^2)^{-1/2}[\mathcal{D}^2, [\mathcal{D}, a]] \otimes \varepsilon'\big)q^\circ.$$

The right hand side is bounded by the QC^∞ property of $(\mathcal{A}, \mathcal{H}, \mathcal{D})$; and the consequent boundedness of the left hand side yields the QC^∞ property for the other spectral triple: see, for instance, [3, Lemma 13.2].

The $\mathcal{J}$-summability of the associated spectral triple has already been established by Proposition 3.27.

If $\mathcal{H}_\infty$ is finitely generated and projective as a left $\mathcal{A}$-module, then so too is $\mathcal{H}_\infty \otimes_{\mathcal{B}} \mathcal{E} \simeq \mathcal{H}^n_\infty q$, which is a direct summand of $\mathcal{H}^n_\infty$. The boundedness of $[\widehat{\mathcal{D}}, c^\circ]$, for $c \in \mathcal{C}$, follows since the relation

$$[[\widehat{\mathcal{D}}, c^\circ], a] = [[\widehat{\mathcal{D}}, a], c^\circ] = 0, \quad \text{for all} \quad a \in \mathcal{A}$$

shows that $[\widehat{\mathcal{D}}, c^\circ]$, which maps $\mathcal{H}_\infty \otimes_{\mathcal{B}} \mathcal{E}$ to itself, is left $\mathcal{A}$-linear. It is easy to check that $\mathcal{H} = L^2(\mathcal{H}_\infty, \psi)$ implies that $\mathcal{H}^n q = L^2(\mathcal{H}_\infty^n q, \psi)$; thus Proposition 2.9 is applicable,[4] and establishes the boundedness of $[\widehat{\mathcal{D}}, c^\circ]$. □

Example 3.29. Let $E \to M$ be a Clifford bundle as in Example 3.2 and, given a Clifford connection ∇^E on E, let $\big(C^\infty(M) \otimes C^\infty(M), L^2(M, E), \mathcal{D} = c_M \circ \nabla^E\big)$ be the associated spectral triple (see Examples 3.4, 3.13, 3.21 and Lemma 3.3). Given another vector bundle $F \to M$, we may form a connection $\nabla^{E\otimes F} := \nabla^E \otimes 1_F + 1_E \otimes \nabla^F$ on $E \otimes F$ and a Clifford action $c_M \otimes 1_F$ on $E \otimes F$. This yields a new "twisted" Clifford bundle and so a spectral triple $\big(C^\infty(M), L^2(M, E\otimes F), (c_M \otimes 1_F) \circ \nabla^{E\otimes F}\big)$.

In [6], Kucerovsky gives sufficient conditions for an unbounded Kasparov module to represent the Kasparov product of two other unbounded Kasparov modules. A spectral triple is tantamount to an unbounded Kasparov module; a finitely generated projective (bi-)module may also be regarded as another, where the zero operator takes the place of the operator $\mathcal{D}$. We use this theory in the next proposition to show that we are computing Kasparov products when we twist our spectral triple by such bimodules. The full force of Kucerovsky's conditions is not needed in this setting, since the product simplifies considerably if one of the operators is zero.

Proposition 3.30. *Assume now that $(\mathcal{A} \otimes \mathcal{B}^\circ, \mathcal{H}_\infty, \mathcal{D})$ is an even spectral triple which is QC^∞ for $\mathcal{A}$, and that $\mathcal{H}_\infty$ is finitely generated and projective over both $\mathcal{A}$ and $\mathcal{B}$. Let $\mathcal{E}$ be a $\mathcal{B}$-$\mathcal{C}$-bimodule, finitely generated and projective over $\mathcal{B}$, and $\mathbb{Z}_2$-graded for $\mathcal{B}$. Then the spectral triple $(\mathcal{A} \otimes \mathcal{C}^\circ, \mathcal{H} \otimes_{\mathcal{B}} \mathcal{E}, \widehat{\mathcal{D}})$ represents the Kasparov product $[\mathcal{E}] \otimes_{B^\circ} [(\mathcal{A} \otimes \mathcal{B}^\circ, \mathcal{H}, \mathcal{D})] \in KK(A \otimes C^\circ, \mathbb{C})$ of the class $[\mathcal{E}] \in KK(C^\circ, B^\circ)$ and the class $[(\mathcal{A} \otimes \mathcal{B}^\circ, \mathcal{H}, \mathcal{D})] \in KK(A \otimes B^\circ, \mathbb{C})$.*

Proof. In order to handle the product of modules, it should be noted that the inner product making a hermitian left $\mathcal{B}$-module $\mathcal{E}$ into a hermitian right $\mathcal{B}^\circ$-module $\mathcal{E}'$ is given by

$$(e \mid f)_{\mathcal{B}^\circ} := \big({}_{\mathcal{B}}(e \mid f)^*\big)^\circ.$$

The module underlying the Kasparov product is $\mathcal{E}' \otimes_{\mathcal{B}^\circ} \mathcal{H}$, but this is cumbersome and hardly intuitive. Unpacking the definitions of the scalar product for $\mathcal{E}' \otimes_{\mathcal{B}^\circ} \mathcal{H}$ yields the formula of (3.7). This means that there is an isomorphism of Hilbert spaces from $\mathcal{E}' \otimes_{\mathcal{B}^\circ} \mathcal{H}$ to $\mathcal{H} \otimes_{\mathcal{B}} \mathcal{E}$ given by $e \otimes_{\mathcal{B}^\circ} \xi \mapsto \xi \otimes_{\mathcal{B}} e$, for $e \in \mathcal{E}$, $\xi \in \mathcal{H}$. It is easy to see that this isomorphism intertwines the actions of $\mathcal{A}$ and $\mathcal{C}$ on these Hilbert spaces. As a result, the module underlying the Kasparov product is $\mathcal{H} \otimes_{\mathcal{B}} \mathcal{E} \simeq \mathcal{H}^n q$.

Having dealt with the underlying module, we are left with checking the three conditions of [6, Thm. 13]. Of these three conditions, the domain compatibility and positivity constraints of that theorem are both trivial, since the operator making the left $\mathcal{B}$-module $\mathcal{E}$ a Kasparov module is zero. Thus we are left with checking the connection condition.

This condition requires that for a dense set of e in (the C^*-completion of) $\mathcal{E}$, with e homogeneous of degree ∂e (i.e., even or odd), the graded commutator

$$\left[\begin{pmatrix} \widehat{\mathcal{D}} & 0 \\ 0 & \mathcal{D} \end{pmatrix}, \begin{pmatrix} 0 & T_e \\ T_e^* & 0 \end{pmatrix}\right]_\pm := \begin{pmatrix} \widehat{\mathcal{D}} & 0 \\ 0 & \mathcal{D} \end{pmatrix} \begin{pmatrix} 0 & T_e \\ T_e^* & 0 \end{pmatrix} - (-1)^{\partial e} \begin{pmatrix} 0 & T_e \\ T_e^* & 0 \end{pmatrix} \begin{pmatrix} \widehat{\mathcal{D}} & 0 \\ 0 & \mathcal{D} \end{pmatrix}$$

should be bounded on $\operatorname{Dom} \widehat{\mathcal{D}} \oplus \operatorname{Dom} \mathcal{D}$, where $T_e \colon \mathcal{H}_\infty \to \mathcal{H}_\infty \otimes_{\mathcal{B}} \mathcal{E}$ is given by $T_e(\rho) := \rho \otimes e$, with adjoint $T_e^*(\xi \otimes f) := \xi\, {}_{\mathcal{B}}(f \mid e)$.

[4] Or rather, we need the obvious right-to-left variant of that Proposition.

With $u_1, \dots, u_n$ the standard basis vectors in $\mathcal{B}^n$; and $\xi \otimes f \in \mathcal{H}_\infty \otimes_{\mathcal{B}} \mathcal{E}$; and $e = \sum_i e_i u_i \in \mathcal{B}^n q$ even or odd; and with $\eta \in \mathcal{H}_\infty$, we compute

$$\left[\begin{pmatrix} \widehat{\mathcal{D}} & 0 \\ 0 & \mathcal{D} \end{pmatrix}, \begin{pmatrix} 0 & T_e \\ T_e^* & 0 \end{pmatrix}\right]_\pm \begin{pmatrix} \xi \otimes f \\ \eta \end{pmatrix} = \begin{pmatrix} \widehat{\mathcal{D}} & 0 \\ 0 & \mathcal{D} \end{pmatrix} \begin{pmatrix} \eta \otimes e \\ \xi\, {}_{\mathcal{B}}(f \mid e) \end{pmatrix} - (-1)^{\partial e} \begin{pmatrix} 0 & T_e \\ T_e^* & 0 \end{pmatrix} \begin{pmatrix} \widehat{\mathcal{D}}(\xi \otimes f) \\ \mathcal{D}\eta \end{pmatrix}$$

$$= \begin{pmatrix} \widehat{\mathcal{D}}(\eta \otimes e) - (-1)^{\partial e} \mathcal{D}\eta \otimes e \\ \mathcal{D}\big(\xi\, {}_{\mathcal{B}}(f \mid e)\big) - (-1)^{\partial e} T_e^* (\widehat{\mathcal{D}}(\xi \otimes f)) \end{pmatrix}$$

$$= \begin{pmatrix} \sum_i \mathcal{D}(\eta e_i) \otimes u_i + (\varepsilon T \otimes \varepsilon' + \widehat{A})(\eta \otimes e) - (-1)^{\partial e} \mathcal{D}\eta \otimes e \\ \mathcal{D}\big(\xi\, {}_{\mathcal{B}}(f \mid e)\big) - (-1)^{\partial e} \sum_i \mathcal{D}(\xi f_i)_{\mathcal{B}}(u_i \mid e) - (-1)^{\partial e} T_e^* (\varepsilon T \otimes \varepsilon' + \widehat{A})(\xi \otimes f) \end{pmatrix}$$

$$= \begin{pmatrix} \sum_i [\mathcal{D}, e_i^\circ]_\pm\, \eta \otimes u_i + (\varepsilon T \otimes \varepsilon' + \widehat{A}) T_e(\eta) \\ \sum_i [\mathcal{D}, e_i^\circ]_\pm\, \xi f_i - (-1)^{\partial e} T_e^* (\varepsilon T \otimes \varepsilon' + \widehat{A})(\xi \otimes f) \end{pmatrix} = \begin{pmatrix} R(\eta) \\ S(\xi \otimes f) \end{pmatrix},$$

where R and S are bounded operators. Thus the graded commutator is bounded for each $e \in \mathcal{B}^n q$. Therefore, the connection condition is satisfied and so $(\mathcal{A} \otimes \mathcal{C}^\circ, \mathcal{H} \otimes_{\mathcal{B}} \mathcal{E}, \widehat{\mathcal{D}})$ represents the Kasparov product. □

Remark 3.31. In the context of Example 3.29, this Proposition asserts that the KK class of the spectral triple $\big(C^\infty(M), L^2(M, E \otimes F), (c_M \otimes 1_F) \circ \nabla^{E \otimes F}\big)$ is precisely the Kasparov product of the classes of F and $\big(C^\infty(M) \otimes C^\infty(M), L^2(M, E), c_M \circ \nabla^E\big)$.

Others have observed the utility of the unbounded version of the Kasparov product [7–10]. The key reason for this utility in each case is the ability to employ connections to explicitly write down representatives of Kasparov products.

4 Manifold structures on spectral triples

In this section, $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ will always denote a spectral triple over a trivially graded unital *-algebra $\mathcal{A}$, topologised as a separable Fréchet algebra for which the operator norm on $\mathcal{B}(\mathcal{H})$ is continuous. (In particular, its norm closure A is a separable C^*-algebra.)

4.1 Spin^c manifolds in NCG

We start with a discussion of some of the conditions proposed by Connes [1, 3] to enable a reconstruction theorem for compact spin^c (and spin) manifolds. We follow the setup of [2], to which we refer for more detail on these conditions.

Condition 1 (Regularity)**.** The spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is QC^∞, as set forth in Definition 3.14. Under our completeness assumption on $\mathcal{A}$, topologised by the seminorms (3.2), $\mathcal{A}$ is then a Fréchet pre-C^*-algebra.

Condition 2 (Dimension)**.** The spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is $\mathcal{Z}_p$-summable for a fixed positive *integer* p. Thus, if Tr_ω denotes any Dixmier trace[5] corresponding to a Dixmier limit ω, the linear functional $\psi_\omega(a) := \mathrm{Tr}_\omega(a \langle \mathcal{D} \rangle^{-p})$ is defined (and positive) on $\mathcal{A}$. Indeed, since every $[\mathcal{D}, a]$ is bounded, the same formula extends ψ_ω to a positive linear functional on $\mathcal{C}_{\mathcal{D}}(\mathcal{A})$.

[5]We may choose and fix a particular Dixmier trace Tr_ω, since none of our results depend on its choice.

Condition 3 (Finiteness and Absolute Continuity)**.** The dense subspace $\mathcal{H}_\infty := \mathrm{Dom}^\infty \mathcal{D}$ of $\mathcal{H}$ is a finitely generated projective left $\mathcal{C}_\mathcal{D}(\mathcal{A})$-module, and the functional ψ_ω is faithful on $\mathcal{C}_\mathcal{D}(\mathcal{A})$ with $\mathcal{H} = L^2(\mathcal{H}_\infty, \psi_\omega)$.

Keeping always the abbreviation $\mathcal{C} \equiv \mathcal{C}_\mathcal{D}(\mathcal{A})$, we can therefore identify $\mathcal{H}_\infty \simeq \mathcal{C}^m q$ where $q \in M_m(\mathcal{C})$ is a selfadjoint idempotent. As a left $\mathcal{C}$-module, $\mathcal{H}_\infty$ has a set of generators $\xi_1, \dots, \xi_m$, and the hermitian pairing of $\xi = \sum_{j=1}^m u_j \xi_j = \sum_{j,k=1}^m u_j q_{jk} \xi_k$ and $\eta = \sum_{j=1}^m v_j \xi_j = \sum_{j,k=1}^m v_j q_{jk} \xi_k$ is given by ${}_\mathcal{C}(\xi \mid \eta) := \sum_{j,k=1}^m u_j q_{jk} v_k^*$.

The identification of $\mathcal{H}$ with $L^2(\mathcal{H}_\infty, \psi_\omega)$ means that the scalar product on $\mathcal{H}$ is given by

$$\langle \eta \mid \xi \rangle = \psi_\omega\big({}_\mathcal{C}(\xi \mid \eta)\big) = \mathrm{Tr}_\omega\big({}_\mathcal{C}(\xi \mid \eta) \, \langle \mathcal{D} \rangle^{-p}\big). \tag{4.1}$$

Remark 4.1. The left-to-right switch of the vectors η and ξ in (4.1) just takes into account that the inner product ${}_\mathcal{C}(\cdot \mid \cdot)$ on the left $\mathcal{A}$-module $\mathcal{H}_\infty$ is linear in the first variable. Compare the proof of Proposition 2.9 above, which deals with right hermitian modules.

Remark 4.2. The reason for asking for $\mathcal{H}_\infty$ to be a finitely generated projective $\mathcal{C}_\mathcal{D}(\mathcal{A})$-module, as opposed to $\mathcal{A}$-module, is to control the representation of $\mathcal{C}_\mathcal{D}(\mathcal{A})$. Classically, $\mathcal{C}_\mathcal{D}(\mathcal{A})$ is the Clifford algebra determined by some Riemannian metric, and various features of manifolds can be characterised in terms of the representation of this algebra. For example, a manifold is spin^c if and only if there is a bundle (of spinors) providing a Morita equivalence between the Clifford algebra and the algebra of functions. Later we shall also look at the representation of the Clifford algebra on the bundle of exterior forms.

Remark 4.3. By Proposition 3.20, if $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ satisfies the previous three conditions and if $\mathcal{H}_\infty$ is (also) finite projective over $\mathcal{A}$, we can amplify its algebra to $\mathcal{A} \otimes \mathcal{B}$, where $\mathcal{B}$ is the algebra of bounded operators mapping $\mathcal{H}_\infty$ to itself and commuting with the whole algebra $\mathcal{C}_\mathcal{D}(\mathcal{A})$ of Definition 3.19 – and the grading, if there is one. Then $(\mathcal{A} \otimes \mathcal{B}, \mathcal{H}, \mathcal{D})$ is a spectral triple satisfying the first order condition, and is QC^∞ for the algebra $\mathcal{A}$.

It is also $\mathcal{Z}_p$-summable and the absolute continuity (4.1) implies that $\mathrm{Tr}_\omega(w \, \langle \mathcal{D} \rangle^{-p}) > 0$ whenever $w \in \mathcal{C}_\mathcal{D}(\mathcal{A})$ is a nonzero positive operator on $\mathcal{H}$. This condition determines p uniquely; we then say that the critical summability parameter p is the *spectral dimension* of $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ or of $(\mathcal{A} \otimes \mathcal{B}, \mathcal{H}, \mathcal{D})$.

Condition 4 (Orientability)**.** The spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, of spectral dimension p, is called *orientable* if there exists a Hochschild p-cycle

$$\mathbf{c} = \textstyle\sum_{\alpha=1}^n a_\alpha^0 \otimes a_\alpha^1 \otimes \cdots \otimes a_\alpha^p \in Z_p(\mathcal{A}, \mathcal{A}) \tag{4.2}$$

such that the bounded operator

$$C \equiv \pi_\mathcal{D}(\mathbf{c}) := \textstyle\sum_\alpha a_\alpha^0 \, [\mathcal{D}, a_\alpha^1] \dots [\mathcal{D}, a_\alpha^p]$$

is invertible and selfadjoint, has square $C^2 = 1$, and satisfies

$$C\mathcal{D} - (-1)^{p-1}\mathcal{D}C = 0; \quad \text{and} \quad C\,a - a\,C = 0 \quad \text{for} \quad a \in \mathcal{A}.$$

Remark 4.4. The Hochschild class $[\mathbf{c}] \in HH_p(\mathcal{A})$ may be called the *orientation* of $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, and we call $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \mathbf{c})$ an *oriented* spectral triple.

One could ask for an orientation cycle $\mathbf{c} \in Z_p(\mathcal{A}, \mathcal{A} \otimes \mathcal{B})$, where the first tensor factors in (4.2) can be taken in $\mathcal{A} \otimes \mathcal{B}$. This case arises when the algebra and Hilbert space are finite dimensional (where the ordinary trace replaces the Dixmier trace and one sets $p = 0$); see, for instance, [28, 29]. We will not treat this more general definition here.

Remark 4.5. When p is even, the selfadjoint unitary operator $C = \pi_{\mathcal{D}}(\mathbf{c})$ commutes with $\mathcal{A}$ and anticommutes with $\mathcal{D}$; in particular, it cannot be trivial. Thus C is a $\mathbb{Z}_2$-*grading* operator, making $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ an *even* spectral triple. (However, this need not coincide with some previous defined grading Γ, if the spectral triple is already even.)

On the other hand, if p is odd, then C commutes with both $\mathcal{A}$ and $\mathcal{D}$, and is thus a central element of $\mathcal{C}_{\mathcal{D}}(\mathcal{A})$.

The basic property of the space of smooth spinors on a spin^c manifold [4] is encoded in the following condition, in which the regularity and finiteness conditions are assumed.

Condition 5 (Spin^c)**.** The subspace $\mathcal{H}_\infty$ is a *pre-Morita equivalence bimodule* between $\mathcal{C}_{\mathcal{D}}(\mathcal{A})$ and $\mathcal{A}$. That is to say: $\mathcal{H}_\infty$ is already a left module for the given action of $\mathcal{C}_{\mathcal{D}}(\mathcal{A})$ on $\mathcal{H}$, and there is a commuting right action of $\mathcal{A}$ on $\mathcal{H}_\infty$, with hermitian structures satisfying Definition 2.2.

Remark 4.6. The spin^c condition essentially fixes the "commuting algebra" $\mathcal{B}$ of (3.5) to be $\mathcal{A}^\circ$. Indeed, Proposition 2.5 establishes that the C^*-completion of $\mathcal{B}$ is A°; and certainly $\mathcal{A}^\circ \subseteq \mathcal{B}$. Any remaining ambiguity concerns smoothness of the "commuting algebra".

Remark 4.7. Some further properties of spectral triples are listed in [2]. We briefly mention them here, though they are not essential to our present purposes. The "first-order condition" is here absorbed by our Proposition 3.20. Other possible requirements are as follows.

(a) *Closedness*: Assuming Conditions 1–4, for any $a_1, \dots, a_p \in \mathcal{A}$ and $b \in \mathcal{B}$, the vanishing relation $\operatorname{Tr}_\omega(\Gamma\, [\mathcal{D}, a_1] \cdots [\mathcal{D}, a_p]\, b\, \langle\mathcal{D}\rangle^{-p}) = 0$ holds.

(b) *Connectivity*: There is an orthogonal family of projectors $p_j \in \mathcal{A}$ such that $\sum_j p_j = 1$ and $[\mathcal{D}, a] = 0$ for $a \in \mathcal{A}$ if and only if $a = \sum_j \lambda_j p_j$ for some scalars $\lambda_j \in \mathbb{C}$.

(c) *Reality*: Assuming Conditions 1–5, there is an antiunitary operator $J \colon \mathcal{H} \to \mathcal{H}$ such that $Ja^*J^{-1} = a^\circ$ for all $a \in \mathcal{A}$ (i.e., $J(\cdot)^*J^{-1}$ exchanges the left and right actions of a by bounded operators on $\mathcal{H}$). Moreover, $J^2 = \pm 1$, $J\mathcal{D}J^{-1} = \pm\mathcal{D}$ and also $J\Gamma J^{-1} = \pm\Gamma$ in the even case; these signs depend only on p mod 8 and coincide with those of the charge conjugation on spin manifolds – see [1] or [13, Sect. 9.5].

Definition 4.8. A *noncommutative oriented spin^c manifold* is an oriented spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \mathbf{c})$ satisfying Conditions 1 to 5. It may be called *spin*, rather than spin^c if it also has the reality property.

By Remark 4.6, a p-dimensional noncommutative oriented spin^c manifold $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \mathbf{c})$ defines a spectral triple $(\mathcal{A} \otimes \mathcal{A}^\circ, \mathcal{H}, \mathcal{D})$ which is p-dimensional.

We now want to characterise the inner product on $\mathcal{H}_\infty$. To do this we need to assume that the only operators on $\mathcal{H}$ commuting with both $\mathcal{A}$ and $\mathcal{D}$ are scalars. (This can be weakened by assuming instead the connectivity condition above; then we can run the following proof on each connected piece to get a similar statement.)

Proposition 4.9. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a spectral triple satisfying Conditions 1, 2 and 3. Suppose moreover that $\mathcal{H}_\infty$ is finite projective over $\mathcal{A}$, so that $\mathcal{H}_\infty \simeq p\mathcal{A}^n$ for some projector $p \in M_n(\mathcal{A})$. Assume also that $\mathcal{A}$ commutes with p and that only scalars commute with all $a \in \mathcal{A}$ and $\mathcal{D}$. Then any right Hermitian pairing on $\mathcal{H}_\infty \simeq p\mathcal{A}^n$ satisfying (4.1) is a positive multiple of the standard one.*

Proof. From Lemma 2.6, there is a positive invertible element $r \in pM_n(\mathcal{A})p$ such that the given pairing is of the form (2.5), i.e., $(\xi \mid \eta)_r \equiv \sum_{j,k} a_j^* r_{jk} b_k$ for $\xi = \sum_j \xi_j a_j$, $\eta = \sum_k \xi_k b_k$ in $\mathcal{H}_\infty$. If $a \in \mathcal{A}$, then since $ap = pa$ we get $a^*\xi = \sum_j \xi_j a^* a_j$ and $a\eta = \sum_k \xi_k a b_k \in \mathcal{H}_\infty$. The formula (4.1) then implies

$$\langle \xi \mid a\eta \rangle = \langle a^*\xi \mid \eta \rangle = \operatorname{Tr}_\omega\big((a^*\xi \mid \eta)_r \,\langle \mathcal{D} \rangle^{-p}\big) = \operatorname{Tr}_\omega\big((\xi \mid r^{-1} a r \eta)_r \,\langle \mathcal{D} \rangle^{-p}\big) = \langle \xi \mid r^{-1} a r \eta \rangle.$$

Hence $[r, a] = 0$ for all $a \in \mathcal{A}$.

Now since $\mathcal{D}$ is a selfadjoint operator on $\mathcal{H}$, we obtain, for $\xi, \eta \in \mathcal{H}_\infty$:

$$\begin{aligned} 0 = \langle \mathcal{D}\xi \mid \eta \rangle - \langle \xi \mid \mathcal{D}\eta \rangle &= \operatorname{Tr}_\omega\big(\big((\mathcal{D}\xi \mid \eta)_r - (\xi \mid \mathcal{D}\eta)_r\big) \,\langle \mathcal{D} \rangle^{-p}\big) \\ &= \operatorname{Tr}_\omega\big(\big((\mathcal{D}\xi \mid r\eta)_\mathcal{A} - (\xi \mid r\mathcal{D}\eta)_\mathcal{A}\big) \,\langle \mathcal{D} \rangle^{-p}\big) =: \langle\!\langle \mathcal{D}\xi \mid r\eta \rangle\!\rangle - \langle\!\langle r\xi \mid \mathcal{D}\eta \rangle\!\rangle \end{aligned} \tag{4.3}$$

where $\langle\!\langle \xi \mid \eta \rangle\!\rangle := \langle r^{-1}\xi \mid \eta \rangle$ defines a new Hilbert space scalar product. Since r^{-1} is bounded with bounded inverse, this new scalar product $\langle\!\langle \cdot \mid \cdot \rangle\!\rangle$ is topologically equivalent to the old one $\langle \cdot \mid \cdot \rangle$, and so $\mathcal{H}$ coincides with the completion of $\mathcal{H}_\infty$ with respect to either scalar product.

Now the right hand side of (4.3) is the quadratic form defining the commutator $[\mathcal{D}, r]$ with respect to the scalar product $\langle\!\langle \cdot \mid \cdot \rangle\!\rangle$. It vanishes on $\mathcal{H}_\infty$ and thus $[\mathcal{D}, r] = 0$. The irreducibility condition now implies that r is (a positive multiple of) the identity p in $pM_n(\mathcal{A})p$, represented by a scalar operator on $\mathcal{H}$. ◻

Proposition 4.10. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \mathbf{c})$ be a noncommutative oriented spin^c manifold such that only scalars commute with all $a \in \mathcal{A}$ and $\mathcal{D}$. Then $\mathcal{H}_\infty$ is finite projective as both a left and a right $\mathcal{A}$-module, and $\mathcal{C}_\mathcal{D}(\mathcal{A})$ is finite projective as a left or right $\mathcal{A}$-module. Moreover, the relations*

$$\langle \eta \mid \xi \rangle = \psi_\omega\big((\eta \mid \xi)_\mathcal{A}\big) = \operatorname{Tr}_\omega\big((\eta \mid \xi)_\mathcal{A} \,\langle \mathcal{D} \rangle^{-p}\big)$$

hold for all $\xi, \eta \in \mathcal{H}_\infty$. In particular, ψ_ω is faithful on $\mathcal{A}^\circ$.

Proof. Let $\mathcal{H}_\infty = \mathcal{C}^m q$, as in Condition 3. Then by Condition 5 and Proposition 2.5, we also get $\mathcal{H}_\infty \simeq p\mathcal{A}^n$ and $\mathcal{C} = pM_n(\mathcal{A})p$. Since $\mathcal{A} \subset \mathcal{C} = \mathcal{C}_\mathcal{D}(\mathcal{A})$, $\mathcal{A}$ commutes with the projector p.

That property ensures that the partial trace $\operatorname{tr}\colon \mathcal{C} \to \mathcal{A}$, defined on $T = [t_{ij}] \in pM_n(\mathcal{A})p$ by $\operatorname{tr}(T) := \sum_{i=1}^n t_{ii}$, is a well-defined operator-valued weight. It also shows that $\mathcal{C}$ is finite projective as a left or right $\mathcal{A}$-module, because $pM_n(\mathcal{A})p \subset M_n(\mathcal{A})$ is a direct summand *as an $\mathcal{A}$-module* precisely because p commutes with the action of $\mathcal{A}$. It now follows immediately that $\mathcal{H}_\infty$ is a finite projective left module over $\mathcal{A}$, since $\mathcal{H}_\infty = \mathcal{C}^m q$ is a direct summand in $\mathcal{C}^m$, which by the previous discussion is a direct summand in $\mathcal{A}^{mn^2}$.

We now observe that by Proposition 4.9 the right $\mathcal{A}$-valued inner product on $\mathcal{H}_\infty$ is given on $\xi = \sum_j \xi_j a_j$ and $\eta = \sum_k \xi_k b_k$ by

$$(\xi \mid \eta)_\mathcal{A} = \lambda \sum_{j,k} a_i^* p_{ij} b_j, \quad \text{for some} \quad \lambda > 0,$$

and we verify that

$$\begin{aligned}\operatorname{Tr}_\omega\big((\xi \mid \eta)_{\mathcal{A}}\,\langle\mathcal{D}\rangle^{-p}\big) &= \lambda\sum_{ij}\operatorname{Tr}_\omega\big(a_i^* p_{ij} b_j \langle\mathcal{D}\rangle^{-p}\big) = \lambda\sum_{ij}\operatorname{Tr}_\omega\big(p_{ij} b_j a_k^* p_{ki}\langle\mathcal{D}\rangle^{-p}\big)\\ &= \operatorname{Tr}_\omega\big(\operatorname{tr}\big({}_{\mathcal{C}}(\eta \mid \xi)\big)\langle\mathcal{D}\rangle^{-p}\big),\end{aligned}$$

the $\mathcal{C}_{\mathcal{D}}(\mathcal{A})$ inner product being determined by the Morita equivalence condition. We have used the tracial property of ψ_ω, which is due to the condition $\langle\mathcal{D}\rangle^{-1} \in \mathcal{Z}_p$ [24].

As a consequence of this calculation and of the positivity of tr, we see that $\mathcal{H} \simeq pL^2(\mathcal{A}, \psi_\omega)^n$. In this picture, the operator trace is precisely $\operatorname{Tr}_{\mathcal{H}} = \operatorname{Tr}_{L^2(\mathcal{A},\psi_\omega)} \circ \operatorname{tr}$, at least when restricted to $\mathcal{C}_{\mathcal{D}}(\mathcal{A})$. So finally we obtain

$$\operatorname{Tr}_\omega\big((\xi \mid \eta)_{\mathcal{A}}\,\langle\mathcal{D}\rangle^{-p}\big) = \operatorname{Tr}_\omega\big({}_{\mathcal{C}}(\eta \mid \xi)\,\langle\mathcal{D}\rangle^{-p}\big) = \langle\xi \mid \eta\rangle. \qquad \square$$

4.2 Riemannian manifolds in NCG

The basic structures which all oriented Riemannian manifolds share are (a) the exterior algebra of differential forms, and (b) the representation on forms of the Clifford algebra determined by the metric. Other features such as the Hodge $*$-operator, the Hodge–de Rham operator $d + d^*$ and so on, can all be obtained from these structures. Most importantly, the Clifford algebra is linearly isomorphic to the algebra of differential forms. Indeed, the differential forms provide a bimodule for the Clifford algebra. This bimodule is always $\mathbb{Z}_2$-graded by parity of forms, irrespective of whether the dimension of the manifold is even or odd. This information is captured in the following condition.

Condition 6 (Riemannian)**.** The vector space $\mathcal{H}_\infty := \operatorname{Dom}^\infty \mathcal{D}$ of $\mathcal{H}$ contains a cyclic and separating vector Φ for the action of the algebra $\mathcal{C} = \mathcal{C}_{\mathcal{D}}(\mathcal{A})$, in the algebraic sense, that is, $\mathcal{H}_\infty = \{\, w\Phi : w \in \mathcal{C}_{\mathcal{D}}(\mathcal{A})\,\}$; and $w = 0$ in $\mathcal{C}_{\mathcal{D}}(\mathcal{A})$ if and only if $w\Phi = 0$ in $\mathcal{H}_\infty$. In particular, $\mathcal{H}_\infty$ is a free left $\mathcal{C}_{\mathcal{D}}(\mathcal{A})$-module of rank one. Moreover, there exists a Hermitian pairing ${}_{\mathcal{C}}(\cdot \mid \cdot)$ on $\mathcal{H}_\infty$ such that[6]

$$\langle\eta \mid \xi\rangle = \psi_\omega\big({}_{\mathcal{C}}(\xi \mid \eta)\big) = \operatorname{Tr}_\omega\big({}_{\mathcal{C}}(\xi \mid \eta)\,\langle\mathcal{D}\rangle^{-p}\big) \quad \text{for} \quad \xi, \eta \in \mathcal{H}_\infty; \tag{4.4}$$

and such that $z = {}_{\mathcal{C}}(\Phi \mid \Phi)$ is a strictly positive central element of $\mathcal{C}_{\mathcal{D}}(\mathcal{A})$. We demand that $\mathcal{H}_\infty$ be finite projective as a left $\mathcal{A}$-module. There is also a grading operator ε on $\mathcal{H}$ such that $\varepsilon\Phi = \Phi$, making $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ an even spectral triple. (It follows that $\varepsilon(\cdot)\varepsilon$ is the parity grading of $\mathcal{C}_{\mathcal{D}}(\mathcal{A})$.) Without loss of generality, we assume that $\|\Phi\| = 1$.

Definition 4.11. A *noncommutative oriented Riemannian manifold* is an oriented spectral triple with a distinguished vector $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \mathbf{c}, \Phi)$, satisfying Conditions 1 to 4 and Condition 6.

For instance, in the commutative case of a compact oriented Riemannian manifold (M, g) without boundary, we take $\mathcal{A} = C^\infty(M)$; $\mathcal{H}$ is the Hilbert space of square-integrable differential forms of all degrees; and $\mathcal{D} = d + d^*$ is the Hodge–de Rham (or Hodge–Dirac) operator. Here $\mathcal{H}_\infty$ is the space of smooth differential forms $\Omega^\bullet(M)$, and we take Φ to be the 0-form (constant function) 1; and $\mathcal{C}_{\mathcal{D}}(\mathcal{A})$ is the complexified Clifford algebra, whose C^*-completion is $\mathbb{C}\ell(M)$.

► Let us examine some of the implications of Condition 6.

[6] In this subsection, for notational convenience, we write $\mathcal{C} = \mathcal{C}_{\mathcal{D}}(\mathcal{A})$ in all subscripts.

Remark 4.12. We have already remarked, using (3.3), that any operator in $\mathrm{Dom}^\infty \delta$ preserves $\mathcal{H}_\infty = \mathrm{Dom}^\infty \mathcal{D}$. Thus, under the assumption of regularity, Condition 1, all elements of $\mathcal{C}_\mathcal{D}(\mathcal{A})$ map $\mathcal{H}_\infty$ to itself, so $\mathcal{H}_\infty$ is indeed a left $\mathcal{C}_\mathcal{D}(\mathcal{A})$-module.

Remark 4.13. Unlike the spin^c case, we must insert finite projectivity of $\mathcal{H}_\infty$ over $\mathcal{A}$ by hand. Omitting this condition would mean that we do not obtain Kasparov's fundamental class, and we would be much more limited in the Kasparov products we could take. On the other hand, it is interesting to speculate whether continuous trace C^*-algebras over compact manifolds give (nonunital) 'Riemannian manifolds' which fail to have this finite projective property. This is related to the work of [8].

There is a positive linear functional $\sigma_\Phi \colon \mathcal{C}_\mathcal{D}(\mathcal{A}) \to \mathbb{C}$ given by

$$\sigma_\Phi(w) := \psi_\omega\big({}_\mathcal{C}(w\Phi \mid \Phi)\big) = \psi_\omega\big(w\, {}_\mathcal{C}(\Phi \mid \Phi)\big) = \psi_\omega(wz). \tag{4.5}$$

By equation (4.4) in Condition 6, σ_Φ is a *vector state*:

$$\sigma_\Phi(w) = \langle \Phi \mid w\Phi \rangle, \quad \text{for all} \quad w \in \mathcal{C}_\mathcal{D}(\mathcal{A}).$$

Lemma 4.14. *Suppose that* $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ *is a spectral triple satisfying Conditions 1, 2 and 6. Then* $\psi_\omega, \sigma_\Phi \colon \mathcal{C}_\mathcal{D}(\mathcal{A}) \to \mathbb{C}$ *are faithful traces; and* $\mathcal{H} = L^2(\mathcal{H}_\infty, \psi_\omega)$.

Proof. We have already remarked that ψ_ω is a trace on $\mathcal{C} = \mathcal{C}_\mathcal{D}(\mathcal{A})$, in view of the tracial properties of $\mathcal{Z}_p$-summable QC^∞ spectral triples [24, 25, 30].

The central element $z = {}_\mathcal{C}(\Phi \mid \Phi)$ of $\mathcal{C}$ is positive, with $\psi_\omega(z) = \|\Phi\|^2 = 1$. From (4.5),

$$\sigma_\Phi(uw) = \psi_\omega(uwz) = \psi_\omega(wzu) = \psi_\omega(wuz) = \sigma_\Phi(wu) \quad \text{for all} \quad u, w \in \mathcal{C},$$

so that σ_Φ is a trace, too.

Let $w \in \mathcal{C}$. Since $\sigma_\Phi(w^*w) = \langle \Phi \mid w^*w\Phi \rangle = \|w\Phi\|^2$, then $\sigma_\Phi(w^*w) = 0$ if and only if $w\Phi = 0$. This implies $w = 0$, since Φ is separating for $\mathcal{C}$. Hence σ_Φ is faithful.

We recall that if $a, b, c \in B(\mathcal{H})$ with $a \leqslant b$, then $c^*ac \leqslant c^*bc$. Now notice, for $0 < w \in \mathcal{C}$, that

$$\sigma_\Phi(w) = \psi_\omega(wz) = \psi_\omega(w^{1/2}zw^{1/2}) \leqslant \psi_\omega(w^{1/2}\|z\|w^{1/2}) = \|z\|\, \psi_\omega(w),$$

since ψ_ω is positive. The faithfulness of σ_Φ shows that $0 < \sigma_\Phi(w) \leqslant \|z\|\, \psi_\omega(w)$ for each positive $w \in \mathcal{C}$ with $w \neq 0$. Hence ψ_ω is faithful, too.

The equality $\mathcal{H} = L^2(\mathcal{H}_\infty, \psi_\omega)$ now follows from Definition 2.8, since $\mathcal{H}_\infty$ is a free rank-one $\mathcal{C}$-module and ψ_ω is faithful on $\mathcal{C}$. ∎

Remark 4.15. In Condition 3, the functional ψ_ω is required to be faithful. The previous lemma shows that this requirement is redundant in the presence of Condition 6.

It is immediate that $\sigma_\Phi : \mathcal{C}_\mathcal{D}(\mathcal{A}) \to \mathbb{C}$ has a normal extension $\tilde\sigma_\Phi$ to the double commutant $\mathcal{C}_\mathcal{D}(\mathcal{A})''$ given by

$$\tilde\sigma_\Phi(T) := \langle \Phi \mid T\Phi \rangle \quad \text{for all} \quad T \in \mathcal{C}_\mathcal{D}(\mathcal{A})''.$$

Lemma 4.16. *Suppose that* $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ *is a spectral triple satisfying Conditions 1, 2 and 6.*

(a) *The vector* $\Phi \in \mathcal{H}$ *is cyclic and separating for the double commutant* $\mathcal{C}_\mathcal{D}(\mathcal{A})''$.

(b) *The functional $\tilde\sigma_\Phi$ is a faithful normal finite trace on $\mathcal{C}_\mathcal{D}(\mathcal{A})''$.*

Proof. Ad (a): The dense subspace $\mathcal{H}_\infty$ of $\mathcal{H}$ equals $\mathcal{C}_\mathcal{D}(\mathcal{A})\Phi$ by hypothesis, so that $\mathcal{C}_\mathcal{D}(\mathcal{A})''\Phi$ is also dense in $\mathcal{H}$, i.e., Φ is a cyclic vector for $\mathcal{C}_\mathcal{D}(\mathcal{A})''$ in the topological sense.

Any cyclic vector associated to a tracial vector state is separating; as we show. Assume that $0 < T \in \mathcal{C}_\mathcal{D}(\mathcal{A})''$ with $T\Phi = 0$, and that $0 < T_n \in \mathcal{C}_\mathcal{D}(\mathcal{A})$ converge strongly to T. Then $T_n\Phi \to T\Phi = 0$. Hence, for each $w \in \mathcal{C}_\mathcal{D}(\mathcal{A})$,

$$\begin{aligned}\|T_n^{1/2} w\Phi\|^2 &= \langle \Phi \mid w^* T_n w \Phi\rangle = \langle \Phi \mid T_n w w^* \Phi\rangle \\ &= \langle T_n\Phi \mid w w^* \Phi\rangle \leqslant \|T_n\Phi\|^2 \|w w^*\Phi\|^2.\end{aligned}$$

The second equality uses Lemma 4.14, the tracial property of σ_Φ. Hence $\lim_n \|T_n^{1/2} w\Phi\| = 0$. As $\mathcal{C}_\mathcal{D}(\mathcal{A})\Phi$ is dense in $\mathcal{H}$, $T_n^{1/2}$ converges strongly to 0. Hence T_n converges strongly to 0. By uniqueness of strong limits in $\mathcal{C}_\mathcal{D}(\mathcal{A})''$, $T = 0$. Thus Φ is separating for $\mathcal{C}_\mathcal{D}(\mathcal{A})''$.

Ad (b): Since $\langle \Phi \mid T^*T\Phi\rangle = \|T\Phi\|^2$, it follows from the separating property of Φ on $\mathcal{C}_\mathcal{D}(\mathcal{A})''$ in (a) that $\tilde\sigma_\Phi(T^*T) = 0$ if and only if $T = 0$. Hence $\tilde\sigma_\Phi$ is faithful. It is straightforward to show that any normal extension of a trace on $\mathcal{C}_\mathcal{D}(\mathcal{A})$ is a trace on $\mathcal{C}_\mathcal{D}(\mathcal{A})''$. Hence $\tilde\sigma_\Phi$ is a faithful trace. It is evidently finite. □

Remark 4.17. Since $z = \mathbb{c}(\Phi \mid \Phi)$ is central in $\mathcal{C}_\mathcal{D}(\mathcal{A})$, it lies in $\mathcal{C}_\mathcal{D}(\mathcal{A})'$, hence z is also central in $\mathcal{C}_\mathcal{D}(\mathcal{A})''$. It is therefore pertinent to ask whether the formula (4.5) can be extended to the bicommutant, that is, whether there exists a (unique) faithful normal trace $\tau\colon \mathcal{C}_\mathcal{D}(\mathcal{A})'' \to \mathbb{C}$, such that $\tilde\sigma_\Phi(T) = \tau(Tz)$ for all $T \in \mathcal{C}_\mathcal{D}(\mathcal{A})''$, which extends the faithful trace ψ_ω on $\mathcal{C}_\mathcal{D}(\mathcal{A})$,

Such an extension to a normal trace is not trivial, since ψ_ω may fail to be strongly continuous on $\mathcal{C}_\mathcal{D}(\mathcal{A})$ because it arises from a Dixmier trace Tr_ω. It is, however, possible to construct such a τ provided one assumes explicitly that z is invertible in $\mathcal{C}_\mathcal{D}(\mathcal{A})$. We do not require this extended trace for the present purposes, so we leave it aside.

In view of Lemma 4.16, from now on we shall write simply σ_Φ, rather than $\tilde\sigma_\Phi$, to denote the vector state $\langle\Phi \mid (\cdot)\Phi\rangle$ on $\mathcal{C}_\mathcal{D}(\mathcal{A})''$ as well as on $\mathcal{C}_\mathcal{D}(\mathcal{A})$.

As a consequence of the Tomita–Takesaki theory [31] there exists an antiunitary operator $J = J_\Phi$ such that $J(\cdot)^*J : \mathcal{C}_\mathcal{D}(\mathcal{A})'' \to \mathcal{C}_\mathcal{D}(\mathcal{A})'$ is an antiisomorphism of von Neumann algebras and

$$J(T\Phi) = T^*\Phi, \quad \text{for all} \quad T \in \mathcal{C}_\mathcal{D}(\mathcal{A})''. \tag{4.6}$$

The assignment $w^\circ := Jw^*J$ enables a commuting right action of the algebra $\mathcal{C}_\mathcal{D}(\mathcal{A})$ on $\mathcal{H}$. *The first order condition is automatically satisfied*:

$$[a, b^\circ] = 0, \quad [[\mathcal{D}, a], b^\circ] = 0, \quad \text{for all} \quad a, b \in \mathcal{A}.$$

Lemma 4.18. *Suppose* $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \mathbf{c}, \Phi)$ *satisfies Conditions 1, 2, 4 and 6. Write $C = \pi_\mathcal{D}(\mathbf{c})$, let ε be the grading operator, and let J be the Tomita conjugation* (4.6)*. Then*

(a) $[J, \varepsilon] = 0$*;*

(b) $\varepsilon = CJCJ = CC^\circ$ *when p is even;*

(c) $\varepsilon C = -C\varepsilon$ *and $C = JCJ = C^\circ$, when p is odd;*

(d) *J maps $\mathcal{H}_\infty$ to $\mathcal{H}_\infty$ bijectively;*

(e) *$\mathcal{H}_\infty$ is a pre-Morita equivalence bimodule between $\mathcal{C}_\mathcal{D}(\mathcal{A})$ and itself.*

Proof. The Tomita conjugation J for the vector Φ is the unique antilinear operator defined by $J(T\Phi) := T^*\Phi$ for all $T \in \mathcal{C}_\mathcal{D}(\mathcal{A})''$. This operator is bounded and antiunitary since the vector state σ_Φ is a trace. Note that $J\Phi = \Phi$ since $\mathcal{C}_\mathcal{D}(\mathcal{A})'' \supset \mathcal{C}_\mathcal{D}(\mathcal{A})$ is unital. The additional properties that $J^2 = 1$, and that $w = Jw^*J$ if and only if $w \in \mathcal{C}_\mathcal{D}(\mathcal{A})$ is central, then follow from the cyclic and separating properties of the vector Φ.

Ad (a): To see that $[J, \varepsilon] = 0$, take $w = w_\mathrm{e} + w_\mathrm{o} \in \mathcal{C}_\mathcal{D}(\mathcal{A})$, where $\varepsilon\, w_\mathrm{e}\, \varepsilon = w_\mathrm{e}$ and $\varepsilon\, w_\mathrm{o}\, \varepsilon = -w_\mathrm{o}$. Since ε is selfadjoint, we get $w^* = (w^*)_\mathrm{e} + (w^*)_\mathrm{o} = (w_\mathrm{e})^* + (w_\mathrm{o})^*$. Then

$$\varepsilon J w\Phi = \varepsilon w^*\Phi = (w_\mathrm{e}^* - w_\mathrm{o}^*)\Phi = J(w_\mathrm{e} - w_\mathrm{o})\Phi = J\varepsilon w\Phi.$$

Hence $[J, \varepsilon]w\Phi = 0$ for all $w \in \mathcal{C}_\mathcal{D}(\mathcal{A})$; equivalently, $[J, \varepsilon]\xi = 0$ for all $\xi \in \mathcal{H}_\infty$, or $\xi \in \mathcal{H}$ for that matter. Hence $[J, \varepsilon] = 0$.

Ad (b): Let p be even. Then $C\,\varepsilon = \varepsilon\, C$ as $C \in C_\mathcal{D}(\mathcal{A})$ is of even parity by Condition 4. Moreover $U := C\,\varepsilon$ commutes with $\mathcal{D}$ and $\mathcal{A}$ by Conditions 4 and 6. Hence $U \in \mathcal{C}_\mathcal{D}(\mathcal{A})'$ and it is a selfadjoint unitary. Note also, since $C \in \mathcal{C}_\mathcal{D}(\mathcal{A})''$, that $JCJ \in \mathcal{C}_\mathcal{D}(\mathcal{A})'$ and Φ is cyclic and separating for $\mathcal{C}_\mathcal{D}(\mathcal{A})'$. Then $(JCJ - U)\Phi = JCJ\Phi - C\varepsilon\Phi = JC\Phi - C\Phi = C\Phi - C\Phi = 0$. Hence $U = JCJ$ by the separating property of Φ. It follows that $\varepsilon = CU = CJCJ$.

Ad (c): Let p be odd. Then C commutes with $\mathcal{D}$ and $\mathcal{A}$ by Condition 4. In this case C is central in $C_\mathcal{D}(\mathcal{A})$, which occurs if and only if $C = JC^*J = JCJ$. It is immediate that $C\varepsilon = -\varepsilon C$ since C has odd parity by Condition 4.

Ad (d): This is clear, since $Jw\Phi = w^*\Phi$ for all $w \in \mathcal{C}_\mathcal{D}(\mathcal{A})$.

Ad (e): We define a *right action* of $\mathcal{C} = \mathcal{C}_\mathcal{D}(\mathcal{A})$ on $\mathcal{H}_\infty$ by

$$w^\circ\xi \equiv \xi \cdot w := Jw^*J\xi, \quad \text{for all} \quad \xi \in \mathcal{H}_\infty,\ w \in \mathcal{C}.$$

Note that, in particular $w\Phi = J(w^*\Phi) = w^\circ J\Phi = w^\circ\Phi = \Phi \cdot w$ for all $w \in \mathcal{C}$. The right-module inner product on $\mathcal{H}_\infty$ is defined by

$$(\xi \mid \eta)_\mathcal{C} := {}_\mathcal{C}(J\xi \mid J\eta),$$

which works because J is antilinear. We now check that this agrees with the natural hermitian pairing of (compact) endomorphisms of $\mathcal{H}_\infty$ as a left $\mathcal{C}$-module, in order that $\mathcal{H}_\infty$ satisfy the requirements of a pre-Morita equivalence bimodule.

For $\xi, \eta, \zeta \in \mathcal{H}_\infty$ we write $\xi = w\,\Phi$, $\eta = v\,\Phi$, $\zeta = u\,\Phi$ where $w, v, u \in \mathcal{C}$. The relationship between the inner products follows from the centrality of $z = {}_\mathcal{C}(\Phi \mid \Phi)$ and the computation

$$\begin{aligned}\xi\,\Theta_{\eta,\zeta} &= {}_\mathcal{C}(\xi \mid \eta)\,\zeta = {}_\mathcal{C}(w\Phi \mid v\Phi)\,u\Phi = wzv^*u\Phi = w(v^*zu)\Phi \\ &= w(v^*zu)^\circ\Phi = (v^*zu)^\circ w\Phi = w\Phi \cdot {}_\mathcal{C}(v^*\Phi \mid u^*\Phi) = \xi \cdot {}_\mathcal{C}(J\eta \mid J\zeta).\end{aligned}$$

Hence every rank-one operator is contained in the range of the inner product and in the range of the right action of $\mathcal{C}_\mathcal{D}(\mathcal{A})$.

The proof is completed by checking the estimates (2.2). For the right-module inner product, we get, for all $\xi = w\Phi \in \mathcal{H}_\infty$ and $v \in \mathcal{C}$:

$$\begin{aligned}(v\xi \mid v\xi)_\mathcal{C} &= {}_\mathcal{C}(Jvw\Phi \mid Jvw\Phi) = w^*v^*zvw = z^{1/2}w^*v^*vwz^{1/2} \\ &\leqslant z^{1/2}w^*\|v^*v\|wz^{1/2} = \|v\|^2\, w^*zw = \|v\|^2\, {}_\mathcal{C}(Jw\Phi \mid Jw\Phi) = \|v\|^2\, (\xi \mid \xi)_\mathcal{C}\,.\end{aligned}$$

The estimate for the left-module inner product is similar. $\square$

In odd dimensions the centrality of $C = JCJ$ gives us a second $\mathbb{Z}_2$-grading.

Corollary 4.19. *Suppose $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ satisfies Conditions 1, 2, 4 and 6, and that the spectral dimension p is an odd integer. Then $P_\pm := \frac{1}{2}(1 \pm C)$ are two complementary projectors commuting with both actions of $\mathcal{C}_\mathcal{D}(\mathcal{A})$, thereby determining a second $\mathbb{Z}_2$-grading on $\mathcal{C}_\mathcal{D}(\mathcal{A}) = \mathcal{C}_\mathcal{D}(\mathcal{A})^+ \oplus \mathcal{C}_\mathcal{D}(\mathcal{A})^-$ where $\mathcal{C}_\mathcal{D}(\mathcal{A})^+ := P_+ \,\mathcal{C}_\mathcal{D}(\mathcal{A})\, P_+$ and $\mathcal{C}_\mathcal{D}(\mathcal{A})^- := P_- \,\mathcal{C}_\mathcal{D}(\mathcal{A})\, P_-$. Moreover $\mathcal{C}_\mathcal{D}(\mathcal{A})^+ = \varepsilon \mathcal{C}_\mathcal{D}(\mathcal{A})^- \varepsilon$.*

Finally we reconcile the two descriptions of the Hilbert space coming from Conditions 6 and 3. Since Lemma 4.14 shows that ψ_ω is faithful on $\mathcal{A} \subset \mathcal{C}_\mathcal{D}(\mathcal{A})$, assuming Condition 6 of course, that part of Condition 3 is redundant.

Corollary 4.20. *Suppose $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ satisfies Conditions 1, 2, and 6. Then $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ satisfies Condition 3 if and only if:*

(i) $\mathcal{C}_\mathcal{D}(\mathcal{A})$ *is a finite projective left $\mathcal{A}$-module (treating $\mathcal{A} \subset \mathcal{C}_\mathcal{D}(\mathcal{A})$);*

(ii) *there is some operator-valued weight $\Psi \colon \mathcal{C}_\mathcal{D}(\mathcal{A}) \to \mathcal{A}$ such that $\psi_\omega = \psi_\omega \circ \Psi$ for all Dixmier limits ω.*

Proof. Ad ($\Rightarrow$): By Lemma 3.23, there is an operator valued weight $\Psi \colon \mathcal{C}_\mathcal{D}(\mathcal{A}) \to \mathcal{A}$ such that ${}_\mathcal{A}(\xi \mid \eta) = \Psi\big({}_\mathcal{C}(\xi \mid \eta)\big)$ for all $\xi, \eta \in \mathcal{H}_\infty$. Then Condition 6 together with (i) and (ii) imply that $\mathcal{H}_\infty$ is a finite projective left $\mathcal{A}$-module and $\langle \eta \mid \xi \rangle = \psi_\omega({}_\mathcal{A}(\xi \mid \eta))$.

Ad ($\Leftarrow$): Conditions 3 and 6 together imply that $\mathcal{C}_\mathcal{D}(\mathcal{A})\Phi$, and also $\mathcal{C}_\mathcal{D}(\mathcal{A})$ by $\mathcal{A}$-linear isomorphism, are finitely generated and projective over $\mathcal{A}$ – the existence of Ψ then follows from Lemma 3.23 – and that $\psi_\omega(\Psi({}_\mathcal{C}(\xi \mid \eta))) = \psi_\omega({}_\mathcal{C}(\xi \mid \eta))$, for $\xi, \eta \in \mathcal{H}_\infty$. Fullness of the hermitian pairing on $\mathcal{C}_\mathcal{D}(\mathcal{A})$ and norm continuity of ψ_ω on $\mathcal{C}_\mathcal{D}(\mathcal{A})$ then entails $\psi_\omega \circ \Psi = \psi_\omega$. $\square$

4.3 Kasparov's fundamental class

In [32, Defn.-lemma 4.2], Kasparov showed that a Riemannian manifold (M, g) has a fundamental class $\lambda \in KK(C(M) \otimes \mathbb{C}\ell(M), \mathbb{C})$. The Kasparov product with this class provides isomorphisms [32, Thms. 4.9, 4.10]:

$$
\begin{aligned}
- \,\widehat{\otimes}_{C(M)}\, \lambda &: K_*(C(M)) = KK(\mathbb{C}, C(M)) \to KK(\mathbb{C}\ell(M), \mathbb{C}), \\
- \,\widehat{\otimes}_{\mathbb{C}\ell(M)}\, \lambda &: KK(\mathbb{C}, \mathbb{C}\ell(M)) \to KK(C(M), \mathbb{C}) = K^*(C(M)).
\end{aligned}
$$

Here we have been careful to use KK notation (rather than just K-homology or K-theory notation) where the algebra is regarded as $\mathbb{Z}_2$-graded.

The class λ can be represented by an unbounded Kasparov module [5] given by the Hilbert space $\mathcal{H} = L^2(\Lambda^\bullet T^*_\mathbb{C} M, g)$, the Hodge–de Rham operator $d + d^*$, and the representation of $C(M)$ given by multiplication operators. The representation of the Clifford algebra $\mathbb{C}\ell(M)$ is given as follows (see [2, Appendix A] for a similar discussion).

Observe that $\mathcal{D} = d + d^*$ is the Dirac operator associated to the left action of the Clifford algebra and the Levi-Civita connection. Let $\mathcal{D}'$ be the Dirac operator associated to the right action of the Clifford algebra, and let ε denote the grading of forms by degree. On k-forms in $\Lambda^k T^*_\mathbb{C} M$, where $\varepsilon|_{\Lambda^k} = (-1)^k$, the operator $\mathcal{D}'$ is given by

$$
\mathcal{D}'\big|_{\Lambda^k} := (-1)^k (d - d^*).
$$

Let $\widetilde{\mathcal{D}} := i\mathcal{D}'\varepsilon = i(d - d^*)$. One can check (see, for instance, [13, Sect. 9.B] or the arguments below) that $df \mapsto [\widetilde{\mathcal{D}}, f]$ provides a representation of the algebra of sections of the (real) Clifford algebra bundle for the quadratic form $-g$ on T^*M.

Passing to the complexification, we get a representation of the complex Clifford algebra $\mathbb{C}\ell(M)$, which graded-commutes with the left action of the Clifford algebra, and so graded-commutes with the symbol of $d + d^*$. Standard techniques, as described in [32, Defn.-lemma 4.2], now show that there is a graded, even Kasparov module for $C(M) \otimes \mathbb{C}\ell(M)$, with this representation of the Clifford algebra.

This complicated construction is necessary for the following reason. The right action of $\mathbb{C}\ell(M)$ has bounded commutators with $d + d^*$ because their principal symbols commute. However, the right action does not commute with the grading; rather, it graded-commutes, so we need to employ graded commutators, which are not bounded. Thus to get an honest Kasparov module for the Clifford algebra we must construct this new representation.[7]

Concretely, the standard left Clifford action of a 1-form α on forms, coming from $\mathcal{D}$, is given by $c(\alpha) := \varepsilon(\alpha) - \iota(\alpha^\sharp)$, where $\varepsilon(\alpha)\colon \omega \mapsto \alpha \wedge \omega$ is exterior multiplication, and $\iota(\alpha^\sharp)$ is contraction with the g-dual vector field $\alpha^\sharp$ of α. One checks that the the representation coming from $\mathcal{D}'$ is given for 1-forms by $c'(\beta) := \varepsilon(\beta) + \iota(\beta^\sharp)$. The graded commutation $[c(\alpha), c'(\beta)]_+ = 0$ is now immediate.

What we will now show is that our characterisation of the Riemannian structure of a manifold allows for the construction of an analogous class, even in the noncommutative case.

Definition 4.21. Let $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \mathbf{c}, \Phi)$ be a noncommutative Riemannian manifold. We define $\mathcal{D}' := J\mathcal{D}J$, and set $\widetilde{\mathcal{D}} := i\mathcal{D}'\varepsilon$. As before, we write $b^\circ = Jb^*J$ for the right action of $\mathcal{A}$.

Lemma 4.22. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \mathbf{c}, \Phi)$ be a noncommutative Riemannian manifold, and let $a, b \in \mathcal{A}$. Then*

$$[\mathcal{D}, a]\,[\widetilde{\mathcal{D}}, b^\circ] + [\widetilde{\mathcal{D}}, b^\circ]\,[\mathcal{D}, a] = 0. \tag{4.7}$$

Proof. This is an easy computation. First note that $[\widetilde{\mathcal{D}}, b^\circ] = iJ[\mathcal{D}, b^*]J\varepsilon$. Hence

$$[\mathcal{D}, a]\,[\widetilde{\mathcal{D}}, b^\circ] + [\widetilde{\mathcal{D}}, b^\circ]\,[\mathcal{D}, a] = i\big([\mathcal{D}, a]J[\mathcal{D}, b^*]J - J[\mathcal{D}, b^*]J[\mathcal{D}, a]\big)\varepsilon = 0,$$

since $J\mathcal{C}_{\mathcal{D}}(\mathcal{A})J \subset \mathcal{C}_{\mathcal{D}}(\mathcal{A})'$. □

Lemma 4.23. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \mathbf{c}, \Phi)$ be a noncommutative Riemannian manifold. If $b \in \mathcal{A}$, then the linear map $T_b\colon \mathcal{H}_\infty \to \mathcal{H}_\infty$ defined by*

$$T_b := \mathcal{D}\,[\widetilde{\mathcal{D}}, b^\circ] + [\widetilde{\mathcal{D}}, b^\circ]\,\mathcal{D}$$

extends to a bounded operator on $\mathcal{H}$.

Proof. Since $[\widetilde{\mathcal{D}}, b^\circ]$ commutes with the left action of $\mathcal{A}$, the commutator $[T_b, a]$ vanishes for each $a \in \mathcal{A}$:

$$[T_b, a] = [\mathcal{D}, a]\,[\widetilde{\mathcal{D}}, b^\circ] + [\widetilde{\mathcal{D}}, b^\circ]\,[\mathcal{D}, a] = 0,$$

by Lemma 4.22; thus T_b is $\mathcal{A}$-linear on $\mathcal{H}_\infty$. Since $\mathcal{H}_\infty$ is finitely generated and projective over $\mathcal{A}$, by Condition 3, T_b extends to a bounded operator on $\mathcal{H}$ by Proposition 2.9. □

[7] Thanks to Nigel Higson for explaining the solution of this conundrum to us.

Remark 4.24. We have not assumed that J has bounded commutator with $|\mathcal{D}|$ – or equivalently, with $\langle\mathcal{D}\rangle$. However, if $[\langle\mathcal{D}\rangle, J]$ is bounded, we also get additional smoothness. Indeed, in that case, for each $b \in \mathcal{A}$, the commutator

$$[\langle\mathcal{D}\rangle, [\widetilde{\mathcal{D}}, b^\circ]] = iJ[\langle\mathcal{D}\rangle, [\mathcal{D}, b^*]]J\varepsilon + i\,[\langle\mathcal{D}\rangle, J]\,[\mathcal{D}, b^*]J\varepsilon - i\,J[\mathcal{D}, b^*]\,[J, \langle\mathcal{D}\rangle]\varepsilon$$

is bounded, since $[\mathcal{D}, b^*] \in \operatorname{Dom}\tilde{\delta}$. In fact, on replacing $\langle\mathcal{D}\rangle$ by $\langle\mathcal{D}\rangle^k$ or $|\mathcal{D}|^k$ for any k, we see that $[\widetilde{\mathcal{D}}, b^\circ] \in \operatorname{Dom}^\infty \delta$.

Even without the additional smoothness that comes from the boundedness of $[|\mathcal{D}|, J]$, we can prove the existence of Kasparov's Riemannian KK-class in the present context.

Proposition 4.25. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \mathbf{c}, \Phi)$ be a noncommutative Riemannian manifold with spectral dimension p. Then the triple*

$$(\mathcal{A} \otimes \mathcal{C}_{\mathcal{D}}(\mathcal{A})^\circ, \mathcal{H}, \mathcal{D})$$

is a $\mathbb{Z}_2$-graded spectral triple (with $\mathcal{A}\otimes\mathcal{A}^\circ$ even) satisfying the first order condition, and represents a class $\lambda \in KK(A \otimes C^\circ, \mathbb{C})$, where C° is the norm closure of $\mathcal{C}_{\mathcal{D}}(\mathcal{A})^\circ$, regarded as a $\mathbb{Z}_2$-graded C^*-algebra.

Proof. The bounded selfadjoint operator $F_{\mathcal{D}} = \mathcal{D}\,\langle\mathcal{D}\rangle^{-1}$ anticommutes with the grading ε. Also, $1 - F_{\mathcal{D}}^2 = (1 + \mathcal{D}^2)^{-1}$ is compact, so we need only check that the graded commutators of $F_{\mathcal{D}}$ with elements of $\mathcal{A} \otimes \mathcal{C}_{\mathcal{D}}(\mathcal{A})^\circ$ are compact. If instead we consider an element $y \in \mathcal{A} \otimes \mathcal{C}_{\widetilde{\mathcal{D}}}(\mathcal{A}^\circ)$, we get

$$[F_{\mathcal{D}}, y]_\pm = [\mathcal{D}, y]_\pm\,\langle\mathcal{D}\rangle^{-1} + \mathcal{D}\,[\langle\mathcal{D}\rangle^{-1}, y]_\pm.$$

The first summand is compact by the compactness of $\langle\mathcal{D}\rangle^{-1}$ and the boundedness of graded commutators with $\mathcal{D}$, which follows from Lemma 4.23. The second term requires more care. The argument of Lemma 2.3 of [33], lightly adjusted to handle graded commutators, shows that if $[\mathcal{D}, y]_\pm$ is bounded, then for any real $t > 0$, we get

$$[y, (t + \mathcal{D}^2)^{-1}]_\pm = \mathcal{D}(t + \mathcal{D}^2)^{-1}[\mathcal{D}, y]_\pm(t + \mathcal{D}^2)^{-1} \mp (t + \mathcal{D}^2)^{-1}[\mathcal{D}, y]_\pm\mathcal{D}(t + \mathcal{D}^2)^{-1}.$$

Now we can employ [33, Prop. 2.4] to show that $\mathcal{D}[(1 + \mathcal{D}^2)^{-1/2}, y]_\pm$ is compact, and in fact lies in $\mathcal{L}^s(\mathcal{H})$ for all $s > p$.

It remains to show that C° is isomorphic to the norm closure of $\mathcal{C}_{\widetilde{\mathcal{D}}}(\mathcal{A}^\circ)$. We define an algebra homomorphism $\alpha\colon \mathcal{C}_{\mathcal{D}}(\mathcal{A})^\circ \to \mathcal{C}_{\widetilde{\mathcal{D}}}(\mathcal{A}^\circ)$ on generators by

$$\alpha(a^\circ) := a^\circ, \qquad \alpha\big([\mathcal{D}, a]^\circ\big) := [\mathcal{D}, a]^\circ(-i\varepsilon) = [\widetilde{\mathcal{D}}, a^\circ].$$

One checks that α is $*$-preserving on generators, so it extends to a $*$-isomorphism between the norm closures of $\mathcal{C}_{\mathcal{D}}(\mathcal{A})^\circ$ and $\mathcal{C}_{\widetilde{\mathcal{D}}}(\mathcal{A}^\circ)$. Thus we regard the action of $\mathcal{C}_{\mathcal{D}}(\mathcal{A})^\circ$ as being via the representation in $\mathcal{C}_{\widetilde{\mathcal{D}}}(\mathcal{A}^\circ)$. □

Example 4.26. In the classical case of the Clifford algebra acting on the left of the differential forms over a closed C^∞ manifold M, the $\mathbb{Z}_2$-graded spectral triple constructed in Proposition 4.25 yields precisely Kasparov's fundamental class.

Remark 4.27. Given $a \in \mathcal{A}$, the operator $F_{\mathcal{D}}\,[\widetilde{\mathcal{D}}, a^\circ] + [\widetilde{\mathcal{D}}, a^\circ]\,F_{\mathcal{D}}$ is compact. Using $[\widetilde{\mathcal{D}}, a^\circ] = -i[\mathcal{D}, a]^\circ\varepsilon$ and $F_{\mathcal{D}}\varepsilon = -\varepsilon F_{\mathcal{D}}$, we find that the operator $-i\big(F_{\mathcal{D}}\,[\mathcal{D}, a]^\circ - [\mathcal{D}, a]^\circ\,F_{\mathcal{D}}\big)\varepsilon$ is also compact. This mirrors exactly what we see in the classical case, and in that context may be traced to the commuting of the principal symbols of the pseudodifferential operators $F_{\mathcal{D}}$ and $[\mathcal{D}, a]^\circ$.

5 The main theorems: from spin^c to Riemannian and back

The existence of the fundamental class λ for a noncommutative Riemannian manifold allows us to ask about Poincaré duality in this picture. In the spin^c and spin settings, Poincaré duality has been considered as one of the defining, or at any rate desirable, properties of noncommutative manifolds [1–3, 20].

For noncommutative spin^c manifolds, this duality has the following formulation. As before, A and C will denote the C^*-algebras obtained by taking the respective norm closures of $\mathcal{A}$ and $\mathcal{C} = \mathcal{C}_{\mathcal{D}}(\mathcal{A})$.

Condition 7 (Spin^c Poincaré duality)**.** The class $\mu \in KK^p(A \otimes A^\circ, \mathbb{C})$ of the p-dimensional noncommutative spin^c manifold $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \mathbf{c})$ determines for each $j = 0, 1$ an isomorphism

$$- \otimes_A \mu : KK^j(\mathbb{C}, A) \to KK^{j+p}(A^\circ, \mathbb{C}) \simeq KK^{j+p}(A, \mathbb{C}).$$

In the Riemannian but not necessarily spin^c setting, we may formulate it as follows.

Condition 8 (Riemannian Poincaré duality)**.** The class $\lambda \in KK(A \otimes C^\circ, \mathbb{C})$ of the noncommutative Riemannian manifold $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \mathbf{c}, \Phi)$ defines two isomorphisms for each $j = 0, 1$:

$$- \otimes_{C^\circ} \lambda : KK^j(\mathbb{C}, C^\circ) \to KK^j(A, \mathbb{C}), \qquad - \otimes_A \lambda : KK^j(\mathbb{C}, A) \to KK^j(C^\circ, \mathbb{C}).$$

We now state and prove two theorems that produce the desired isomorphisms in each case.

To handle the even and odd cases together, we adopt the following notational convention. The algebra $\mathbb{C}\ell_1^p$ denotes $\mathbb{C}\ell_1$ if p is an odd integer and $\mathbb{C}$ if p is even. Similarly, the vector space $\mathbb{C}_p^2$ will denote $\mathbb{C}^2$ if p is odd and $\mathbb{C}$ if p is even. Thus

$$\mathcal{A} \otimes \mathbb{C}\ell_1^p = \begin{cases} \mathcal{A} \otimes \mathbb{C}\ell_1 & \text{for } p \text{ odd}, \\ \mathcal{A} & \text{for } p \text{ even}, \end{cases} \qquad \mathcal{H} \otimes \mathbb{C}_p^2 \simeq \begin{cases} \mathcal{H} \otimes \mathbb{C}^2 & \text{for } p \text{ odd}, \\ \mathcal{H} & \text{for } p \text{ even}. \end{cases}$$

We also observe that the transpose map gives a $*$-isomorphism from $\mathbb{C}\ell_1 \simeq M_2(\mathbb{C})$ to its opposite. This notation will streamline our discussion of the odd and even cases. To better identify the classes obtained from Kasparov products, some details about odd Kasparov classes are laid out in Appendix A.

Theorem 5.1. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \mathbf{c})$ be a p-dimensional noncommutative spin^c manifold with Kasparov class $\mu \in KK^p(A \otimes A^\circ, \mathbb{C}) \simeq KK(A \otimes A^\circ \otimes \mathbb{C}\ell_1^p, \mathbb{C})$. Regard the conjugate module $(\mathcal{H}_\infty \otimes \mathbb{C}_p^2)^\sharp$ as an $(\mathcal{A} \otimes \mathbb{C}\ell_1^p)$-$\mathcal{C}_{\mathcal{D}}(\mathcal{A})$-bimodule, graded in odd spectral dimensions, with class $\sigma \in KK(C^\circ, A^\circ \otimes \mathbb{C}\ell_1^p)$. Then $\lambda := \sigma \otimes_{A^\circ \otimes \mathbb{C}\ell_1^p} \mu \in KK(A \otimes C^\circ, \mathbb{C})$ is the class of a noncommutative Riemannian manifold. If μ satisfies spin^c Poincaré duality, then λ satisfies Riemannian Poincaré duality.*

Theorem 5.2. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \mathbf{c}, \Phi)$ be a noncommutative Riemannian manifold and let $\lambda \in KK(A \otimes C^\circ, \mathbb{C})$ be its Kasparov class. Let $\mathcal{E}$ be a pre-Morita equivalence bimodule between either $\mathcal{C}_{\mathcal{D}}(\mathcal{A})$ or $\mathcal{C}_{\mathcal{D}}(\mathcal{A})^+$ (according to parity of the spectral dimension) and $\mathcal{A}$, with class $\tau := [\mathcal{E} \otimes \mathbb{C}_p^2] \in KK(A^\circ \otimes \mathbb{C}\ell_1^p, C^\circ)$. Then $\mu := \tau \otimes_{C^\circ} \lambda \in KK(A \otimes A^\circ \otimes \mathbb{C}\ell_1^p, \mathbb{C})$ is the class of a noncommutative spin^c manifold. If λ satisfies Riemannian Poincaré duality, then μ satisfies spin^c Poincaré duality.*

Remark 5.3. As in Proposition 3.30, a few left-right issues must be addressed in order to formulate the Kasparov product correctly. However, just as in that Proposition, we can unpack the definitions to find that, for example,

$$\mathcal{E} \otimes_{C^\circ} \mathcal{H}_\infty \simeq \mathcal{H}_\infty \otimes_C \mathcal{E},$$

where we regard $\mathcal{E}$ on the left hand side as a right C°-module and on the right hand side as a left C-module.

5.1 Proof of Theorem 5.1

We prove the even case first; the odd case needs only a few modifications, to be discussed later.

We begin with the noncommutative spin^c manifold $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \mathbf{c})$ and the pre-Morita equivalence bimodule $\mathcal{H}_\infty$ between $\mathcal{C}_\mathcal{D}(\mathcal{A})$ and $\mathcal{A}$ given by the spin^c condition. As in Proposition 4.10, we identify $\mathcal{H}_\infty = q\mathcal{A}^m$. The conjugate module $\mathcal{H}_\infty^\sharp$ gives a pre-Morita equivalence between $\mathcal{A}$ and $\mathcal{C}_\mathcal{D}(\mathcal{A}) \simeq qM_m(\mathcal{A})q$.

Theorem 3.28 tells us that we may form the spectral triple $(\mathcal{A}, \mathcal{H} \otimes_\mathcal{A} \mathcal{H}_\infty^\sharp, \widehat{\mathcal{D}}) = (\mathcal{A}, \mathcal{H}^m q, \widehat{\mathcal{D}})$, which satisfies the first-order condition, as well as Conditions 1, 2 and 3.

Observe that Proposition 3.27 implies that

$$\langle\widehat{\mathcal{D}}\rangle^{-p} = q^\circ(\langle\mathcal{D}\rangle^{-p} \otimes 1_m)q^\circ + B, \quad \text{where} \quad B \in \mathcal{L}_0^{1,\infty}(\mathcal{H}^m q).$$

To see that B has vanishing Dixmier trace, we write $\mathcal{D}'_m = q^\circ(\mathcal{D} \otimes 1_m)q^\circ$, so that $\widehat{\mathcal{D}} = \mathcal{D}'_m + \widehat{A}$ with $\widehat{A}$ bounded, by Lemma 3.25. Using (3.9), we see that $(i + \widehat{\mathcal{D}})^{-1} \equiv (i + \mathcal{D}'_m)^{-1} \bmod \mathcal{Z}_{p/2}$, and thus $\langle\widehat{\mathcal{D}}\rangle^{-1} \equiv \langle\mathcal{D}'_m\rangle^{-1} \bmod \mathcal{Z}_{p/2}$.

The operator trace over $\mathcal{H}^m q$ is $\mathrm{Tr}_\mathcal{H} \otimes \mathrm{tr}_m(q^\circ(\cdot)q^\circ)$ with tr_m denoting a matrix trace. The left action of $\mathcal{C}_\mathcal{D}(\mathcal{A})$ commutes with q°; thus, if w is an even element of $\mathcal{C}_\mathcal{D}(\mathcal{A})$, we get an equality of Dixmier traces:

$$\mathrm{Tr}_\omega^{\mathcal{H}^m q}\big(w\,\langle\widehat{\mathcal{D}}\rangle^{-p}\big) = \mathrm{Tr}_\omega^{\mathcal{H}^m q}\big(wq^\circ\,\langle\mathcal{D}\rangle^{-p}\big) = \mathrm{Tr}_\omega^{\mathcal{H}}\big(w\,\mathrm{tr}_m(q^\circ)\,\langle\mathcal{D}\rangle^{-p}\big). \tag{5.1}$$

We must now show that the new spectral triple satisfies Condition 6. The spin^c condition, namely that $\mathcal{H}_\infty$ is a pre-Morita equivalence bimodule between $\mathcal{C}_\mathcal{D}(\mathcal{A})$ and $\mathcal{A}$, shows that there are finitely many vectors $\xi_j, \eta_j \in \mathcal{H}_\infty$ such that

$$\Theta_{\xi_1,\eta_1} + \cdots + \Theta_{\xi_m,\eta_m} = 1 \in \mathcal{C}_\mathcal{D}(\mathcal{A}). \tag{5.2}$$

Consider the vector $\Phi \in \mathcal{H} \otimes_\mathcal{A} \mathcal{H}_\infty^\sharp$ defined by

$$\Phi := \xi_1 \otimes \eta_1^\sharp + \cdots + \xi_m \otimes \eta_m^\sharp.$$

We claim that Φ is an algebraically cyclic vector for $\mathcal{C}_\mathcal{D}(\mathcal{A})$, and that the vector state $\sigma_\Phi \colon w \mapsto \langle\Phi \mid w\Phi\rangle$ is of the form

$$\sigma_\Phi(w) = \mathrm{Tr}_\omega^{\mathcal{H}^m q}(wz\langle\widehat{\mathcal{D}}\rangle^{-p})$$

for a central element $z \in \mathcal{C}_\mathcal{D}(\mathcal{A})$.

Under the isomorphism $\Lambda\colon \Theta_{\xi,\eta} \mapsto \xi \otimes \eta^\sharp : \mathcal{C}_\mathcal{D}(\mathcal{A}) \to \mathcal{H}_\infty \otimes_\mathcal{A} \mathcal{H}_\infty^\sharp$, the vector Φ is just the image $\Lambda(1)$ of the unit element of $\mathcal{C}_\mathcal{D}(\mathcal{A})$. Note that

$$1 = \sum_k \Theta_{\xi_k,\eta_k} = 1^2 = \sum_{j,k} \Theta_{\xi_k(\eta_k|\xi_j)_\mathcal{A},\eta_j} = \sum_{j,k} \Theta_{\xi_k,\eta_j(\xi_j|\eta_k)_\mathcal{A}}$$

so that $\eta_k = \sum_j \eta_j (\xi_j \mid \eta_k)_\mathcal{A}$ for each k. Moreover, if $w = \sum_{i,k} \Theta_{\xi_i a_i, \eta_k b_k} \in \mathcal{C}_\mathcal{D}(\mathcal{A})$, then

$$\begin{aligned} w\Phi &= \sum_{i,j,k} \xi_i a_i \, (\eta_k b_k \mid \xi_j)_\mathcal{A} \otimes \eta_j^\sharp = \sum_{i,j,k} \xi_i a_i b_k^* \, (\eta_k \mid \xi_j)_\mathcal{A} \otimes \eta_j^\sharp \\ &= \sum_{i,k} \xi_i a_i b_k^* \otimes \eta_k^\sharp = \sum_{i,k} \xi_i a_i \otimes (\eta_k b_k)^\sharp = \Lambda(w). \end{aligned} \tag{5.3}$$

Thus $w \mapsto w\Phi = \Lambda(w)$ maps $\mathcal{C}_\mathcal{D}(\mathcal{A})$ *onto* $\mathcal{H}_\infty \otimes_\mathcal{A} \mathcal{H}_\infty^\sharp$, so that Φ is algebraically cyclic; and it is separating as well, since the pre-Morita equivalence implies that Λ is bijective.

Using Proposition 4.10, the scalar product on the Hilbert space $\mathcal{H} \otimes_\mathcal{A} \mathcal{H}_\infty^\sharp$ is given on the dense subspace $\mathcal{H}_\infty \otimes_\mathcal{A} \mathcal{H}_\infty^\sharp$ by

$$\begin{aligned} \langle \xi \otimes \eta^\sharp \mid \zeta \otimes \rho^\sharp \rangle &:= \big\langle \xi \, {}_\mathcal{A}(\eta^\sharp \mid \rho^\sharp) \bigm| \zeta \big\rangle = \big\langle \xi \, (\eta \mid \rho)_\mathcal{A} \bigm| \zeta \big\rangle \\ &= \operatorname{Tr}_\omega\big((\xi(\eta \mid \rho)_\mathcal{A} \mid \zeta)_\mathcal{A} \, \langle \mathcal{D} \rangle^{-p} \big) = \operatorname{Tr}_\omega\big((\rho \mid \eta)_\mathcal{A} (\xi \mid \zeta)_\mathcal{A} \, \langle \mathcal{D} \rangle^{-p} \big), \end{aligned} \tag{5.4}$$

for $\xi, \eta, \zeta, \rho \in \mathcal{H}_\infty$.

Lemma 5.4. *Evaluation of the vector state σ_Φ on the operator $\Theta_{\rho,\tau} \otimes 1$ in $\mathcal{C}_\mathcal{D}(\mathcal{A}) \otimes 1$ (acting on $\mathcal{H} \otimes \mathcal{H}_\infty^\sharp$) yields*

$$\sigma_\Phi(\Theta_{\rho,\tau} \otimes 1) \equiv \langle \Phi \mid (\Theta_{\rho,\tau} \otimes 1)\Phi \rangle = \operatorname{Tr}_\omega^\mathcal{H}\big((\tau \mid \rho)_\mathcal{A} \, \langle \mathcal{D} \rangle^{-p} \big).$$

Moreover, σ_Φ is tracial on $\mathcal{C}_\mathcal{D}(\mathcal{A})$.

Proof. We first recall from Proposition 4.10 that we can use the right $\mathcal{A}$-valued inner product on $\mathcal{H}_\infty$ and ψ_ω to compute the scalar product. The evaluation proceeds as follows, for $\rho, \tau \in \mathcal{H}_\infty$:

$$\begin{aligned} \langle \Phi \mid (\Theta_{\rho,\tau} \otimes 1)\Phi \rangle &= \sum_{j,k} \langle \xi_j \otimes \eta_j^\sharp \mid \Theta_{\rho,\tau} \, \xi_k \otimes \eta_k^\sharp \rangle = \sum_{j,k} \big\langle \xi_j \otimes \eta_j^\sharp \bigm| \rho \, (\tau \mid \xi_k)_\mathcal{A} \otimes \eta_k^\sharp \big\rangle \\ &= \sum_{j,k} \big\langle \xi_j \bigm| \rho \, (\tau \mid \xi_k)_\mathcal{A} \, (\eta_k \mid \eta_j)_\mathcal{A} \big\rangle = \sum_{j,k} \big\langle \xi_j \bigm| \rho \, (\tau \mid \Theta_{\xi_k,\eta_k} \eta_j)_\mathcal{A} \big\rangle \\ &= \sum_j \big\langle \xi_j \bigm| \rho \, (\tau \mid \eta_j)_\mathcal{A} \big\rangle = \sum_j \operatorname{Tr}_\omega\big((\xi_j \mid \rho \, (\tau \mid \eta_j)_\mathcal{A})_\mathcal{A} \, \langle \mathcal{D} \rangle^{-p} \big) \\ &= \sum_j \operatorname{Tr}_\omega\big((\xi_j \, (\eta_j \mid \tau)_\mathcal{A} \mid \rho)_\mathcal{A} \, \langle \mathcal{D} \rangle^{-p} \big) \\ &= \sum_j \operatorname{Tr}_\omega\big((\Theta_{\xi_j,\eta_j} \tau \mid \rho)_\mathcal{A} \, \langle \mathcal{D} \rangle^{-p} \big) = \operatorname{Tr}_\omega\big((\tau \mid \rho)_\mathcal{A} \, \langle \mathcal{D} \rangle^{-p} \big). \end{aligned}$$

Here we have used Condition 3 for the spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ and the adjointability of the right action of $\mathcal{A}$.

We can now suppress the tensor factor $\otimes 1$ in the notation for the action of $\mathcal{C}_{\mathcal{D}}(\mathcal{A})$ on $\mathcal{H} \otimes \mathcal{H}_\infty^\sharp$. The tracial property of σ_Φ on $\mathcal{C}_{\mathcal{D}}(\mathcal{A})$ follows at once, since $\mathcal{C}_{\mathcal{D}}(\mathcal{A})$ is spanned by finite-rank endomorphisms:

$$
\begin{aligned}
\langle \Phi \mid \Theta_{\rho,\tau}\Theta_{\alpha,\beta}\Phi\rangle &= \langle \Phi \mid \Theta_{\rho(\tau|\alpha)_{\mathcal{A}},\beta}\,\Phi\rangle = \operatorname{Tr}_\omega\big((\beta \mid \rho\,(\tau \mid \alpha)_{\mathcal{A}})_{\mathcal{A}}\,\langle\mathcal{D}\rangle^{-p}\big) \\
&= \operatorname{Tr}_\omega\big((\beta \mid \rho)_{\mathcal{A}}\,(\tau \mid \alpha)_{\mathcal{A}}\,\langle\mathcal{D}\rangle^{-p}\big) = \operatorname{Tr}_\omega\big((\tau\,(\rho \mid \beta)_{\mathcal{A}} \mid \alpha)_{\mathcal{A}}\,\langle\mathcal{D}\rangle^{-p}\big) \\
&= \operatorname{Tr}_\omega\big((\tau \mid \alpha\,(\beta \mid \rho)_{\mathcal{A}})_{\mathcal{A}}\,\langle\mathcal{D}\rangle^{-p}\big) = \operatorname{Tr}_\omega\big((\tau \mid \Theta_{\alpha,\beta}\,\rho)_{\mathcal{A}}\,\langle\mathcal{D}\rangle^{-p}\big) \\
&= \langle \Phi \mid \Theta_{\alpha,\beta}\Theta_{\rho,\tau}\Phi\rangle. \qquad \square
\end{aligned}
$$

To complete the Riemannian requirements, we need a left-module $\mathcal{C}_{\mathcal{D}}(\mathcal{A})$-valued pairing on $\mathcal{H}_\infty \otimes \mathcal{H}_\infty^\sharp$. The obvious choice is

$$
{}_{\mathcal{C}}(\Lambda(u) \mid \Lambda(v)) := uv^*, \quad \text{for all} \quad u, v \in \mathcal{C}_{\mathcal{D}}(\mathcal{A}).
$$

With this choice, ${}_{\mathcal{C}}(\Phi \mid \Phi) = 1$ is certainly central and strictly positive.

To show that this choice of inner product satisfies the requirements of Conditions 6 and 3, we emulate the identification of the scalar product on the Hilbert space $\mathcal{H} \otimes_{\mathcal{A}} \mathcal{H}_\infty^\sharp$ in (5.4). On the dense subspace $\mathcal{H}_\infty \otimes_{\mathcal{A}} \mathcal{H}_\infty^\sharp$ we get

$$
\begin{aligned}
\langle \xi \otimes \eta^\sharp \mid \zeta \otimes \rho^\sharp\rangle &:= \big\langle \xi\,{}_{\mathcal{A}}(\eta^\sharp \mid \rho^\sharp) \bigm| \zeta\big\rangle = \big\langle \xi\,(\eta \mid \rho)_{\mathcal{A}} \bigm| \zeta\big\rangle \\
&= \operatorname{Tr}_\omega\big({}_{\mathcal{C}}(\zeta \mid \xi(\eta \mid \rho)_{\mathcal{A}})\,\langle\mathcal{D}\rangle^{-p}\big) = \operatorname{Tr}_\omega\big({}_{\mathcal{C}}(\zeta \mid {}_{\mathcal{C}}(\xi \mid \eta)\rho)\,\langle\mathcal{D}\rangle^{-p}\big) \\
&= \operatorname{Tr}_\omega\big({}_{\mathcal{C}}(\zeta \mid \rho)\,{}_{\mathcal{C}}(\eta \mid \xi)\,\langle\mathcal{D}\rangle^{-p}\big) = \operatorname{Tr}_\omega\big(\Theta_{\zeta,\rho}\Theta_{\xi,\eta}^*\,\langle\mathcal{D}\rangle^{-p}\big)
\end{aligned} \tag{5.5}
$$

for $\xi, \eta, \zeta, \rho \in \mathcal{H}_\infty$. Thus the scalar product is the composition of our chosen left $\mathcal{C}_{\mathcal{D}}(\mathcal{A})$-valued inner product and ψ_ω. This gives us the required positivity and faithfulness as well, since these properties hold for the spin^c manifold $(\mathcal{A}, \mathcal{H}, \mathcal{D})$. Also $\mathcal{H}_\infty \otimes_{\mathcal{A}} \mathcal{H}_\infty^\sharp \simeq \mathcal{H}_\infty^m q$ is finite projective as a left $\mathcal{A}$-module, since $\mathcal{H}_\infty$ is finite projective by Proposition 4.10.

The Tomita conjugation $J := J_\Phi$ of (4.6) satisfies $J(T\Phi) = T^*\Phi$ for $T \in \mathcal{C}_{\mathcal{D}}(\mathcal{A})''$. Hence J maps $\mathcal{H}_\infty \otimes_{\mathcal{A}} \mathcal{H}_\infty^\sharp = \Lambda(\mathcal{C}_{\mathcal{D}}(\mathcal{A}))$ onto itself. Indeed, from (5.3) one sees that J is given on that subspace by $\xi \otimes \eta^\sharp \mapsto \eta \otimes \xi^\sharp$.

In the even-dimensional case, the spin^c condition gives

$$
\pi_{\widehat{\mathcal{D}}}(\mathbf{c}) = \pi_{\mathcal{D}}(\mathbf{c}) \otimes 1 = C \otimes 1,
$$

so the Hochschild p-cycle is the same for both spectral triples. We can abbreviate $\widehat{C} := \pi_{\widehat{\mathcal{D}}}(\mathbf{c})$.

We claim that $\varepsilon := \pi_{\widehat{\mathcal{D}}}(\mathbf{c})\,J\pi_{\widehat{\mathcal{D}}}(\mathbf{c})J = \widehat{C}J\widehat{C}J$ anticommutes with $\widehat{\mathcal{D}}$.

To see that, we use the standard isomorphism $\psi\colon q\mathcal{A}^m \to q(\mathbb{C}^m \otimes \mathcal{A})$ presenting elements of $q\mathcal{A}^m$ as column vectors. Notice that $(q(\mathbb{C}^m \otimes \mathcal{A}))^\sharp \simeq (\mathcal{A} \otimes \mathbb{C}^m)q$ with row vectors on the right hand side. Using the standard bases of column vectors u_i and row vectors $v_i = u_i^T$ for $\mathbb{C}^m$, this isomorphism is given by $(q\sum_i u_i \otimes a_i)^\sharp \mapsto \sum_i (a_i^* \otimes v_i)q$.

Since $\mathcal{A}$ and $q = 1_{\mathcal{H}_\infty}$ commute with the grading $C = \pi_{\mathcal{D}}(\mathbf{c})$ of $\mathcal{H}_\infty \simeq q\mathcal{A}^m$, there is a $\mathbb{Z}_2$-grading G of $\mathbb{C}^m$ such that $\psi \circ C = q(1_{\mathcal{A}} \otimes G) \circ \psi$. Using the identifications

$$
\mathcal{H}_\infty \otimes_{\mathcal{A}} (\mathcal{H}_\infty)^\sharp \simeq (\mathcal{H}_\infty \otimes \mathbb{C}^m)q \simeq \mathcal{H}_\infty \otimes_{\mathcal{A}} \psi(\mathcal{H}_\infty)^\sharp
$$

and writing $\Phi = \sum_j \xi_j \otimes_{\mathcal{A}} \eta_j^\sharp = \sum_{i,j} \xi_j (\eta_j^i)^* \otimes v_i$, we get

$$\begin{aligned} J\widehat{C}J\Phi &= \sum_j \xi_j \otimes_{\mathcal{A}} (C\eta_j)^\sharp = \sum_j \xi_j \otimes_{\mathcal{A}} \sum_i (Gu_i \otimes \eta_j^i)^\sharp \\ &= \sum_{i,j} \xi_j \otimes_{\mathcal{A}} ((\eta_j^i)^* \otimes v_i G) = \sum_{i,j} \xi_j (\eta_j^i)^* \otimes v_i G. \end{aligned}$$

The constructions in the proof of Lemma 3.25 show that $\widehat{\mathcal{D}}$, as an operator on $(\mathcal{H}_\infty \otimes \mathbb{C}^m)q$, may be written as $\widehat{\mathcal{D}} = q^\circ(\mathcal{D} \otimes 1_m)q^\circ + q^\circ(M \otimes 1_m)q^\circ$, where M is a right $\mathcal{A}$-linear map on $\mathcal{H}_\infty$. With that, we compute

$$\widehat{\mathcal{D}}J\widehat{C}J\Phi = \sum_{i,j} \mathcal{D}(\xi_j(\eta_j^i)^*) \otimes v_i G + \sum_j M\xi_j \otimes (C\eta_j)^\sharp = J\widehat{C}J\widehat{\mathcal{D}}\Phi.$$

For a general element $S\Phi$ of $\mathcal{H}_\infty \otimes_{\mathcal{A}} \mathcal{H}_\infty^\sharp$ – using (5.3) directly when S is of finite rank – we may simply replace $\xi_j \otimes \eta_j^\sharp$ by $S(\xi_j \otimes \eta_j^\sharp)$ in the above computations. Thus we find that $J\widehat{C}J\widehat{\mathcal{D}} = \widehat{\mathcal{D}}J\widehat{C}J$.

Similarly, using $C\mathcal{D} = -\mathcal{D}C$, we find that $\widehat{C}\widehat{\mathcal{D}} = -\widehat{\mathcal{D}}\widehat{C}$. For $\varepsilon := \widehat{C}J\widehat{C}J$ we arrive at $\widehat{\mathcal{D}}\varepsilon = -\varepsilon\widehat{\mathcal{D}}$. Moreover, since $\varepsilon\Phi = \widehat{C}^2\Phi = \Phi$, Condition 6 is established in the even case.

In the odd-dimensional case, we first use Appendix A to replace our odd spectral triple with an even $\mathbb{Z}_2$-graded spectral triple:

$$\left(\mathcal{A} \otimes \mathcal{A}^\circ \otimes \mathbb{C}\ell_1, \mathcal{H} \otimes \mathbb{C}^2, \mathcal{D}' = \begin{pmatrix} \mathcal{D} & 0 \\ 0 & -\mathcal{D} \end{pmatrix}, \Gamma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \mathbf{c}\right). \tag{5.6}$$

Also we take $\mathcal{H}_\infty \otimes \mathbb{C}^2$ as a graded right module over $\mathcal{A} \otimes \mathbb{C}\ell_1$. Then the Kasparov product over $A \otimes \mathbb{C}\ell_1$ of the spectral triple (5.6) and $(\mathcal{H}_\infty \otimes \mathbb{C}^2)^\sharp$ can be computed similarly to the even case, and is represented by

$$\left(\mathcal{A} \otimes \mathcal{C}_{\mathcal{D}}(\mathcal{A})^\circ, \mathcal{H}_\infty \otimes_{\mathcal{A}} \mathcal{H}_\infty^\sharp \otimes \mathbb{C}^2, \widehat{\mathcal{D}}' = \begin{pmatrix} \widehat{\mathcal{D}} & 0 \\ 0 & -\widehat{\mathcal{D}} \end{pmatrix}, \Gamma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \mathbf{c}\right).$$

The discussion of summability is as in the even case, and we find that $\Phi \oplus \Phi$ is cyclic and separating, where $\Phi := \Lambda(1_{\mathcal{C}})$ as in the even case. Then $\widehat{C}' \equiv \pi_{\widehat{\mathcal{D}}'}(\mathbf{c}) = \begin{pmatrix} \widehat{C} & 0 \\ 0 & -\widehat{C} \end{pmatrix}$ and the grading $\varepsilon' := \widehat{C}'(J \oplus J)\widehat{C}'(J \oplus J)$ satisfy the Riemannian conditions.

Observe that in the odd case, the left action of $\mathcal{C}_{\mathcal{D}}(\mathcal{A})$ is given by the algebra $\mathcal{C}_{\widehat{\mathcal{D}}'}(\mathcal{A}) = (\mathcal{C}_{\mathcal{D}}(\mathcal{A}) \otimes 1) \oplus (\mathcal{C}_{\mathcal{D}}(\mathcal{A}) \otimes 1)$.

► Now assume that the class $\mu \in KK^p(A \otimes A^\circ, \mathbb{C})$ of $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ satisfies the spin^c version of Poincaré duality, Condition 7. We recall that the Morita equivalence bimodule $(\mathcal{H}_\infty \otimes \mathbb{C}_p^2)^\sharp$ defines a class σ in $KK(C^\circ, A^\circ \otimes \mathbb{C}\ell_1^p)$. Taking the Kasparov product with the class σ gives two isomorphisms

$$\begin{aligned} - \otimes_{C^\circ} \sigma &: KK^j(\mathbb{C}, C^\circ) \to KK^j(\mathbb{C}, A^\circ \otimes \mathbb{C}\ell_1^p), \quad \text{and} \\ \sigma \otimes_{A^\circ \otimes \mathbb{C}\ell_1^p} - &: KK^j(A^\circ \otimes \mathbb{C}\ell_1^p, \mathbb{C}) \to KK^j(C^\circ, \mathbb{C}). \end{aligned}$$

Then, combined with $\mu \in KK(A \otimes A^\circ \otimes \mathbb{C}\ell_1^p, \mathbb{C})$, we get another two isomorphisms

$$- \otimes_{C^\circ} \sigma \otimes_{A^\circ \otimes \mathbb{C}\ell_1^p} \mu : KK^j(\mathbb{C}, C^\circ) \to KK^j(A, \mathbb{C}), \quad \text{and}$$
$$\sigma \otimes_{A^\circ \otimes \mathbb{C}\ell_1^p} - \otimes_A \mu : KK^j(\mathbb{C}, A) \to KK^j(C^\circ, \mathbb{C})$$

are isomorphisms. The second follows because

$$(\sigma \otimes_{A^\circ \otimes \mathbb{C}\ell_1^p} -) \otimes_A \mu = \sigma \otimes_{A^\circ \otimes \mathbb{C}\ell_1^p} (- \otimes_A \mu)$$

by associativity of the Kasparov product, and $- \otimes_A \mu$ gives an isomorphism from $KK^j(\mathbb{C}, A) = K_j(A)$ to $K^{j+p}(A^\circ) = KK^j(A^\circ \otimes \mathbb{C}\ell_1^p, \mathbb{C})$. Finally, it can be explicitly checked by unpacking modules isomorphisms as in Proposition 3.30 that

$$- \otimes_A (\sigma \otimes_{A^\circ \otimes \mathbb{C}\ell_1^p} \mu) = \sigma \otimes_{A^\circ \otimes \mathbb{C}\ell_1^p} (- \otimes_A \mu).$$

This shows that the class $\lambda := \sigma \otimes_{A^\circ \otimes \mathbb{C}\ell_1^p} \mu$ satisfies the Riemannian version of Poincaré duality, Condition 8. This completes the proof of Theorem 5.1. □

5.2 Proof of Theorem 5.2

Our starting point here is a noncommutative oriented Riemannian manifold $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \mathbf{c}, \Phi)$ and a pre-Morita equivalence bimodule $\mathcal{E}$ between $\mathcal{C}_{\mathcal{D}}(\mathcal{A})$ and $\mathcal{A}$ (even case); or between $\mathcal{C}_{\mathcal{D}}(\mathcal{A})^+$ and $\mathcal{A}$ (odd case).

Proposition 4.25 yields an unbounded Kasparov module for the $\mathbb{Z}_2$-graded algebra $\mathcal{A} \otimes \mathcal{C}_{\mathcal{D}}(\mathcal{A})^\circ$.

The composition of pre-Morita equivalences is again a pre-Morita equivalence, by a variant of [14, Prop. 3.16] or [34, Prop. 4.5]. Since $\mathcal{H}_\infty$ is a pre-Morita equivalence from $\mathcal{C} = \mathcal{C}_{\mathcal{D}}(\mathcal{A})$ to itself, the bimodule $\mathcal{H}_\infty \otimes_{\mathcal{C}} \mathcal{E}$ provides a pre-Morita equivalence between $\mathcal{C}$ and $\mathcal{A}$ in the even case. Indeed, since $\mathcal{H}_\infty$ is free of rank one over $\mathcal{C}$, it is clear that $\mathcal{H}_\infty \otimes_{\mathcal{C}} \mathcal{E} \simeq \mathcal{E}$ as $\mathcal{C}$-$\mathcal{A}$-bimodules; the isomorphism is given by $\rho(\Phi \otimes e) := e$.

If p is even, the grading operator ε on $\mathcal{H}$ need not extend to a well-defined grading operator on $\mathcal{H} \otimes_{\mathcal{C}} \mathcal{E}$; indeed, if $\mathcal{E} \simeq \mathcal{C}^n q$, then $\mathcal{H} \otimes_{\mathcal{C}} \mathcal{E} \simeq \mathcal{H}^n q$ but $q \in M_n(\mathcal{C}_{\mathcal{D}}(\mathcal{A}))$ need not be ε-even. Instead, we must use $\widehat{C} := \pi_{\widehat{\mathcal{D}}}(\mathbf{c}) = C \otimes 1$ and recall that C anticommutes with $\mathcal{D}$ since p is even.

Now we again use $\widehat{\mathcal{D}} = q^\circ(\mathcal{D} \otimes 1_m)q^\circ + q^\circ(M \otimes 1_m)q^\circ$ as an operator on $(\mathcal{H}_\infty \otimes \mathbb{C}^m)q$, where M is a right $\mathcal{A}$-linear operator on $\mathcal{H}_\infty$. With this description, it is straightforward to check that $\widehat{\mathcal{D}}\widehat{C} = -\widehat{C}\widehat{\mathcal{D}}$. This proves the orientation and all of the spin^c condition in the even case.

To complete the proof of finiteness and absolute continuity, we must display a left $\mathcal{C}_{\mathcal{D}}(\mathcal{A})$-valued inner product on $\mathcal{H}_\infty \otimes_{\mathcal{C}} \mathcal{E} \simeq \mathcal{E}$ which captures the scalar product. We define

$$_{\mathcal{C}}(\Phi \otimes e_1 \mid \Phi \otimes e_2) := {_{\mathcal{C}}(\Phi \mid \Phi)}\, \Theta_{e_1,e_2} = \Theta_{e_1,e_2}\, {_{\mathcal{C}}(\Phi \mid \Phi)}, \quad \text{for} \quad e_1, e_2 \in \mathcal{E},$$

using on the right hand side the given $\mathcal{C}_{\mathcal{D}}(\mathcal{A})$-valued inner product on $\mathcal{H}_\infty$ and $\Theta_{e_1,e_2} \in \mathcal{C}_{\mathcal{D}}(\mathcal{A})$. Since $z = {_{\mathcal{C}}(\Phi \mid \Phi)}$ is central and strictly positive, we see that the new inner product is well defined. To show that this inner product captures the scalar product on $\mathcal{H}_\infty \otimes_{\mathcal{C}} \mathcal{E} \simeq \mathcal{E}$, we compute

$$\begin{aligned}
\langle \Phi \otimes e_1 \mid \Phi \otimes e_2 \rangle_{\mathcal{H} \otimes_{\mathcal{C}} \mathcal{E}} &= \langle \Phi \mid \Phi\Theta_{e_1,e_2} \rangle_{\mathcal{H}} = \langle J(\Phi\Theta_{e_1,e_2}) \mid J\Phi \rangle \quad \text{as } J \text{ is anti-unitary} \\
&= \langle \Theta_{e_2,e_1}\Phi \mid J\Phi \rangle \quad \text{since} \quad w\Phi = wJ\Phi = Jw^{*\circ}\Phi = J(\Phi w^*) \\
&= \langle \Theta_{e_2,e_1}\Phi \mid \Phi \rangle = \psi_\omega\big({_{\mathcal{C}}(\Phi \mid \Theta_{e_2,e_1}\Phi)}\big) \\
&= \psi_\omega\big({_{\mathcal{C}}(\Phi \mid \Phi)}\, \Theta_{e_1,e_2}\big) = \psi_\omega\big({_{\mathcal{C}}(\Phi \otimes e_2 \mid \Phi \otimes e_1)}\big).
\end{aligned}$$

This demonstrates both parts of Condition 3.

As in the proof of Theorem 5.1, $\widehat{\mathcal{D}}$ satisfies the regularity condition for the left action of $\mathcal{A}$. So in the even case, we are done.

► If p is odd, we take $\mathcal{C} := \mathcal{C}_{\mathcal{D}}(\mathcal{A})^+$ as defined in Corollary 4.19. We now write $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ where the splitting of $\mathcal{H}$ is into (± 1)-eigenspaces for $C = \pi_{\mathcal{D}}(\mathbf{c})$. We use the fact that $C \otimes 1$ acts as the identity on $\mathcal{H}^+ \otimes_{\mathcal{C}} \mathcal{E}$, and $\varepsilon C = -C\varepsilon$ in odd dimensions, to deduce that

$$\varepsilon \otimes 1 \quad \text{acts on} \quad \mathcal{H} \otimes_{\mathcal{C}} \mathcal{E} = \begin{pmatrix} \mathcal{H}^+ \otimes_{\mathcal{C}} \mathcal{E} \\ \mathcal{H}^- \otimes_{\mathcal{C}} \mathcal{E} \end{pmatrix} \quad \text{as} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Thus $\varepsilon \otimes 1$ provides an isomorphism from $\mathcal{H}^+ \otimes_{\mathcal{C}} \mathcal{E}$ to $\mathcal{H}^- \otimes_{\mathcal{C}} \mathcal{E}$, and conjugation by ε carries $\mathcal{C}_{\mathcal{D}}(\mathcal{A})^\pm$ onto $\mathcal{C}_{\mathcal{D}}(\mathcal{A})^\mp$, by Corollary 4.19. These facts give us a Hilbert space *and* left $\mathcal{C}_{\mathcal{D}}(\mathcal{A})$-module isomorphism

$$\mathcal{H} \otimes_{\mathcal{C}} \mathcal{E} \simeq \mathcal{H}^+ \otimes_{\mathcal{C}} \mathcal{E} \otimes \mathbb{C}^2. \tag{5.7}$$

The copy of $\mathbb{C}^2$ in Equation (5.7) is graded by the action of $C \otimes 1$ as $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. The two-dimensional algebra $\mathbb{C}\langle 1, \varepsilon\rangle \simeq \mathbb{C}\ell_1$ acts on $\mathbb{C}^2$ with ε acting as $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

The operator $\widehat{\mathcal{D}}$ anticommutes with $\varepsilon \otimes 1$ and commutes with $C \otimes 1$, so in the two-by-two matrix picture, it must be of the form $\widehat{\mathcal{D}} =: \begin{pmatrix} \widehat{\mathcal{D}}' & 0 \\ 0 & -\widehat{\mathcal{D}}' \end{pmatrix}$ for a selfadjoint operator $\widehat{\mathcal{D}}'$. The relations satisfied by $\widehat{\mathcal{D}}$ now imply that we obtain an unbounded even Kasparov module for the algebra $\mathcal{A} \otimes \mathbb{C}\ell_1$.

Now by the discussion in the Appendix, $(\mathcal{A}, \mathcal{H}^+ \otimes_{\mathcal{C}} \mathcal{E}, \widehat{\mathcal{D}}')$ is an odd spectral triple. Similarly to the even case, we find that $\mathcal{H}^+ \otimes_{\mathcal{C}} \mathcal{E} \simeq \mathcal{E}$ via $P_+ w\Phi \otimes e \mapsto P_+ w P_+ e$, since $P_+ = \frac{1}{2}(1 + C)$ is central and $\mathcal{E}$ is a pre-Morita equivalence bimodule from $\mathcal{C}_{\mathcal{D}}(\mathcal{A})^+$ to $\mathcal{A}$. Hence the spectral triple $(\mathcal{A}, \mathcal{H}^+ \otimes_{\mathcal{C}} \mathcal{E}, \widehat{\mathcal{D}}')$ satisfies the spin^c condition.

Using $\lambda \in KK(A \otimes C^\circ, \mathbb{C})$ and $\tau = [\mathcal{E} \otimes \mathbb{C}^2_p] \in KK(A^\circ \otimes \mathbb{C}\ell_1^p, C^\circ)$, we compose the isomorphisms $-\otimes_A \lambda : KK^j(\mathbb{C}, A) \to KK^j(C^\circ, \mathbb{C})$ and $\tau \otimes_{C^\circ} - : KK^j(C^\circ, \mathbb{C}) \to KK^j(A^\circ \otimes \mathbb{C}\ell_1^p, \mathbb{C})$ and thereby get an isomorphism

$$-\otimes_A (\tau \otimes_{C^\circ} \lambda) = \tau \otimes_{C^\circ} (-\otimes_A \lambda) : KK^j(\mathbb{C}, A) \to KK^j(A^\circ \otimes \mathbb{C}\ell_1^p, \mathbb{C}) \simeq KK^{j+p}(A, \mathbb{C}).$$

Thus the class $\mu := \tau \otimes_{C^\circ} \lambda$ satisfies the spin^c Poincaré duality condition. □

A Appendix on odd KK-classes

Suppose that $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is an odd spectral triple for the ungraded algebra $\mathcal{A}$. This defines a class in $KK^1(A, \mathbb{C}) \simeq KK^0(A \otimes \mathbb{C}\ell_1, \mathbb{C})$, and we present a $\mathbb{Z}_2$-graded even spectral triple for $\mathcal{A} \otimes \mathbb{C}\ell_1$ representing $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ in the even KK-group.

The representative we use is

$$\left(\mathcal{A} \otimes \mathbb{C}\ell_1, \mathcal{H} \otimes \mathbb{C}^2, \mathcal{D}' = \begin{pmatrix} \mathcal{D} & 0 \\ 0 & -\mathcal{D} \end{pmatrix}, \Gamma' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right),$$

where $\mathbb{C}\ell_1$ is generated by $\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$. In [20, Prop. IV.A.13], Connes employs a different representative, namely

$$\left(\mathcal{A} \otimes \mathbb{C}\ell_1, \mathcal{H} \otimes \mathbb{C}^2, \mathcal{D}'' = \begin{pmatrix} 0 & -i\mathcal{D} \\ i\mathcal{D} & 0 \end{pmatrix}, \Gamma'' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right),$$

with $\mathbb{C}\ell_1$ generated by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

These two representatives define the same KK-class. To see this, first employ the unitary equivalence defined by $U = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$, which conjugates Connes' representative to

$$\left(\mathcal{A} \otimes \mathbb{C}\ell_1, \mathcal{H} \otimes \mathbb{C}^2, \mathcal{D}''' = \begin{pmatrix} 0 & -\mathcal{D} \\ -\mathcal{D} & 0 \end{pmatrix}, \Gamma'' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right),$$

with $\mathbb{C}\ell_1$ now generated by $\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$. Next, we employ the homotopy

$$\left(\mathcal{A} \otimes \mathbb{C}\ell_1, \mathcal{H} \otimes \mathbb{C}^2, \mathcal{D}_t = \begin{pmatrix} \mathcal{D}\sin t & -\mathcal{D}\cos t \\ -\mathcal{D}\cos t & -\mathcal{D}\sin t \end{pmatrix}, \Gamma_t = \begin{pmatrix} \cos t & \sin t \\ \sin t & -\cos t \end{pmatrix}\right), \quad 0 \leqslant t \leqslant \frac{\pi}{2}.$$

This homotopy of $\mathbb{Z}_2$-graded spectral triples takes us from $\mathcal{D}'''$, Γ'' to $\mathcal{D}'$, Γ', and so the equality of the KK-classes is established. The same argument can be carried through unchanged for the associated Kasparov modules defined by applying the real function $x \mapsto x(1+x^2)^{-1/2}$ to the various operators $\mathcal{D}$, $\mathcal{D}'$, $\mathcal{D}''$, $\mathcal{D}'''$.

Now given an even ($\mathbb{Z}_2$-graded) Kasparov module for the $\mathbb{Z}_2$-graded algebra $A \otimes \mathbb{C}\ell_1$ and $\mathbb{C}$, we can take a representative of the form

$$\left(A \otimes \mathbb{C}\ell_1, \mathcal{H} \otimes \mathbb{C}^2, \begin{pmatrix} F & 0 \\ 0 & -F \end{pmatrix}, \Gamma' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right), \tag{A.1}$$

by taking $\mathbb{C}\ell_1$ to be generated by $\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$. Then an odd Kasparov module for the ungraded algebra A, representing the class in $KK^1(A, \mathbb{C})$ obtained from the class of the Kasparov module (A.1) via the isomorphism $KK^0(A \otimes \mathbb{C}\ell_1, \mathbb{C}) \simeq KK^1(A, \mathbb{C})$, is given by $(A, \mathcal{H}, F)$.

Replacing the bounded operator F by an unbounded operator $\mathcal{D}$ does not alter the discussion.

References

[1] A. Connes, "Gravity coupled with matter and foundation of noncommutative geometry", Commun. Math. Phys. **182** (1996), 155–176.

[2] A. Rennie and J. C. Várilly, "Reconstruction of manifolds in noncommutative geometry", arXiv:math.oa/0610418, San José, 2006.

[3] A. Connes, "On the spectral characterization of manifolds", J. Noncommut. Geom. **7** (2013), 1–82.

[4] R. J. Plymen, "Strong Morita equivalence, spinors and symplectic spinors", J. Oper. Theory **16** (1986), 305–324.

[5] S. Baaj and P. Julg, "Théorie bivariante de Kasparov et opérateurs non bornées dans les C^*-modules hilbertiens", C. R. Acad. Sci. Paris **296** (1983), 875–878.

[6] D. Kucerovsky, "The KK-product of unbounded modules", K-Theory **11** (1997), 17–34.

[7] B. Mesland, "Unbounded bivariant K-theory and correspondences in noncommutative geometry", J. reine angew. Math. **691** (2014), 101–172.

[8] D. Zhang, "Projective Dirac operators, twisted K-theory and local index formula", J. Noncommut. Geom. **8** (2014), 179–215.

[9] B. Ćaćić, "A reconstruction theorem for almost-commutative spectral triples", Lett. Math. Phys. **100** (2012), 181–202.

[10] J. Boeijink and W. D. van Suijlekom, "The noncommutative geometry of Yang–Mills fields", J. Geom. Phys. **61** (2011), 1122–1134.

[11] S. Lord, "Riemannian noncommutative geometry", Ph. D. thesis, University of Adelaide, 2004.

[12] J. Fröhlich, O. Grandjean and A. Recknagel, "Supersymmetric quantum theory and noncommutative geometry", Commun. Math. Phys. **203** (1999), 119–184.

[13] J. M. Gracia-Bondía, J. C. Várilly and H. Figueroa, *Elements of Noncommutative Geometry*, Birkhäuser, Boston, 2001.

[14] I. Raeburn and D. P. Williams, *Morita Equivalence and Continuous-Trace C^*-algebras*, Mathematical Surveys and Monographs **60**, American Mathematical Society, Providence, RI, 1998.

[15] M. A. Rieffel, "Induced representations of C^*-algebras", Adv. Math. **13** (1974), 176–257.

[16] G. G. Kasparov, "The operator K-functor and extensions of C^*-algebras", Math. USSR Izv. **16** (1981), 513–572.

[17] L. B. Schweitzer, "A short proof that $M_n(A)$ is local if A is local and Fréchet", Int. J. Math. **3** (1992), 581–589.

[18] B. Blackadar, *K-Theory for Operator Algebras*, 2nd edition, Cambridge University Press, Cambridge, 1998.

[19] A. Ya. Helemskii, *The Homology of Banach and Topological Algebras*, Kluwer, Dordrecht, 1989.

[20] A. Connes, *Noncommutative Geometry*, Academic Press, London and San Diego, 1994.

[21] N. Berline, E. Getzler and M. Vergne, *Heat Kernels and Dirac Operators*, Springer, Berlin, 1992.

[22] A. L. Carey, J. Phillips, A. Rennie and F. A. Sukochev, "The Hochschild class of the Chern character for semifinite spectral triples", J. Funct. Anal. **213** (2004), 111–153.

[23] A. Rennie, "Smoothness and locality for nonunital spectral triples", K-Theory **28** (2003), 127–165.

[24] A. L. Carey, V. Gayral, A. Rennie, F. Sukochev, "Integration on locally compact noncommutative spaces", J. Funct. Anal. **263** (2012), 383–414.

[25] A. L. Carey, A. Rennie, F. A. Sukochev, A. Sedaev, "The Dixmier trace and asymptotics of zeta functions", J. Funct. Anal. **249** (2007), 253–283.

[26] J.-L. Loday, *Cyclic Homology*, 2nd edition, Springer, Berlin, 1996.

[27] H. Kosaki, *Type III Factors and Index Theory*, RIM–GARC Lecture Notes **43**, Seoul National University, Seoul, 1998.

[28] T. Krajewski, "Classification of finite spectral triples", J. Geom. Phys. **28** (1998), 1–30.

[29] M. Paschke and A. Sitarz, "Discrete spectral triples and their symmetries", J. Math. Phys. **39** (1998), 6191–6205.

[30] F. Cipriani, D. Guido, S. Scarlatti, "A remark on trace properties of K-cycles", J. Oper. Theory **35** (1996), 179–189.

[31] M. Takesaki, *Theory of Operator Algebras II*, Encyclopaedia of Mathematical Sciences **125**, Springer, Berlin, 2003.

[32] G. G. Kasparov, "Equivariant KK-theory and the Novikov conjecture", Invent. Math. **91** (1988), 147–201.

[33] A. L. Carey and J. Phillips, "Unbounded Fredholm modules and spectral flow", Can. J. Math. **50** (1998), 673–718.

[34] E. C. Lance, *Hilbert C^*-modules*, London Mathematical Society Lecture Notes **210**, Cambridge University Press, Cambridge, 1995.