ON THE SUPREMUM OF A FAMILY OF SET FUNCTIONS

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ABSTRACT. The concept of supremum of a family of set functions was introduced by M. Veraar and I. Yaroslavtsev (2016) for families of measures defined on a measurable space. We expand this concept to include families of set functions in a very general setting. The case of families of signed measures is widely discussed and exploited.

CONTENTS

1. INTRODUCTION

Given a family of sets $\mathcal{A}$ and a family $\{f_j\}_{j\in J}$ of real extended set functions defined on $\mathcal{A}$, the classical supremum defined as $f(A) := \sup\{f_j(A) : A \in \mathcal{A}\}$ gives the least upper bound of the family under the natural order (see Definition 2.8).

In practical situations, one might need to work with a smaller class of set functions, as for instance the class of measures over a sigma-algebra, or the class of finitely additive set functions over an algebra. The classical definition of supremum given above might produce an object which is not in the class.

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In [9], they show that, in the class of (non-negative) measures on a measurable space $(\Omega, \mathcal{F})$, a suitable definition for the supremum of a family of measures $(\nu_j)_{j\in J}$ is

$$\mu(A) = \sup_{\Pi\in\mathscr{P}(A)} \sum_{C\in\Pi} \sup_{j\in J} \nu_j(C),$$

where $\mathscr{P}(A)$ is the family of all finite partitions of A by elements of $\mathcal{F}$.

In this paper, we show that this definition works perfectly well in larger classes of set functions and explore some nice consequences of this concept.

Although the literature provides plenty of concepts and results related to that of the supremum of measures, we could only find the first reference in [9]. Some works like [5] and [7] have mentioned the existence of a supremum in the case of a family of measures, but not in a constructive way. In [9], they do provide a constructive definition for this particular kind of families, developing a few tools they needed in their construction of the quadratic variation of a cylindrical local martingale. When trying to apply this concept to our recent work on cylindrical martingale-valued measures [1], we have found it necessary to extend it to families of a more general class of set functions, for instance to families of random measures, signed measures or even random sub-additive set functions.

In Section 2 we give the definition of the supremum of a family of set functions in the most general possible setting. We establish three levels of *admissibility*, that will be exploited according to the kind of properties we need the supremum to inherit from the given family. The weak admissibility is the least we can ask in order to define the set function supremum, but we need admissibility if we want this function to inherit at least finitely sub-additivity from the given family.

In Section 3 we work with set functions defined on rings. The main result on this section, Theorem 3.4, shows that the set function supremum of an admissible family of sub-additive set functions is a signed measure, whenever we can control it from below. We also introduce the concept of set function infimum.

Results in Section 4 can be considered as the main contributions of the paper. We concentrate on families of signed measures, for which the concepts of admissibility and strong admissibility are equivalent. We use the set function supremum to express the positive and negative part of signed measure, as well as the total variation. The main result, Theorem 4.11, generalizes a result stated on [9] for families of (non-negative) measures. We finish by giving some very pleasant applications of this result.

2. The set function supremum

Throughout this section we consider a non-empty set S and a family $\mathcal{A}$ of subsets of S. Unless otherwise stated, no particular structure is assumed on $\mathcal{A}$. We denote by $\mathcal{C}$ the class of all set functions $f : \mathcal{A} \to \overline{\mathbb{R}}$.

Definition 2.1. Consider a family $(f_j)_{j\in J}$ on $\mathcal{C}$. We say that this family is:

(i) *weakly admissible* if, for any pair of disjoint sets $C, D \in \mathcal{A}$,
$$\sup_{j\in J} f_j(D) = \infty \Rightarrow \sup_{j\in J} f_j(C) > -\infty.$$

(ii) *admissible* if for each $C \in \mathcal{A}$ we have
$$\sup_{j\in J} f_j(C) > -\infty.$$

(iii) *strongly admissible* if there is $a \in \mathbb{R}$ such that
$$\forall\, C \in \mathcal{A}, \quad \sup_{j\in J} f_j(C) > a.$$

It is evident that every strongly admissible family is admissible, and every admissible family is weakly admissible. The importance of these properties will become clear as we establish and discuss the definition of set function supremum. Before that, we have the following example.

Example 2.2. *For $A \in \mathcal{B}(\mathbb{R})$ let $|A|$ denote its Lebesgue measure.*

(i) For every $n \in \mathbb{N}$ let $\alpha_n : \mathcal{B}(\mathbb{R}) \to \overline{\mathbb{R}}$ be defined by
$$\alpha_n(A) := |A \cap (0,n]| - |A \cap (-n,0]|\,.$$
The family $(\alpha_n)_{n\geq 1}$ is strongly admissible.

(ii) Let $g : \mathcal{B}(\mathbb{R}) \to \overline{\mathbb{R}}$ be defined by
$$g(A) = |A \cap (0,\infty)| - |A \cap (-\infty,0)|\,, \quad \forall A \in \mathcal{B}(\mathbb{R}),\ A \neq \mathbb{R},$$
and $g(\mathbb{R}) = 0$. The single element family (g) is not weakly admissible since for $C = (-\infty, 0)$ and $D = (0,\infty)$ we have $g(D) = \infty$ but $g(C) = -\infty$.

The following results state useful sufficient conditions for admissibility and for strong admissibility. Their proofs are simple so we omit them.

Lemma 2.3. *Any family $(f_j)_{j\in J}$ of functions from $\mathcal{A}$ into $(-\infty,\infty]$ is admissible.*

Lemma 2.4. *Consider a family $(f_j)_{j\in J}$ on $\mathcal{C}$. Assume that there exists $a \in \mathbb{R}$ and $j_0 \in J$ such that $f_{j_0}(A) \geq a$ for every $A \in \mathcal{A}$. Then the family $(f_j)_{j\in J}$ is strongly admissible. In particular, any family on $\mathcal{C}$ for which at least one of its members is non-negative, is strongly admissible.*

Definition 2.5. Consider a weakly admissible family $(f_j)_{j\in J}$ on $\mathcal{C}$. We define its *set function supremum* as
$$\left(\operatorname*{ssup}_{j\in J} f_j\right)(A) := \sup_{\Pi\in\mathscr{P}(A)} \sum_{C\in\Pi} \sup_{j\in J} f_j(C), \quad \forall A \in \mathcal{A},$$

where $\mathscr{P}(A)$ is the family of all finite partitions of A by elements of $\mathcal{A}$.

Remark 2.6. The weak admissibility condition avoids having undetermined sums in the last definition. When the family is admissible, the supremum does not achieve the value $-\infty$.

Example 2.7. *Consider $S=\{0,1\}$ and $\mathcal{A}=2^S$. For $n=1,2,\dots$ define $\alpha_n=n\cdot\delta_{\{0\}}-\infty\cdot\delta_{\{1\}}$ (here $\infty\cdot 0=0$). Each α_n is clearly a signed measure on $(S,2^S)$. But notice that $\sup_{n\geq 1}\alpha_n(\{0\})=+\infty$ and $\sup_{n\geq 1}\alpha_n(\{1\})=-\infty$. This family is not weakly admissible so we are not able to apply the previous definition for $A=S$.*

It is clear from the definition that $\operatorname{ssup}_{j\in J} f_j\in\mathcal{C}$. In order to explore more properties of the set function supremum we will need to be able to compare elements in $\mathcal{C}$.

Definition 2.8. Given $f,g\in\mathcal{C}$, we write $f\leq g$ whenever $f(A)\leq g(A)$, $\forall A\in\mathcal{A}$. If this is the case, we say that g *dominates* f.

It is clear that the relation "$\leq$" introduces a partial order on $\mathcal{C}$. We will show (see Corollary 2.13 below) that the set function supremum of an admissible family is the smallest finitely super-additive set function that dominates each f_j. Here, a function $g\in\mathcal{C}$ is called *finitely super-additive* if whenever $A,B\in\mathcal{A}$ are disjoint and $A\cup B\in\mathcal{A}$, then $g(A)+g(B)$ is a well defined extended real number and $g(A)+g(B)\leq g(A\cup B)$.

Lemma 2.9. *Let $(f_j)_{j\in J}$ be weakly admissible and $f=\operatorname{ssup}_{j\in J} f_j$. If $g\in\mathcal{C}$ is finitely super-additive and dominates each f_j, then it also dominates f.*

Proof. Let $A\in\mathcal{A}$. For any $\Pi\in\mathscr{P}(A)$ we have

$$\sum_{C\in\Pi}\sup_{j\in J} f_j(C)\leq\sum_{C\in\Pi} g(C)\leq g(A),$$

the last inequality is a consequence of the finite super-additivity of g. Since $f(A)$ is the supremum of the left-hand side, this implies $f(A)\leq g(A)$. $\square$

In order to talk about super-additivity of the set function supremum, weakly admissibility of the family is not enough.

Example 2.10. *Consider $S=\{1,2,3,...\}$ and $\mathcal{A}=2^S$. Define f on $\mathcal{A}$ by $f(\emptyset)=0$, $f(A)=1$ if $1\notin A\neq\emptyset$, $f(A)=-\infty$ if $1\in A$. Let $A=\{1\}$ and $B=\{2,3,\dots\}$. The unitary family $\{f\}$ is weakly admissible but not admissible. The supremum $\check{f}=\operatorname{ssup}\{f\}$, though well defined, is not supper-additive. In fact, notice that $\check{f}(A)=-\infty$ and $\check{f}(B)=+\infty$.*

Lemma 2.11. *If $(f_j)_{j\in J}$ is admissible, then $f=\operatorname{ssup}_{j\in J} f_j$ is finitely super-additive and dominates each f_j.*

Proof. Let $A, B \in \mathcal{A}$ be disjoint and such that $A \cup B \in \mathcal{A}$. Because of the admissibility, $f(A), f(B) \in (-\infty, \infty]$. Let $c \in \mathbb{R}$ such that $c < f(A) + f(B)$. Let $a < f(A)$ and $b < f(B)$ such that $a + b = c$. Choose partitions $\Pi_A \in \mathscr{P}(A)$ and $\Pi_B \in \mathscr{P}(B)$ such that

$$a < \sum_{C \in \Pi_A} \sup_{j \in J} f_j(C), \quad b < \sum_{C \in \Pi_B} \sup_{j \in J} f_j(C).$$

Then for each $C \in \Pi_A \cup \Pi_B$, there exists $j_C \in J$ such that

$$a < \sum_{C \in \Pi_A} f_{j_C}(C), \qquad b < \sum_{C \in \Pi_B} f_{j_C}(C).$$

Since $\Pi_A \cup \Pi_B \in \mathscr{P}(A \cup B)$, it follows that

$$c = a + b < \sum_{C \in \Pi_A \cup \Pi_B} f_{j_C}(C) \leq f(A \cup B).$$

The inequality

$$f(A) + f(B) \leq f(A \cup B)$$

follows from the fact that $c < f(A) + f(B)$ was arbitrary. $\square$

The results of these lemmas can be applied to quite general situations. For instance, $\mathcal{A}$ could be so small that no partition other that $\Pi = \{A\}$ exists for each set A.

Example 2.12. *Consider $S = \{0, 1, 2, 3\}$ and $\mathcal{A} = \{\{0\}, \{0, 1\}, S\}$. Any set function f defined on $\mathcal{A}$ is additive. In fact, there are no disjoint sets on $\mathcal{A}$ to test. In this case, $\left(\operatorname{ssup}_{j \in J} f_j\right)(A) = \sup_{j \in J} f_j(A)$. We get a similar situation with $\mathcal{A} = \{\{0\}, \{1, 2\}, S\}$. In this case we have two disjoint sets, but their union is not in $\mathcal{A}$.*

By combining the results of Lemmas 2.9 and 2.11 we conclude the following:

Corollary 2.13. *For an admissible family $(f_j)_{j \in J}$, the supremum $\operatorname{ssup}_{j \in J} f_j$ is the smallest finitely super-additive set function that dominates each f_j.*

Remark 2.14. Notice the special case of a unitary family $\{f\}$, with $f > -\infty$. In this case, $\operatorname{ssup}\{f\}$ is the smallest finitely super-additive set function defined on $\mathcal{A}$ that dominates f. In particular, $f = \operatorname{ssup}\{f\}$ if and only if f is finitely super-additive.

Remark 2.15. Consider an admissible family of set functions $(\nu_j)_{j \in J}$, defined on $\mathcal{A} \subseteq 2^S$.

(i) If for each $C \in \mathcal{A}$ there exists $j \in J$ such that $\nu_j(C) \geq 0$, then

$$\operatorname*{ssup}_{j \in J} \nu_j = \operatorname*{ssup}_{j \in J} (\nu_j)_+ = \operatorname{ssup}\{\nu_j : j \in J\} \cup \{0\},$$

where $(\nu_j)_+(A) = \max\{\nu_j(A), 0\}$.

(ii) Given $k \in J$ define $\mu_j(A) := \max\{\nu_j(A), \nu_k(A)\}$ and $\lambda_j := \operatorname{ssup}\{\nu_j, \nu_k\}$. Then

$$\operatorname*{ssup}_{j \in J} \nu_j = \operatorname*{ssup}_{j \in J} \mu_j = \operatorname*{ssup}_{j \in J} \lambda_j.$$

(iii) If for each $C \in \mathcal{A}$

$$\sup_{j\in J} \nu_j(C) \geq -\inf_{j\in J} \nu_j(C),$$

then,

$$\operatorname*{ssup}_{j\in J} \nu_j = \operatorname*{ssup}_{j\in J} |\nu_j| \, .$$

3. Sub-additive set functions on rings

In this section we assume that the family $\mathcal{A}$ is a ring. More precisely, $\mathcal{A}$ is closed under finite unions and difference of sets, and therefore it is also closed under finite intersections. We will try to prove that $\operatorname{ssup} f_j$ inherits the sub-additivity from the f_j's. Since negative values are allowed, in the definition of sub-additivity we must assume that the sets are disjoint. We say that a function $f \in \mathcal{C}$ is *sub-additive* if

$$f\left(\bigcup_{n=1}^{\infty} B_n\right) \leq \sum_{n=1}^{\infty} f(B_n) \tag{3.1}$$

for any countable family (B_n) of pairwise disjoint sets on $\mathcal{A}$ such that $\cup B_n \in \mathcal{A}$. As part of the definition, the sum on the right-hand side must make sense independently of any reordering. *Finite sub-additivity* is defined analogously.

Lemma 3.1. *Consider an admissible family $(f_j)_{j\in J}$ of functions defined on a ring $\mathcal{A}$. If each f_j is sub-additive, then $f := \operatorname{ssup} f_j$ is also sub-additive.*

Proof. Let (B_n) be a sequence of pairwise disjoint sets on $\mathcal{A}$ such that $B = \cup B_n \in \mathcal{A}$. If $\Pi \in \mathscr{P}(B)$, it follows that $C \cap B_n \in \mathcal{A}$ for each $C \in \Pi$. By applying the sub-additivity of f_j, and the super-additivity of f one gets

$$\sum_{C\in\Pi} \sup_{j\in J} f_j(C) \leq \sum_{C\in\Pi} \sup_{j\in J} \sum_{n=1}^{\infty} f_j(C\cap B_n) \leq \sum_{n=1}^{\infty} \sum_{C\in\Pi} f(C\cap B_n) \leq \sum_{n=1}^{\infty} f(B_n).$$

Taking the supremum on the left we get the result. □

As the next example illustrates, in Lemma 3.1 the admissibility assumption can not be replaced by weak admissibility.

Example 3.2. *Consider $S = \{1, 2, \ldots\}$, $\mathcal{A} = 2^S$ and define*

$$\alpha_n(A) = \sum_{k\in A} p_{nk},$$

where $p_{nk} = (-1)^k$ for $k \leq n$ and $p_{nk} = -1$ for $k > n$. The sum is always well defined and gives $\alpha_n(A) = -\infty$ for any infinite set A. Each α_n is sub-additive, and better yet, a signed measure. The family (α_n) is weakly admissible but not admissible. For $\mu = \operatorname{ssup} \alpha_n$ we have $\mu(\{k\}) = (-1)^k$, consequently μ is not sub-additive.

Lemma 3.3. *Suppose the family $(f_j)_{j\in J}$ is admissible and each f_j is finitely sub-additive on a ring $\mathcal{A}$. Then $f := \operatorname{ssup}_{j\in J} f_j$ is finitely additive.*

Proof. The argument of Lemma 3.1 clearly works for finite families of disjoint sets. This gives the finite sub-additivity, and Lemma 2.11 finishes the work. □

Recall that, if a set function defined on a measurable space $(S, \mathcal{F})$ is sub-additive, non-negative and finitely additive, then it is a measure. This fact is used in the proof of the following useful result.

Theorem 3.4. *Consider an admissible family of sub-additive set functions defined on a measurable space $(S, \mathcal{F})$. Let μ be the set function supremum of that family, and suppose there is a finite measure ν such that $\mu \geq -\nu$. Then μ is a signed measure. In particular, if $\mu \geq 0$, then it is a measure.*

Proof. By Lemmas 3.1 and 3.3, μ is finitely additive and sub-additive, then so is the non-negative set function $\mu + \nu$. We conclude that $\mu + \nu$ is a measure, so $\mu = (\mu + \nu) - \nu$ is a signed measure. □

Corollary 3.5. *Let $(\nu_j)_{j\in J}$ be a family of sub-additive set functions defined on a measurable space $(S, \mathcal{F})$. If for some $j \in J$ we have $\nu_j \geq 0$, then $\nu := \operatorname{ssup}_{j\in J} \nu_j$ is a measure. In particular, the set function supremum of a family of measures on (S, Σ), is itself a measure.*

Proof. Since the family (n_j) has a non-negative element, then it is strongly admissible (Lemma 2.4) and clearly its supremum is non-negative. Hence by Theorem 3.4 its supremum is a measure. The second assertion is an easy consequence of the former. □

Remark 3.6. The result in Corollary 3.5 generalizes Lemma 2.6 in [9].

We finish this section by introducing the concept of set function infimum. Consider a family $(\alpha_j)_{j\in J}$ of set functions defined on a family of sets $\mathcal{A}$. If the family $(-\alpha_j)$ is weakly admissible, we can define

$$\operatorname*{iinf}_{j\in J} \alpha_j := -\operatorname*{ssup}_{j\in J}(-\alpha_j).$$

We immediately get

$$\left(\operatorname*{iinf}_{j\in J} \alpha_j\right)(A) = \inf_{\Pi\in\mathscr{P}(A)} \sum_{C\in\Pi} \inf_{j\in J} \alpha_j(C).$$

If $(-\alpha_j)$ is admissible, then $\operatorname{iinf}_{j\in J} \alpha_j$ is the biggest finitely sub-additive set function that is dominated by each α_j. If $\mathcal{A}$ is a ring and each α_j is finitely super-additive, then $\operatorname{iinf}_{j\in J} \alpha_j$ is finitely additive; if each α_j is super-additive, then $\operatorname{iinf}_{j\in J} \alpha_j$ is super-additive.

4. Supremum of signed measures

We now consider the supremum $\mu := \operatorname{ssup}_{j\in J} \alpha_j$ of an admissible family of signed measures defined on a σ-algebra $\mathcal{A}$.

Example 4.1. *Consider the (admissible) family of finite signed measures $(\alpha_n)_{n\geq 1}$ defined on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ by*

$$\alpha_n(A) := |A \cap (0, n]| - |A \cap (-n, 0]| \,.$$

Since $\check{\mu} = \operatorname{ssup}_{n\geq 1} \alpha_n$ is finitely additive we have

$$\check{\mu}(A) = \check{\mu}(A \cap \mathbb{R}_-) + \check{\mu}(A \cap \mathbb{R}_+) = |A \cap \mathbb{R}_+| - |A \cap (-1, 0]| \,.$$

This is clearly a signed measure.

Example 4.2. *Consider now the (strongly admissible) family of finite signed measures $(\alpha_n)_{n\geq 1}$ given by*

$$\alpha_n(A) := |A \cap (n-1, n]| - |A \cap (-n, -n+1]| \,.$$

In this case $\check{\mu} = \operatorname{ssup}_{n\geq 1} \alpha_n$ is the measure given by $\check{\mu}(A) = |A \cap \mathbb{R}_+|$.

Since signed measures are sub-additive and finitely additive, by Theorem 3.4 μ is a signed measure if (and only if) there exists a finite measure ν such that $\mu \geq -\nu$. In particular, if some member of the family misses the value $-\infty$, then μ is a signed measure on $\mathcal{A}$. The following lemma shows that in the present case, admissibility does the whole work.

For a signed measure α, we consider the Jordan decomposition $\alpha = \alpha^+ - \alpha^-$.

Lemma 4.3. *Consider a family $(\nu_j)_{j\in J}$ of signed measures defined on a measurable space $(S, \mathcal{F})$. The following are equivalent:*

(i) The family (ν_j) is admissible

(ii) The family (ν_j) is strongly admissible

(iii) There exists $k \in J$ such that $\nu_k > -\infty$.

Proof. If the family is admissible, by definition there exists $k \in J$ such that $\nu_k(S) > -\infty$. The additivity implies that ν_k misses the value $-\infty$. This proves that (i) implies (iii). To see that (iii) implies (ii) it is enough to take

$$a = -\nu_k^-(S).$$

Finally, it is clear that (ii) implies (i). □

Last lemma, combined with Theorem 3.4, gives us the following result.

Theorem 4.4. *If $(\nu_j)_{j\in J}$ is an admissible family of signed measures defined on a measurable space $(S, \mathcal{F})$, then $\mu := \operatorname{ssup} \nu_j$ is a signed measure with $\mu^- < \infty$.*

We now use the concept of set function supremum to provide alternative expressions for α^+, α^- and $\mathrm{var}(\alpha) = \alpha^+ + \alpha^-$. We start by noticing that, for a signed measure α, $\mathrm{ssup}\,\{\alpha, 0\}$ is the measure defined by

$$\mathrm{ssup}\,\{\alpha, 0\}(A) := \sup_{\Pi} \sum_{C\in\Pi} \max\{\alpha(C), 0\} = \sup_{\Pi} \sum_{C\in\Pi} \alpha(C)_+.$$

Lemma 4.5. *For a signed measure α we have*

$$\alpha^+ = \mathrm{ssup}\{\alpha, 0\}, \quad \alpha^- = \mathrm{ssup}\,\{-\alpha, 0\}.$$

Proof. In fact, let Ω^+ and Ω^- be respectively the positive and negative sets of a Hahn decomposition for α. By considering the partition $\Pi = \{A \cap \Omega^+, A \cap \Omega^-\}$ we have

$$\mathrm{ssup}\{\alpha, 0\}(A) \geq \alpha(A \cap \Omega^+) = \alpha^+(A).$$

On the other hand, since for every $A \in \mathcal{A}$ we have $\alpha^+(A) \geq 0$ and also

$$\alpha(A) = \alpha^+(A) - \alpha^-(A) \leq \alpha^+(A),$$

we must have $\mathrm{ssup}\{\alpha, 0\}(A) \leq a^+(A)$, and thus the desired equality. To get the second identity, we apply what we have just proven to $-\alpha$. □

Remark 4.6. If $|\alpha|$ denotes the absolute value of α, which is a sub-additive set function, then $\mathrm{var}(\alpha) = \alpha^+ + \alpha^-$ coincides with $\mathrm{ssup}\,\{|\alpha|\}$ (the supremum of a unitary family). This readily follows from the definition of supremum of measures and the general definition of variation for a signed measure.

Lemma 4.7. *For a signed measure α we have*

$$\mathrm{var}(\alpha) = \mathrm{ssup}\{\alpha, -\alpha\}$$

Proof. In fact, by definition we have

$$\mathrm{ssup}\{\alpha, -\alpha\}(A) = \sup_{\Pi} \sum_{C\in\Pi} \max\{\alpha(C), -\alpha(C)\} = \sup_{\Pi} \sum_{C\in\Pi} |\alpha(C)| = \mathrm{var}(\alpha). \qquad \square$$

Because of Theorem 3.4, when $(-\alpha_j)$ is an admissible family of sub-additive set functions, then $\mu := \mathrm{iinf}_{j\in J}\, \alpha_j$ is a signed measure if and only if there exists a finite measure ν such that $\mu \leq \nu$. In particular, if there exists j such that $\alpha_j < \infty$, then μ is a signed measure. More particularly, if $(\mu_j)_{j\in J}$ is a family of measures and $(-\mu_j)$ is admissible, according to Theorem 4.4, $\mathrm{iinf}_{j\in J}\, \nu_j$ is also a measure.

Theorem 4.8. *Consider a family $(\nu_j)_{j\in J}$ of finite signed measures on the measurable space $(\Omega, \mathcal{F})$. The signed measure $\mu := \mathrm{ssup}_{j\in J}\, \nu_j$ satisfies*

$$\mu^+ = \mathrm{ssup}_{j\in J}\, \nu_j^+, \quad \mu^- \leq \mathrm{iinf}_{j\in J}\, \nu_j^-.$$

If J is countable, we also have

$$\mu^- = \operatorname*{iinf}_{j\in J} \nu_j^- .$$

Proof. It is clear that $\mu \leq \operatorname{ssup} \nu_j^+$. Let Ω^+ and Ω^- form a Hahn decomposition of μ. We have $\mu \geq 0$ and $\mu \geq \nu_j$ on Ω^+ and consequently $\mu \geq \operatorname{ssup}\{\nu_j, 0\} = \nu_j^+$ on Ω^+. This shows that on Ω^+ we have $\mu^+ = \mu = \operatorname{ssup} \nu_j^+$. On Ω^- we have $\nu_j \leq \mu \leq 0$ and consequently $\nu_j^+ = 0$ for each j. We conclude $\mu = \operatorname{ssup} \nu_j^+$ on Ω^+ and on Ω^-, so all over Ω.
Now, on Ω^- we also have $\nu_j^- = \operatorname{ssup}\{-\nu_j, 0\} = -\nu_j$ and then

$$\mu^- = -\mu = -\operatorname{ssup} \nu_j = \operatorname{iinf}(-\nu_j) = \operatorname{iinf} \nu_j^-$$

while on Ω^+

$$0 = \mu^- \leq \operatorname*{iinf}_{j\in J} \nu_j^- .$$

It remains to show that, for the countable case, $\operatorname{iinf} \nu_j = 0$ on Ω^+. In that case, we can choose a Hahn decomposition $\{\Omega_j^+, \Omega_j^-\}$ for each ν_j, it is easy to check that $\Omega^+ = \cup\Omega_j^+$ and $\Omega^- = \cap\Omega_j^-$ is a Hahn decomposition for μ. Since $\operatorname{iinf} \nu_j^- \leq \nu_k^- = 0$ on Ω_k^+ for each $k \in J$, the result follows. □

Using an analogous argument, we can infer the next result.

Corollary 4.9. *Consider a family $(\nu_j)_{j\in J}$ of finite signed measures on the measurable space $(\Omega, \mathcal{F})$. The signed measure $\mu := \operatorname{iinf}_{j\in J} \nu_j$ satisfies*

$$\mu^- = \operatorname*{ssup}_{j\in J} \nu_j^-, \quad \mu^+ \leq \operatorname*{iinf}_{j\in J} \nu_j^+.$$

If J is countable, we also have

$$\mu^+ = \operatorname*{iinf}_{j\in J} \nu_j^+.$$

Theorem 4.10. *Consider a family $(\nu_j)_{j\in J}$ of finite signed measures on the measurable space $(\Omega, \mathcal{F})$, and let $\mu = \operatorname{ssup}_{j\in J} \nu_j$. Then*

$$\operatorname{var}(\mu) \leq \operatorname*{ssup}_{j\in J} |\nu_j| = \operatorname*{ssup}_{j\in J} \operatorname{var}(\nu_j).$$

Proof. Since for all $A \in \mathcal{F}$ we have

$$\begin{aligned} \mu(A) &= (\operatorname*{ssup}_{j\in J} \nu_j)(A) \leq (\operatorname*{ssup}_{j\in J} |\nu_j|)(A), \\ -\mu(A) &= (\operatorname*{iinf}_{j\in J}(-\nu_j))(A) \leq (\operatorname*{iinf}_{j\in J} |\nu_j|)(A) \leq (\operatorname*{ssup}_{j\in J} |\nu_j|)(A), \end{aligned}$$

then $|\mu| \leq \operatorname{ssup}_{j\in J} |\nu_j|$. It follows that $\operatorname{var}(\mu) = \operatorname{ssup}\{|\mu|\} \leq \operatorname{ssup}_{j\in J} |\nu_j|$. Now, since for every $j \in J$ we have $|\nu_j| \leq \operatorname{var}(\nu_j)$ then $\operatorname{ssup}_{j\in J} |\nu_j| \leq \operatorname{ssup}_{j\in J} \operatorname{var}(\nu_j)$. On the other hand, for every $j \in J$, $|\nu_j| \leq \operatorname{ssup}_{j\in J} |\nu_j|$, so $\operatorname{var}(\nu_j) = \operatorname{ssup}\{|\nu_j|\} \leq \operatorname{ssup}_{j\in J} |\nu_j|$. By definition of supremum of measures we get

$$\operatorname*{ssup}_{j\in J} \operatorname{var}(\nu_j) \leq \operatorname*{ssup}_{j\in J} |\nu_j|$$

and therefore we have the desired equality. □

4.1. **Signed measures with density.** We extend Lemma 2.8 on [9] to the case of signed measures.

Theorem 4.11. *Let $(S, \mathcal{A}, \nu)$ be a σ-finite measure space. Let $\mathfrak{F}$ be a family of measurable functions from S into $(-\infty, \infty]$, such that*

$$\int f^- d\nu < \infty \text{ for each } f \in \mathfrak{F}.$$

Let $(f_j)_{j\in\mathbb{N}}$ be a sequence in $\mathfrak{F}$. Define $\check{f} = \sup_{j\geq 1} f_j$ and assume that $\sup_{f\in\mathfrak{F}} f = \check{f}$. For each $f \in \mathfrak{F}$ let μ_f be the signed measure defined by

$$\mu_f(A) = \int_B f d\nu, \quad A \in \mathcal{A}.$$

If we define $\check{\mu} := \operatorname{ssup}_{f\in\mathfrak{F}} \mu_f$, then $\check{\mu} = \operatorname{ssup}_{j\geq 1} \mu_{f_j}$ and

$$\check{\mu}(A) = \int_A \check{f} d\nu. \tag{4.1}$$

Proof. Clearly $\check{f}$ is measurable, so $A \mapsto \int_A \check{f} d\nu$ defines a signed measure that dominates each μ_f. This gives the inequality "$\leq$" in (4.1).

For the other inequality, we adapt the proof of Lemma 2.8 in [9], which is established for non-negative functions and measures. Let $A \in \mathcal{A}$, $n \in \mathbb{N}$ and $0 < \varepsilon < 1$. Define

$$A_1 = \{s \in A : f_1(s) > (\check{f}(s) \wedge n)(1 - \varepsilon) > 0\}$$

and

$$A_{j+1} = \{s \in A : f_{j+1}(s) > (\check{f}(s) \wedge n)(1 - \varepsilon) > 0\} \setminus \bigcup_{i=1}^{j} A_i.$$

The sets A_j, $j \geq 1$ are disjoint and cover $A \cap \{\check{f} > 0\}$, so

$$\check{\mu}(A \cap \{\check{f} > 0\}) = \sum_{j\geq 1} \check{\mu}(A_j) \geq \sum_{j\geq 1} \mu_{f_j}(A_j) = \sum_{j\geq 1} \int_{A_j} f_j d\nu.$$

It follows that

$$\check{\mu}(A \cap \{\check{f} > 0\}) \geq (1 - \varepsilon) \sum_{j\geq 1} \int_{A_j} (\check{f} \wedge n) d\nu = (1 - \varepsilon) \int_{A\cap\{\check{f}>0\}} (\check{f} \wedge n) d\nu.$$

Since ε and n are arbitrary, we obtain

$$\check{\mu}(A \cap \{\check{f} > 0\}) \geq \int_{A\cap\{\check{f}>0\}} \check{f} d\nu.$$

If we now use

$$A_1 = \{s \in A : \check{f}(s) < 0, f_1(s) > \check{f}(s)(1+\varepsilon)\}$$

and

$$A_{j+1} = \{s \in A : \check{f}(s) < 0, f_{j+1}(s) > \check{f}(s)(1+\varepsilon)\} \setminus \bigcup_{i=1}^{j} A_i$$

we get

$$\check{\mu}(A \cap \{\check{f} < 0\}) \geq \int_{A\cap\{\check{f}<0\}} \check{f} d\nu.$$

Now with $A_1 = \{s \in A : -\varepsilon < f_1(s) \leq 0\}$ and

$$A_{j+1} = \{s \in A : -\varepsilon < f_{j+1}(s) \leq 0\} \setminus \bigcup_{i=1}^{j} A_i$$

we get

$$\check{\mu}(A \cap \{\check{f} = 0\}) \geq -\varepsilon\nu(A \cap \{\check{f} = 0\}).$$

Since ε is arbitrary and (by localization) ν may be assumed finite, we get $\check{\mu}(A\cap\{\check{f} = 0\}) \geq 0$. Finally

$$\check{\mu}(A) = \check{\mu}(A \cap \{\check{f} > 0\}) + \check{\mu}(A \cap \{\check{f} = 0\}) + \check{\mu}(A \cap \{\check{f} < 0\}) \geq \int_A \check{f} d\nu.$$

We have proven (4.1). If we replace $\mathfrak{F}$ with (f_j), we obtain

$$\operatorname*{ssup}_{j\geq 1} \mu_{f_j} = \int_A \check{f} d\nu = \check{\mu}$$

and this completes the proof. □

Corollary 4.12. *Consider a signed measure μ given by*

$$\mu(A) := \int_A f(s)\, d\nu(s),$$

where ν is a σ-finite measure. Then

$$\mu^+(A) = \int_A f^+(s)\, d\nu(s), \quad \mu^-(A) = \int_A f^-(s)\, d\nu(s), \quad \operatorname{var}(\mu)(A) = \int_A |f(s)|\, d\nu(s).$$

Proof. By Theorem 4.11 we have

$$\mu^+(A) = \operatorname{ssup}\{\mu, 0\}(A) = \int_A \sup\{f(s), 0\}\, d\nu(s) = \int_A f^+(s)\, d\nu(s).$$

For μ^- we proceed similarly. Finally

$$\operatorname{var}(\mu)(A) = \operatorname{ssup}\{\mu, -\mu\}(A) = \int_A \sup\{f(s), -f(s)\}\, d\nu(s) = \int_A |f(s)|\, d\nu(s)$$

and the proof is complete. □

Example 4.13. *Consider $f : \mathbb{R} \to \mathbb{R}$ integrable and define α on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ by*

$$\alpha(A) = \int_A f(t)dt.$$

By Corollary 4.12, the total variation of α is given by

$$\operatorname{var}(\alpha)(A) = \int_A |f(t)|\, dt.$$

When applying this with $A = (a, x]$, we can use Riemann-type partitions in the definition of the supremum (see for instance Remark 2.10 in [9]). We obtain a very direct proof of the identity

$$V[F; a, x] = \int_a^x |f(t)|\, dt, \qquad \text{for } F(x) := \int_a^x f(t)dt.$$

4.2. **Application: Vector measures on a space of bilinear forms.** Consider a σ-finite measure space (S, Σ, ν) and a separable Banach space X. We denote by $\mathfrak{Bil}(X)$ the space of bounded bilinear forms defined on $X \times X$, and by $\mathcal{L}(X, X^*)$ the space of bounded linear operators from X to X^*. In both spaces, we use the σ-algebra of Borel sets. Consider $Q : S \to \mathcal{L}(X, X^*)$ Bochner-integrable and define a $\mathfrak{Bil}(X)$-valued measure α by

$$\alpha(A)(x, y) = \int_A \langle Q(s)x, y\rangle\, d\nu(s), \quad A \in \Sigma, \quad (x, y) \in X \times X.$$

Theorem 4.14. *The vector measure α is of bounded variation and its variation is given by*

$$\operatorname{var}(\alpha)(A) = \int_A \|Q(s)\|_{\mathcal{L}(X,X^*)}\, d\nu(s), \quad A \in \Sigma.$$

In other words, $\operatorname{var}(\alpha) \ll \nu$ *and*

$$\frac{d}{d\nu}\operatorname{var}(\alpha) = \|Q\|\,.$$

Proof. Notice that

$$\int_A \|Q(s)\|\, d\nu(s) = \int_A \sup_{\|x\|=\|y\|=1} \langle Q(s)x, y\rangle\, d\nu(s),$$

where the supremum can be computed over (x_n, x_m), for a dense sequence (x_n) on the unit sphere S_X. We do not use absolute value here because, when $\langle Q(s)x, y\rangle$ is negative, we can replace x with $-x$. We apply Theorem 4.11 with $f_{(x,y)} = \langle Q(\cdot)x, y\rangle$. We obtain

$$\int_A \|Q\|\, d\nu = \left(\operatorname*{ssup}_{\|x\|=\|y\|=1} \alpha(\cdot)(x, y)\right)(A) = \sup_{\Pi \in \mathscr{P}(A)} \sum_{C \in \Pi} \sup_{\|x\|=\|y\|=1} \alpha(C)(x, y).$$

We conclude that

$$\int_A \|Q\|\, d\nu = \sup_{\Pi \in \mathscr{P}(A)} \sum_{C \in \Pi} \|\alpha(C)\|_{\mathfrak{Bil}} = \mathrm{var}(\alpha)(A)$$

and, since the left-hand side is finite, α is of bounded variation. $\square$

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