Metric properties of the fuzzy sphere

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Abstract

The fuzzy sphere, as a quantum metric space, carries a sequence of metrics which we describe in detail. We show that the Bloch coherent states, with these spectral distances, form a sequence of metric spaces that converge to the round sphere in the high-spin limit.

1   Introduction

It is common practice in several fields to "approximate" a manifold with a finite or countable subset of points. A typical example in is the study of quantum field theories on a lattice. One drawback is the absence of some of the symmetries of the continuous theory it purports to approximate (e.g., Poincaré symmetries in flat Minkowski space).

Take the simple example of a unit two-sphere $\mathbb{S}^2$. On replacing $\mathbb{S}^2$ with a subset of N points, rotational symmetry is lost. In algebraic language: the algebra $\mathbb{C}^N$ of functions on N points is not an $\mathcal{U}(\mathfrak{su}(2))$-module $*$-algebra. There are no nontrivial $SU(2)$-orbits with finitely many points; to preserve the symmetries and keep the algebra finite dimensional, one may replace the function algebra $\mathbb{C}^N$ with a noncommutative one, provided that the noncommutativity be suppressed as $N \to \infty$. This is the idea behind the fuzzy sphere (and more general fuzzy spaces), put forward in [26], as well as in [21, 35, 36].

Let x_1, x_2, x_3 be Cartesian coordinates on $\mathbb{S}^2$, and $\mathcal{A}(\mathbb{S}^2)$ be the $*$-algebra of polynomials in these. As an abstract $*$-algebra, this is the complex unital commutative $*$-algebra with three self-adjoint generators x_1, x_2, x_3 subject only to the relation $x_1^2 + x_2^2 + x_3^2 = 1$. As an $\mathcal{U}(\mathfrak{su}(2))$-module $*$-algebra,

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$\mathcal{A}(\mathbb{S}^2)$ decomposes into a direct sum of irreducible representations $\mathcal{A}(\mathbb{S}^2) \simeq \bigoplus_{\ell=0}^{\infty} V_\ell$. Here V_ℓ is the vector space underlying the irreducible representation of $\mathcal{U}(\mathfrak{su}(2))$ with highest weight $\ell \in \mathbb{N}$, and is spanned by Laplace spherical harmonics $Y_{\ell,m}$.

In the spirit of [3, 4], we introduce a cut-off in the energy spectrum, i.e., we neglect all but the first $(N + 1)$ representations in the decomposition of $\mathcal{A}(\mathbb{S}^2)$. One cannot simply take the linear span of $Y_{\ell,m}$ for $\ell = 0, 1, \ldots, N$, as this is not a subalgebra of $\mathcal{A}(\mathbb{S}^2)$. To proceed, we write $N = 2j$ and denote by $\pi_j \colon \mathcal{U}(\mathfrak{su}(2)) \to M_{2j+1}(\mathbb{C})$ the spin j representation of $\mathcal{U}(\mathfrak{su}(2))$; the action (using Sweedler notation for the coproduct):

$$h \triangleright a := \pi_j(h_{(1)}) \, a \, \pi_j(S(h_{(2)})), \qquad h \in \mathcal{U}(\mathfrak{su}(2)), \ a \in M_{2j+1}(\mathbb{C})$$

makes the matrix algebra $\mathcal{A}_N := M_{N+1}(\mathbb{C})$ an $\mathcal{U}(\mathfrak{su}(2))$-module $*$-algebra. There is a decomposition into irreducible representations:

$$\mathcal{A}_N \simeq V_j \otimes V_j^* \simeq \bigoplus_{\ell=0}^{2j} V_\ell$$

and a surjective homomorphism $\mathcal{A}(\mathbb{S}^2) \to \mathcal{A}_N$ of $\mathcal{U}(\mathfrak{su}(2))$-modules (but not of module algebras), given on generators by

$$x_k \mapsto \hat{x}_k := \frac{1}{\sqrt{j(j+1)}} \, \pi_j(J_k),$$

where the J_k are the standard real generators of $\mathcal{U}(\mathfrak{su}(2))$. The map $x_k \mapsto \hat{x}_k$ does not extend to an algebra morphism, but can be extended in a unique way, using coherent-state quantization, to an isometry between $*$-representations of $\mathcal{U}(\mathfrak{su}(2))$ sending the spherical harmonic $Y_{\ell,m}$, for $\ell \leq 2j$, into a matrix $\widehat{Y}_{\ell,m}^{(j)}$ sometimes called a "fuzzy spherical harmonic" (details at the end of Sect. 3.3). Since an infinite-dimensional vector space is mapped onto a finite-dimensional one, information is lost and the space becomes "fuzzy".

The matrices $\hat{x}_k$ are normalized in such a way that the spherical relation still holds: $\hat{x}_1^2 + \hat{x}_2^2 + \hat{x}_3^2 = 1$, but their commutators are clearly not zero [36]:

$$[\hat{x}_k, \hat{x}_l] \simeq \frac{1}{\sqrt{j(j+1)}} \, \mathrm{i} \, \varepsilon_{klm} \hat{x}_m.$$

Since the coefficient in the commutator vanishes for $N = 2j \to \infty$, the naïve idea is that the fuzzy sphere "converges", as $N \to \infty$, to a unit sphere. It is clear that the notion of convergence must involve the Riemannian metric of $\mathbb{S}^2$.

The correct mathematical framework for the convergence of matrix algebras to algebras of functions on Riemannian manifolds (or more generally, on metric spaces) was developed by Rieffel in a series of seminal papers, where he introduced the notion of (compact) quantum metric spaces and quantum Gromov–Hausdorff convergence [29–31]. The convergence of the fuzzy sphere to $\mathbb{S}^2$ was established in [32]. However, there the metrics are dealt with globally and the proof does not indicate how to choose a sequence of elements approximating a given point of $\mathbb{S}^2$. In this paper we approximate the points of $\mathbb{S}^2$ by the corresponding (Bloch) coherent states of $\mathcal{A}_N$.

A distance d_N on the state space of $\mathcal{A}_N$ can be defined via a generalized Dirac operator. Since, for any N, the set of coherent states is labelled by $\mathbb{S}^2$, this gives a distance on $\mathbb{S}^2$ depending on the

deformation parameter N. Denoting by d_{geo} the geodesic distance of the round sphere, we prove that
$$\lim_{N\to\infty} d_N(p,q) = d_{\mathrm{geo}}(p,q), \quad \text{for all} \quad p,q \in \mathbb{S}^2.$$

Another noncommutative space where the distance between coherent states has already been studied is the Moyal plane [10,27,37]. In contrast with that example, whose distance is independent of the deformation parameter, here the distance depends on N.

Sect. 2 briefly recalls the basics of noncommutative spaces. In Sect. 3, we introduce our spectral triples for the sphere and compare them with other proposals in the literature. In Sect. 4, we recall the Bloch coherent states [5] and compute some particular distances between them. Then we prove that the spectral distance is $SU(2)$-invariant, nondecreasing with N, and converges to the geodesic distance on $\mathbb{S}^2$ when $N \to \infty$.

2 Preliminaries on noncommutative manifolds

Material in this section is mainly taken from [15, 18]. In the spirit of Connes' noncommutative geometry, manifolds are replaced by spectral triples. A *unital spectral triple* $(\mathcal{A}, \mathcal{H}, D)$ has the following data: (i) a separable complex Hilbert space $\mathcal{H}$; (ii) a complex associative involutive unital algebra $\mathcal{A}$ with a faithful unital $*$-representation $\mathcal{A} \to \mathcal{B}(\mathcal{H})$, the representation symbol usually being omitted; (iii) a self-adjoint operator D on $\mathcal{H}$ with compact resolvent such that $[D, a]$ is a bounded operator for all $a \in \mathcal{A}$.

A spectral triple is *even* if there is a *grading* γ on $\mathcal{H}$, i.e., a bounded operator satisfying $\gamma = \gamma^*$ and $\gamma^2 = 1$, commuting with any $a \in \mathcal{A}$ and anticommuting with D.

A spectral triple is *real* if there is an antilinear isometry $J \colon \mathcal{H} \to \mathcal{H}$ (the "real structure"), such that $J^2 = \pm 1$, $JD = \pm DJ$ and $J\gamma = \pm \gamma J$ in the even case, with the signs related to the KO-dimension of the triple [16]; and

$$[a, JbJ^{-1}] = 0, \qquad [[D, a], JbJ^{-1}] = 0, \qquad \text{for all} \quad a, b \in \mathcal{A}. \tag{2.1}$$

This shows that $b \mapsto Jb^*J^{-1}$ is an injective homomorphism of $\mathcal{A}$ into its commutant.

For the notion of *equivariant* spectral triple, we refer to [34]. A group, or more generally a Hopf algebra, acts on $\mathcal{A}$ and on $\mathcal{H}$, intertwining the operator D with itself.

Remark 2.1. Note that if $(\mathcal{A}, \mathcal{H}, D, \gamma)$ is an even spectral triple and v an eigenvector of D with eigenvalue λ, then γv is an eigenvector of D with eigenvalue $-\lambda$. Thus, the eigenvalues λ and $-\lambda$ have the same multiplicity.

We use the following notations and conventions. $\mathcal{B}(\mathcal{H})$ is the algebra of all bounded linear operators on $\mathcal{H}$. The set of all states of (the norm completion of) $\mathcal{A}$ is denoted by $\mathcal{S}(\mathcal{A})$. We denote by $\|\cdot\|$ the operator norm of $\mathcal{B}(\mathcal{H})$; by $\|v\|_{\mathcal{H}}^2 = \langle v \mid v \rangle$ the norm-squared of a vector $v \in \mathcal{H}$, writing $\langle \cdot \mid \cdot \rangle$ for scalar products. By $\mathrm{C}\ell(\mathfrak{g})$ we mean the Clifford algebra over a semisimple Lie algebra with its Killing form.

Recall that $\mathcal{S}(\mathcal{A})$ is a convex set, compact in the weak* topology, whose extremal points are the *pure states* of $\mathcal{A}$. $\mathcal{S}(\mathcal{A})$ is an extended metric space (allowing distances to be $+\infty$), with distance function given by

$$d_{\mathcal{A},D}(\omega, \omega') := \sup_{a = a^* \in \mathcal{A}} \{ |\omega(a) - \omega'(a)| : \|[D, a]\| \le 1 \} \tag{2.2}$$

for all $\omega, \omega' \in \mathcal{S}(\mathcal{A})$. This is usually called *Connes' metric* or *spectral distance* [14]. The supremum is usually taken over all $a \in \mathcal{A}$ obeying the side condition; but it was noted in [23] that the supremum is always attained on self-adjoint elements. More generally, when defining a metric, one can replace $\|[D, a]\|$ by $L(a)$ where L is a Leibniz seminorm on $\mathcal{A}$. The structure $(\mathcal{A}, d_{\mathcal{A},L})$ so obtained is a "compact quantum metric space" [30, 33].

3   Dirac operators for the fuzzy sphere

The classical Dirac operator $\not{D}$ on a compact semisimple Lie group G with Lie algebra $\mathfrak{g}$ can be seen as a purely algebraic object $\mathfrak{D}$ living in the noncommutative Weil algebra $U(\mathfrak{g}) \otimes \mathrm{C}\ell(\mathfrak{g})$, see [22,25]. It is equivariant in the sense that there exists a Lie algebra homomorphism $\mathfrak{g} \to U(\mathfrak{g}) \otimes \mathrm{C}\ell(\mathfrak{g})$ with whose range $\mathfrak{D}$ commutes. The spinor bundle of G is parallelizable: $L^2(G, S) \simeq L^2(G) \otimes \Sigma$, where Σ is an irreducible $\mathrm{C}\ell(\mathfrak{g})$-module. The algebra $U(\mathfrak{g}) \otimes \mathrm{C}\ell(\mathfrak{g})$ acts on the Hilbert space $L^2(G) \otimes \Sigma$ making $\mathfrak{D}$ into the "concrete" Dirac operator $\not{D}$ of G, an unbounded first-order elliptic operator. Using the injection $\mathfrak{g} \hookrightarrow \mathrm{C}\ell(\mathfrak{g})$ we can also think of $\mathfrak{D}$ as an element of $U(\mathfrak{g}) \otimes U(\mathfrak{g})$, equivariant in the sense that it commutes with the range of the coproduct Δ in $U(\mathfrak{g}) \otimes U(\mathfrak{g})$. On a compact Riemannian symmetric space G/U, this construction also applies (indeed, it works on G as a symmetric space of $G \times G$), although the spinor bundle is not always parallelizable. This is the point of view that we shall adopt for the fuzzy sphere.

3.1   An abstract Dirac operator

We begin with the two-sphere $\mathbb{S}^2$. The abstract *Dirac element* $\mathfrak{D} \in \mathcal{U}(\mathfrak{su}(2)) \otimes \mathcal{U}(\mathfrak{su}(2))$ is defined as

$$\mathfrak{D} := 1 \otimes 1 + 2 \sum_k J_k \otimes J_k . \tag{3.1}$$

Since $\sum_k [J_k \otimes J_k, J_l \otimes 1 + 1 \otimes J_l] = 0$, this element commutes with the range of the coproduct $\Delta \colon \mathcal{U}(\mathfrak{su}(2)) \to \mathcal{U}(\mathfrak{su}(2)) \otimes \mathcal{U}(\mathfrak{su}(2))$. That is an equivariance property of $\mathfrak{D}$.

The corresponding element of $\mathcal{U}(\mathfrak{su}(2)) \otimes \mathrm{C}\ell_{20}$ is

$$\mathfrak{D}_S := (\mathrm{id} \otimes \pi_{\frac{1}{2}})(\mathfrak{D}) = 1 \otimes 1 + \sum_k J_k \otimes \sigma_k = \begin{pmatrix} 1+H & F \\ E & 1-H \end{pmatrix}. \tag{3.2}$$

where $H = J_3$, $E = J_1 + iJ_2$, $F = E^*$. The square of $\mathfrak{D}$ is $\mathfrak{D}_S^2 = C_{SU(2)} + \frac{1}{4}(1 \otimes 1)$, where $C_{SU(2)} := \sum_k (J_k \otimes 1 + 1 \otimes \frac{1}{2}\sigma_k)^2$ is the Casimir operator and $1/4 = R/8$ is the scalar curvature term ($R = 2$ being the scalar curvature of $\mathbb{S}^2$). This is the symmetric space version, $D^2 = C_G + R/8$, of the Schrödinger–Lichnerowicz formula for equivariant Dirac operators [17, p. 87].

Lemma 3.1. *For any $\ell \neq 0$ in $\frac{1}{2}\mathbb{N}$, the operator $(\pi_\ell \otimes \pi_{\frac{1}{2}})(\mathfrak{D}^2)$ has eigenvalues ℓ^2 with multiplicity 2ℓ and $(\ell+1)^2$ with multiplicity $2\ell+2$. For $\ell = 0$, $(\pi_0 \otimes \pi_{\frac{1}{2}})(\mathfrak{D}^2)$ has eigenvalue 1 with multiplicity 2.*

Proof. With $J^2 = \sum_k J_k^2$, it follows from $\Delta(J^2) = \sum_k \Delta(J_k)^2 = \sum_k (J_k \otimes 1 + 1 \otimes J_k)^2$ that $C_{SU(2)} = (\mathrm{id} \otimes \pi_{\frac{1}{2}})\Delta(J^2)$. Since $\Delta(1) = 1 \otimes 1$, this yields $\mathfrak{D}_S^2 = (\mathrm{id} \otimes \pi_{\frac{1}{2}})\Delta(J^2 + \frac{1}{4})$. Therefore,

$$(\pi_\ell \otimes \pi_{\frac{1}{2}})(\mathfrak{D}^2) = (\pi_\ell \otimes \mathrm{id})(\mathfrak{D}_S^2) = (\pi_\ell \otimes \pi_{\frac{1}{2}})\Delta(J^2 + \tfrac{1}{4}).$$

4

Now $(\pi_\ell \otimes \pi_{\frac{1}{2}})\Delta$ is the Hopf tensor product of the representations π_ℓ and $\pi_{\frac{1}{2}}$. From

$$V_\ell \otimes V_{\frac{1}{2}} \simeq V_{\ell+\frac{1}{2}} \oplus V_{\ell-\frac{1}{2}} \tag{3.3}$$

it follows that $(\pi_\ell \otimes \pi_{\frac{1}{2}})(\mathfrak{D}^2)$ is unitarily equivalent to $\pi_{\ell+\frac{1}{2}}(J^2 + \frac{1}{4}) \oplus \pi_{\ell-\frac{1}{2}}(J^2 + \frac{1}{4})$, and hence has eigenvalues

$$(\ell \pm \tfrac{1}{2})(\ell \pm \tfrac{1}{2} + 1) + \tfrac{1}{4} = \begin{cases} (\ell+1)^2 & \text{on } V_{\ell+\frac{1}{2}}, \\ \ell^2 & \text{on } V_{\ell-\frac{1}{2}}. \end{cases}$$

If $\ell = 0$, the summand $V_{\ell-\frac{1}{2}}$ in (3.3) is missing, so the only eigenvalue is 1 on $V_{\frac{1}{2}}$. $\qquad\square$

3.2 The Dirac operator of $\mathbb{S}^2$

The natural representation of $\mathcal{U}(\mathfrak{su}(2))$ on $\mathbb{S}^2$ as vector fields yields the Dirac operator $\slashed{D}$ of the unit sphere (with round metric). The spinor bundle $S \to \mathbb{S}^2$ is trivial of rank 2, so the spinor space is $L^2(\mathbb{S}^2, S) \simeq L^2(\mathbb{S}^2) \otimes \mathbb{C}^2$.

Modulo the identification $L^2(\mathbb{S}^2) \simeq \bigoplus_{\ell \in \mathbb{N}} V_\ell$, the operator $\slashed{D}$ is given by

$$\slashed{D} = \bigoplus_{\ell \in \mathbb{N}} (\pi_\ell \otimes \pi_{\frac{1}{2}})(\mathfrak{D}),$$

with $\mathfrak{D}$ as in (3.1). It follows from Lemma 3.1 that $\slashed{D}^2$ has eigenvalues $\lambda_\ell = \ell^2$ with multiplicity $m_\ell = 4\ell$, for every integer $\ell \geq 1$. The spectral triple of $\mathbb{S}^2$ is even, using the grading that exchanges the two half-spinor line bundles [18]. From Remark 2.1 it follows that $\slashed{D}$ has eigenvalues $\pm\ell$ with multiplicities $\frac{1}{2}m_\ell = 2\ell$.

3.3 Dirac operators on the fuzzy sphere

We require an equivariant Dirac operator whose spectrum is that of $\slashed{D}$, truncated at $\ell = N + 1$. Let $N = 2j \geq 1$ be a fixed integer. The *fuzzy sphere* (labelled by N) is the "noncommutative $SU(2)$ coset space" described by the algebra $\mathcal{A}_N := M_{N+1}(\mathbb{C})$ with the $SU(2)$ left action $(g, a) \mapsto a^g :=$ $\pi_j(g)\, a\, \pi_j(g)^*$, for $g \in SU(2)$, $a \in \mathcal{A}_N$.

Definition 3.2. The *irreducible* spectral triple on $\mathcal{A}_N$, that we denote by $(\mathcal{A}_N, \mathcal{H}_N, D_N)$, is given by $\mathcal{H}_N := V_j \otimes \mathbb{C}^2$, with the natural representation of $\mathcal{A}_N$ via row-by-column multiplication on the factor $V_j \simeq \mathbb{C}^{N+1}$, and $D_N := (\pi_j \otimes \pi_{\frac{1}{2}})(\mathfrak{D})$, where $\mathfrak{D}$ is the abstract Dirac element in (3.1).

Proposition 3.3. *The irreducible spectral triple on $\mathcal{A}_N = \mathcal{A}_{2j}$ has these properties:*

(i) *It is equivariant with respect to the $SU(2)$ representation $\pi_j \otimes \pi_{\frac{1}{2}}$.*

(ii) *D_N has eigenvalues $j + 1$ and $(-j)$, with respective multiplicities $2j + 2$ and $2j$.*

(iii) *No grading or real structure is compatible with this spectral triple.*

Proof. Equivariance comes from the commuting of $\mathfrak{D}_S$ with the range of the coproduct, so that D_N commutes with the representation $\pi_j \otimes \pi_{\frac{1}{2}}$ of $\mathcal{U}(\mathfrak{su}(2))$ —or the corresponding representation of $SU(2)$— and from the intertwining relation:

$$(\pi_j \otimes \pi_{\frac{1}{2}})(g)\,(a \otimes 1)\,(\pi_j \otimes \pi_{\frac{1}{2}})(g)^* = \pi_j(g)\,a\,\pi_j(g)^* \otimes \pi_{\frac{1}{2}}(g)\pi_{\frac{1}{2}}(g)^* = a^g \otimes 1, \quad a \in \mathcal{A}_N\,.$$

From Lemma 3.1 follows that D_N^2 has eigenvalues j^2 and $(j+1)^2$. However, the spectrum of D_N is *not* symmetric about 0. Indeed, an explicit computation shows that $\mathcal{H}_N$ has the following orthonormal basis of eigenvectors for D_N:

$$|j,m\rangle\!\rangle_+ := \sqrt{\tfrac{j+m+1}{2j+1}}\,|j,m\rangle \otimes \tbinom{1}{0} + \sqrt{\tfrac{j-m}{2j+1}}\,|j,m+1\rangle \otimes \tbinom{0}{1}, \qquad m = -j-1,\ldots,j\,;$$

$$|j,m\rangle\!\rangle_- := -\sqrt{\tfrac{j-m}{2j+1}}\,|j,m\rangle \otimes \tbinom{1}{0} + \sqrt{\tfrac{j+m+1}{2j+1}}\,|j,m+1\rangle \otimes \tbinom{0}{1}, \qquad m = -j,\ldots,j-1\,. \tag{3.4}$$

One easily checks that

$$D_N|j,m\rangle_+ = (j+1)|j,m\rangle_+, \qquad D_N|j,m\rangle_- = -j|j,m\rangle_-\,.$$

Therefore, D_N has eigenvalue $j+1$ with multiplicity $2j+2$, and eigenvalue $-j$ with multiplicity $2j$, as claimed. This asymmetry of the spectrum of D_N and Remark 2.1 rule out any grading for this spectral triple.

If there were a real structure, the commutant $\mathcal{A}_N'$ of $\mathcal{A}_N$ would contain $J\mathcal{A}_N J^{-1}$, whose dimension is $(N+1)^2 \geq 4$. But $\dim \mathcal{A}_N' = 2$; hence, no real structure can exist. $\qquad\square$

Definition 3.4. The *full* spectral triple on $\mathcal{A}_N$, that we denote by $(\mathcal{A}_N, \widetilde{\mathcal{H}}_N, \widetilde{\mathcal{D}}_N, \widetilde{\mathcal{J}}_N)$, is given by $\widetilde{\mathcal{H}}_N \simeq \mathcal{A}_N \otimes \mathbb{C}^2$, where the first factor carries the left regular representation of $\mathcal{A}_N$, i.e., the GNS representation associated to the matrix trace; and the Dirac operator and real structure are defined by:

$$\widetilde{\mathcal{D}}_N(a \otimes v) := a \otimes v + \sum_k [\pi_j(J_k), a] \otimes \sigma_k v,$$
$$\widetilde{\mathcal{J}}_N(a \otimes v) := a^* \otimes \sigma_2 \bar{v},$$

for any $a \in \mathcal{A}_N$ and $v \in \mathbb{C}^2$ (a column vector). For $v = (v_1, v_2)^t \in \mathbb{C}^2$, $\bar{v} := (v_1^*, v_2^*)^t$ is again a column vector.

The nuance between D_N and $\widetilde{\mathcal{D}}_N$ is that π_j is replaced by its adjoint action on the space $\mathcal{A}_N = \mathrm{End}(V_j) \simeq V_j \otimes V_j^*$.

Proposition 3.5. *The full spectral triple on $\mathcal{A}_N$ has the following properties:*

(i) *It is a real spectral triple.*

(ii) *It is equivariant with respect to the $SU(2)$ representation given by the product of the action $a \mapsto a^g$ on $\mathcal{A}_N$ and the spin-$\frac{1}{2}$ representation.*

(iii) *$\widetilde{\mathcal{D}}_N$ has integer eigenvalues $\pm \ell$ with multiplicity 2ℓ, for every $\ell = 1, \ldots, N$; and eigenvalue $N+1$ with multiplicity $2N+2$.*

(iv) *This spectral triple carries no grading.*

Proof. Clearly $\widetilde{\mathfrak{J}}_N$ is antilinear, indeed antiunitary, since

$$\langle\widetilde{\mathfrak{J}}_N(a\otimes v)\mid\widetilde{\mathfrak{J}}_N(b\otimes w)\rangle = \mathrm{Tr}(a^*b)\langle\sigma_2\bar{v}\mid\sigma_2\bar{w}\rangle = \mathrm{Tr}(a^*b)\langle\bar{v}\mid\bar{w}\rangle$$
$$= \overline{\mathrm{Tr}(b^*a)}\ \overline{\langle w\mid v\rangle} = \overline{\langle b\otimes w\mid a\otimes v\rangle}.$$

We need to check the conditions (2.1). The equality $\bar{\sigma}_2 = -\sigma_2$ shows that $(\widetilde{\mathfrak{J}}_N)^2 = -1$. Using $\widetilde{\mathfrak{J}}_N^{-1} = -\widetilde{\mathfrak{J}}_N$, we find that

$$\widetilde{\mathfrak{J}}_N\,b\,\widetilde{\mathfrak{J}}_N^{-1}(a\otimes v) = -\widetilde{\mathfrak{J}}_N(ba^*\otimes\sigma_2\bar{v}) = ab^*\otimes v,\quad\text{for all}\quad a,b\in\mathcal{A}_N,\ v\in\mathbb{C}^2.$$

Since left and right multiplication on $\mathcal{A}_N$ commute, $\widetilde{\mathfrak{J}}_N\,b\,\widetilde{\mathfrak{J}}_N^{-1}$ lies in the commutant of $\mathcal{A}_N\otimes M_2(\mathbb{C})$, and both conditions in (2.1) are satisfied.

Since $\sigma_2\bar{\sigma}_k = -\sigma_k\sigma_2$ for $k = 1, 2, 3$, and $[\pi_j(J_k), a]^* = -[\pi_j(J_k), a^*]$, we obtain

$$\widetilde{\mathfrak{J}}_N\widetilde{\mathfrak{D}}_N(a\otimes v) = a^*\otimes\sigma_2 v - \sum_k[\pi_j(J_k), a^*]\otimes\sigma_2\bar{\sigma}_k v = \widetilde{\mathfrak{D}}_N\widetilde{\mathfrak{J}}_N(a\otimes v),$$

for any $a\in\mathcal{A}_N$ and $v\in\mathbb{C}^2$. Hence $\widetilde{\mathfrak{J}}_N\widetilde{\mathfrak{D}}_N = \widetilde{\mathfrak{D}}_N\widetilde{\mathfrak{J}}_N$.

Equivariance follows again from the commuting of $\mathfrak{D}_S$ with the range of the coproduct, since the representation $J_k\mapsto[\pi_j(J_k),\cdot]$ is the derivative of the adjoint action $a\mapsto a^g = \pi_j(g)\,a\,\pi_j(g)^*$ of $SU(2)$.

Writing $\mathrm{ad}\,\pi_j(h)\colon a\mapsto\pi_j(h_{(1)})\,a\,\pi_j(S(h_{(2)}))$ for $h\in\mathcal{U}(\mathfrak{su}(2))$ and $a\in\mathcal{A}_N$, we see that

$$\widetilde{\mathfrak{D}}_N = (\mathrm{ad}\,\pi_j\otimes\pi_{\frac{1}{2}})(\mathfrak{D}),$$

In view of the unitary $\mathcal{U}(\mathfrak{su}(2))$-module isomorphism

$$\mathcal{A}_N\simeq V_j\otimes V_j^*\simeq\bigoplus_{\ell=0}^{2j}V_\ell,$$

$\widetilde{\mathfrak{D}}_N$ is unitarily equivalent to the operator $\bigoplus_{\ell=0}^{2j}(\pi_\ell\otimes\pi_{\frac{1}{2}})(\mathfrak{D})$. Replacing $N = 2j$ by 2ℓ in Prop. 3.3(ii), we see that $(\pi_\ell\otimes\pi_{\frac{1}{2}})(\mathfrak{D})$ has eigenvalues $\ell+1$ and $(-\ell)$, with respective multiplicities $2\ell+2$ and 2ℓ (but if $\ell = 0$ the eigenvalue $-\ell$ is missing). Hence $\widetilde{\mathfrak{D}}_N$ has the eigenvalues $\pm\ell$, each with multiplicity 2ℓ for $\ell = 1,\ldots,N$; and $N+1$ with multiplicity $2N+2$.

Lastly, since the spectrum of $\widetilde{\mathfrak{D}}_N$ is not symmetric about 0, there can exist no grading for this spectral triple. $\qquad\square$

Proposition 3.6. *The irreducible and full spectral triples induce the same metric on the state space $\mathcal{S}(\mathcal{A}_N)$ of the fuzzy sphere.*

Proof. This follows from the calculation:

$$[\widetilde{\mathfrak{D}}_N, a](b\otimes v) = \sum_k\big([\pi_j(J_k), ab] - a\,[\pi_j(J_k), b]\big)\otimes\sigma_k v$$
$$= \sum_k[\pi_j(J_k), a]\,b\otimes\sigma_k v = [D_N, a](b\otimes v).$$

Hence $[\widetilde{\mathfrak{D}}_N, a]$ is the operator of left multiplication by the matrix $[D_N, a]\in\mathcal{A}_N\otimes M_2(\mathbb{C})$, so its operator norm coincides with the norm of the matrix. Therefore, since $\big\|[\widetilde{\mathfrak{D}}_N, a]\big\| = \big\|[D_N, a]\big\|$ for each $a\in\mathcal{A}_N$, it follows that the two spectral triples induce the same metric (2.2) on the state space of $\mathcal{A}_N$. $\qquad\square$

It is useful to give a more explicit presentation of the full spectral triple by exhibiting its eigen-spinors. Recall that the polynomial algebra $\mathcal{A}(\mathbb{S}^2)$ is linearly spanned by the spherical harmonics $Y_{\ell,m}$, each of which is a homogeneous polynomial in Cartesian coordinates of degree ℓ, with the multiplication rule

$$Y_{\ell',m'} Y_{\ell'',m''} = \sum_{\ell=|\ell'-\ell''|}^{\ell=\ell'+\ell''} \sum_{m=-\ell}^{\ell} \sqrt{\frac{(2\ell'+1)(2\ell''+1)}{4\pi(2\ell+1)}} \, C_{\ell'0,\ell''0}^{\ell 0} C_{\ell'm',\ell''m''}^{\ell m} Y_{\ell,m} \, ,$$

involving $SU(2)$ Clebsch–Gordan coefficients. From there it is clear that the subspace spanned by the $Y_{\ell,m}$ for $\ell = 0, 1, \ldots, N$ does not close under multiplication. To replace them, while keeping $SU(2)$ symmetry, one can make use of the irreducible tensor operators at level $N = 2j$ [1,8]. These are elements $\widehat{T}_{\ell,m}^{(j)} \in M_{N+1}(\mathbb{C})$ whose matrix elements are

$$\langle jm'' \mid \widehat{T}_{\ell,m}^{(j)} \mid jm' \rangle := \sqrt{\frac{2\ell+1}{2j+1}} \, C_{jm',\ell m}^{jm''} \, .$$

They transform like the $Y_{\ell,m}$ under $SU(2)$, but still require an appropriate normalization. For any $-1 \le s \le 1$, one can define a matrix $\widehat{Y}_{\ell,m}^{(j,s)} \in M_{N+1}(\mathbb{C})$ as follows [2,9,13,24]:

$$\widehat{Y}_{\ell,m}^{(j,s)} := \sqrt{\frac{4\pi}{2j+1}} \left(C_{jj,\ell 0}^{jj} \right)^s \widehat{T}_{\ell,m}^{(j)}. \tag{3.5}$$

We omit the precise multiplication rules for these operators, see [24]; but in any case it is clear, by working backwards, that the ordinary spherical harmonic $Y_{\ell,m}$ can be regarded as a "symbol" of the operator $\widehat{Y}_{\ell,m}^{(j,s)}$ for fixed j and s. The cases $s = 1$, $s = 0$ and $s = -1$ correspond respectively to the Husimi Q-function, the Moyal–Wigner W-function and the Glauber P-function [9]. Here we put $s = 1$ in (3.5), omit the superscripts, and call these operators the *fuzzy harmonics* $\widehat{Y}_{\ell,m} \in \mathcal{A}_N$. The commutation rules for the irreducible tensor operators and the fuzzy harmonics come directly from their symmetries [1,8,36]:

$$[\pi_j(J_3), \widehat{Y}_{\ell,m}] = [\widehat{Y}_{1,0}, \widehat{Y}_{\ell,m}] = m \, \widehat{Y}_{\ell,m} \, ,$$
$$[\pi_j(J_1 \pm \mathrm{i}J_2), \widehat{Y}_{\ell,m}] = [\widehat{Y}_{1,\pm 1}, \widehat{Y}_{\ell,m}] = \sqrt{(\ell \mp m)(\ell \pm m + 1)} \, \widehat{Y}_{\ell,m\pm 1} \, .$$

Adopting a 2×2 block matrix notation, as in (3.2), we can write

$$\widetilde{\mathcal{D}}_N = \begin{pmatrix} 1 + \mathcal{L}_3 & \mathcal{L}_- \\ \mathcal{L}_+ & 1 - \mathcal{L}_3 \end{pmatrix}, \quad \text{where} \quad \mathcal{L}_3 = \mathrm{ad}\,\pi_j(J_3), \ \mathcal{L}_\pm = \mathrm{ad}\,\pi_j(J_1 \pm \mathrm{i}J_2). \tag{3.6}$$

Then the normalized eigenspinors for the operators $\widetilde{\mathcal{D}}_N$ are

$$|\ell, m\rangle\!\rangle_+ := \frac{1}{\sqrt{2\ell+1}} \begin{pmatrix} \sqrt{\ell+m+1}\,\widehat{Y}_{\ell,m} \\ \sqrt{\ell-m}\,\widehat{Y}_{\ell,m+1} \end{pmatrix}, \quad |\ell, m\rangle\!\rangle_- := \frac{1}{\sqrt{2\ell+1}} \begin{pmatrix} -\sqrt{\ell-m}\,\widehat{Y}_{\ell,m} \\ \sqrt{\ell+m+1}\,\widehat{Y}_{\ell,m+1} \end{pmatrix}$$

for $\ell = 0, 1, \ldots, N$; whereby

$$\widetilde{\mathcal{D}}_N |\ell, m\rangle\!\rangle_+ = (\ell+1)\,|\ell, m\rangle\!\rangle_+ \quad \text{for} \quad m = -\ell-1, \ldots, \ell \, ;$$
$$\widetilde{\mathcal{D}}_N |\ell, m\rangle\!\rangle_- = (-\ell)\,|\ell, m\rangle\!\rangle_- \quad \text{for} \quad m = -\ell, \ldots, \ell-1 \, .$$

The full spectral triple on $\mathcal{A}_N$ is thus a truncation of the standard spectral triple over $\mathbb{S}^2$, in the following sense. The Hilbert space of spinors $L^2(\mathbb{S}^2) \otimes \mathbb{C}^2$, generated by pairs of spherical harmonics $Y_{\ell,m}$, is truncated at $l \leq N$. On replacing these by pairs of fuzzy harmonics $\widehat{Y}_{\ell,m}$, the resulting spectrum of $\widetilde{\mathcal{D}}_N$ is a truncation of the spectrum of $I\!\!\!\!D$ to the range $\{-N, \ldots, N+1\}$, unavoidably breaking the parity symmetry.

3.4 Comparison with the literature

Two spectral triples on the fuzzy sphere algebra $\mathcal{A}_n$ have been introduced, one constructed with the irreducible $\mathfrak{su}(2)$-module V_j and the other with the left regular or GNS representation. Neither one is even (there exists no grading); although this could be remedied by allowing $\mathcal{A}_N$ to act trivially on a supplementary vector space. The first carries no real structure but the second one does, because the reducible action of the algebra on the Hilbert space allows for a large enough commutant. The crucial point here, however, is Prop. 3.6, showing that both spectral triples give the same metric. Other Dirac operators for the fuzzy sphere have been proposed in [6, 7, 11, 12, 19] and are recalled below.

In [19], $\mathcal{A}_N$ is obtained as the even part of a truncated supersphere, and the Dirac operator is defined as the odd part of a truncated superfield. Reformulating the result of Sect. 4.3 of [19] in our language, the Hilbert space is taken to be

$$\mathcal{H}'_N := \bigoplus_{\ell=\frac{1}{2},\ldots,N-\frac{1}{2}} V_\ell \oplus V_\ell \, .$$

Note that to get our $\mathcal{A}_N \otimes V_{\frac{1}{2}}$ one must add an extra $V_{N+\frac{1}{2}}$ subspace. The algebra $\mathcal{A}_N$ is generated by the three matrices $\hat{x}_k$, proportional to $\pi_j(J_k)$, which can be represented on $\mathcal{H}'_N$ using a suitable direct sum of irreducible representations of $\mathfrak{su}(2)$. The Dirac operator can be defined by representing the abstract Dirac element (3.1) on $\mathcal{H}'$ using the same representation of $\mathfrak{su}(2)$; it is proportional to the identity on each subspace V_ℓ and its spectrum is given by the eigenvalues $\pm\ell$, for $\ell = 1, \ldots, N$ (restricted to $V_\ell \oplus V_\ell$ their Dirac operator is the operator $\ell \oplus -\ell$). Compared to our full spectral triple, the eigenvalue $N+1$ is missing. Since the two copies of V_ℓ carry the same representation of $\mathcal{A}_N$, the operator γ_N that exchanges these copies commutes with $\mathcal{A}_N$ (and anticommutes with the Dirac operator): therefore, one obtains an even spectral triple.

This construct is still metrically equivalent to the spectral triples of subsection 3.3. Here $\widetilde{\mathcal{H}}_N \simeq \mathcal{H}'_N \oplus V_{N+\frac{1}{2}}$; but the additional term $V_{N+\frac{1}{2}}$ carries a nontrivial subrepresentation of $\mathcal{A}_N$, and the Dirac operator $\widetilde{D}_N$ is proportional to the identity on such a subspace: hence $[\widetilde{D}_N, a]$ vanishes on the subspace $V_{N+\frac{1}{2}}$ for any $a \in \mathcal{A}_N$. Therefore the two spectral triples induce the same seminorm on $\mathcal{A}_N$, and hence the same distance.

The authors of [6, 7] take another approach. Given any finite-dimensional $\mathfrak{su}(2)$-module Σ, one can construct a Dirac-like operator on $L^2(\mathbb{S}^2) \otimes \Sigma$ by using the appropriate representation of the abstract Dirac element (3.1). If Σ is the spin j representation, this can be called a "spin-j" Dirac operator. For $j = \frac{1}{2}$ we recover the ordinary Dirac operator acting on 2-spinors.

A spin-$\frac{1}{2}$ Dirac operator for the fuzzy sphere is discussed in [6], and is generalized to arbitrary spin j in [7]. These are constructed using the Ginsparg–Wilson algebra, namely, the free algebra generated by two grading operators Γ and Γ'. The linear combinations $\Gamma_1 = \frac{1}{2}(\Gamma+\Gamma')$, $\Gamma_2 = \frac{1}{2}(\Gamma-\Gamma')$ anticommute, and the proposal is to realize them as operators on a suitable Hilbert space, interpreting

Γ_1 as the Dirac operator and Γ_2 as the chirality operator. In the spin-$\frac{1}{2}$ case, the Hilbert space is taken to be $\mathcal{A}_N \otimes V_{\frac{1}{2}}$. From equation (2.20) of [7], or equivalently from (8.29) of [6], we see that the Dirac operator is the same as the operator (3.6) of our full spectral triple. The chirality operator, (2.21) of [7], in contrast with the Dirac operator, is constructed using the anticommutator with $\pi_j(J_k)$, i.e., $L_k^L + L_k^R$ in the notation of [7].

The asymmetry of the Dirac operator spectrum was already noticed in [6]. At the end of subsection 8.3.2 we read:

> *For $j = 2L + 1$ [$\ell = N + 1$ in our notations here] we get the positive eigenvalue correctly, but the negative one is missing. That is an edge effect caused by cutting off the angular momentum at 2L.*

And in the same subsection, after equation (8.30):

> *As mentioned earlier, use of Γ_2 as chirality resolves a difficulty addressed elsewhere [80], where $\mathrm{sign}(\Gamma_2)$ was used as chirality. That necessitates projecting out V_{+1} and creates a very inelegant situation.*

In other words, Γ_2 is not a true grading operator. Since Γ_2 anticommutes with the Dirac operator, it must vanish on $V_{N+\frac{1}{2}}$ (otherwise, the Dirac operator would have an eigenvector $\Gamma_2 v$ for $v \in V_{N+\frac{1}{2}}$, with eigenvalue $-N - 1$); which entails $(\Gamma_2)^2 \neq 1$.

A third proposal is that of [11, 12]. It starts by constructing, on the Hilbert space $\mathcal{A}_N$, a square 1 chirality operator that is a genuine $\mathbb{Z}_2$-grading, then finding a Dirac-like operator $\mathbf{D}$ by imposing anticommutation with the grading, arriving at an even spectral triple. It follows that this operator cannot be isospectral to our $\widetilde{\mathcal{D}}_N$. The earlier paper uses a chirality operator γ_χ, see (5) of [12], that does not commute with the algebra $\mathcal{A}_N$. Later, in (6) of [12], this is corrected to γ_χ° by replacing left with right multiplication. On imposing anticommutation of $\mathbf{D}$ with that grading, one arrives at a "second order" operator, (8) of [12], that in our notations is $\mathbf{D}(a \otimes v) := c\, \gamma_\chi^\circ \sum_{klm} \varepsilon_{klm}\, \pi_j(J_k)\, a\, \pi_j(J_l) \otimes \sigma_m v$, where c is a normalization constant.

From (17) of [12], relabelling with $\ell = j + \frac{1}{2}$, we see that the spectrum of $\mathbf{D}$ is given by the eigenvalues $\pm\lambda_\ell$, for $\ell = 1, \ldots, N + 1$, with

$$\lambda_\ell^2 := \frac{\ell^2((N + 1)^2 - \ell^2)}{N(N + 2)} \ .$$

Note that λ_ℓ is nonlinear in ℓ, and that $\lambda_{N+1} = 0$, i.e., this operator has a kernel $V_{N+\frac{1}{2}}$.

The mentioned proposals, and other variants such as [20], begin with a chirality operator and then find an anticommuting self-adjoint Dirac-like operator with a plausible spectrum. Our approach, in contrast, starts from $SU(2)$-equivariance and arrives at a neater truncation of the classical spectrum, paying the price of spectral asymmetry.

4 Spectral distance between coherent states

Having reduced the problem of computing distances on the fuzzy sphere, via Prop. 3.6, to the use of the irreducible spectral triple $(\mathcal{A}_N, \mathcal{H}_N, D_N)$, we now compute the distance between particular

pairs of pure states in $S(A_n)$. Using (3.2), we know that

$$D_N = \begin{pmatrix} 1 + \pi_j(H) & \pi_j(F) \\ \pi_j(E) & 1 - \pi_j(H) \end{pmatrix} \tag{4.1}$$

where again $2j = N$. From now on we omit the representation symbol π_j and use the matrix of (3.2) instead, by an abuse of notation. The spectral distance is denoted by d_N.

Lemma 4.1. *For any $a \in A_N$, the following inequalities hold:*

$$\big\|[H, a]\big\| \leq \big\|[D_N, a]\big\|, \qquad \big\|[E, a]\big\| \leq \big\|[D_N, a]\big\|, \qquad \big\|[F, a]\big\| \leq \big\|[D_N, a]\big\|.$$

Moreover, if a is a diagonal hermitian matrix, then $\|[D_N, a]\| = \|[E, a]\|$.

Proof. Using the expression

$$[D_N, a]^*[D_N, a] = \begin{pmatrix} [H, a]^*[H, a] + [E, a]^*[E, a] & \cdots \\ \cdots & \cdots \end{pmatrix},$$

we find a lower bound for $\|[D_N, a]\|$ taking the supremum over unit vectors of the form $(x, 0)^t$, with $x \in V_j$:

$$\big\|[D_N, a]\big\|^2 \geq \sup_{\|x\|=1} \big\langle x \mid ([H, a]^*[H, a] + [E, a]^*[E, a])x \big\rangle$$

$$= \sup_{\|x\|=1} \big(\|[H, a]\, x\|^2 + \|[E, a]\, x\|^2 \big) = \|[H, a]\|^2 + \|[E, a]\|^2.$$

Thus $\|[H, a]\| \leq \big\|[D_N, a]\big\|$ and $\|[E, a]\| \leq \big\|[D_N, a]\big\|$, Since $[F, a] = -[E, a^*]^*$, we also get $\|[F, a]\| \leq \|[D_N, a^*]\| = \|[D_N, a]^*\| = \|[D_N, a]\|$.

If $a \in A_N$ is a diagonal matrix, then $[H, a] = 0$, so that

$$[D_N, a]^*[D_N, a] = \begin{pmatrix} [E, a]^*[E, a] & 0 \\ 0 & [F, a]^*[F, a] \end{pmatrix},$$

thus $\|[D_N, a]\|$ is the greater of $\|[E, a]\|$ and $\|[F, a]\|$. Furthermore, if $a = a^*$, then $[F, a] = -[E, a]^*$ and $\|[E, a]\| = \|[F, a]\|$, so that $\|[D_N, a]\| = \|[E, a]\| = \|[F, a]\|$. $\qquad\square$

The $SU(2)$-coherent states on A_N were introduced in [5], under the names Bloch or atomic coherent states, by applying the rotation $R_{(\varphi,\theta)}$ to the "ground" state $|j, -j\rangle \in V_j$. The coherent-state vectors are [5]:

$$|\varphi, \theta)_N := \sum_{m=-j}^{j} \binom{2j}{j+m}^{\frac{1}{2}} e^{-im\varphi} (\sin \tfrac{\theta}{2})^{j+m} (\cos \tfrac{\theta}{2})^{j-m} |j, m\rangle. \tag{4.2}$$

The corresponding vector states are denoted by

$$\psi^N_{(\varphi,\theta)}(a) := (\varphi, \theta \mid a \mid \varphi, \theta)_N .$$

These Bloch coherent states are for the group $SU(2)$ what the usual harmonic oscillator coherent states are for the Heisenberg group [28]. In particular, they are minimum uncertainty states. The map $\mathbb{S}^2 \to V_j$, sending the point $(\varphi, \theta) \in \mathbb{S}^2$ to the vector $|\varphi, \theta\rangle$, intertwines the rotation action of $SU(2)$ on $\mathbb{S}^2$ with the irrep π_j on V_j. At the infinitesimal level, this is expressed by the next lemma, whose proof is a simple direct computation.

Lemma 4.2. *Regarding $\psi^N_{(\varphi,\theta)}$ as a vector state on $\mathcal{B}(V_j)$, we find that*

$$\psi^N_{(\varphi,\theta)}([H,a]) = -i \frac{\partial}{\partial\varphi}\, \psi^N_{(\varphi,\theta)}(a), \tag{4.3a}$$

$$\psi^N_{(\varphi,\theta)}([E,a]) = e^{i\varphi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right)\psi^N_{(\varphi,\theta)}(a), \tag{4.3b}$$

$$\psi^N_{(\varphi,\theta)}([F,a]) = -e^{-i\varphi}\left(\frac{\partial}{\partial\theta} - i\cot\theta\frac{\partial}{\partial\varphi}\right)\psi^N_{(\varphi,\theta)}(a). \tag{4.3c}$$

4.1 The $N = 1$ case

We write the general hermitian element $a = a^* \in M_2(\mathbb{C})$ as

$$a = \begin{pmatrix} a_0 + a_3 & a_1 + ia_2 \\ a_1 - ia_2 & a_0 - a_3 \end{pmatrix} = a_0\, 1_2 + \vec{a} \cdot \vec{\sigma},$$

with a_0 real and $\vec{a} = (a_1, a_2, a_3) \in \mathbb{R}^3$. Arbitrary (not necessarily pure) states on $M_2(\mathbb{C})$ are given by $\omega_{\vec{x}}(a) := a_0 + \vec{x} \cdot \vec{a}$, with $\vec{x}$ in the closed unit ball $B^3 \subset \mathbb{R}^3$. This state is pure if and only if $\vec{x} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$ lies on the boundary $\mathbb{S}^2$ of the ball, in which case it coincides with the coherent state $\psi^1_{(\varphi,\theta)}$. Note that *for $N = 1$, all pure states are coherent states*.

The next proposition shows that the distance among states is half of the Euclidean distance in the ball; thus, for coherent states, *half of the chordal distance* on the sphere.

Proposition 4.3. *For all $\vec{x}, \vec{y} \in B^3$, the distance between the corresponding states is*

$$d_1(\omega_{\vec{x}}, \omega_{\vec{y}}) = \frac{1}{2}\,|\vec{x} - \vec{y}|. \tag{4.4}$$

In particular, $d_1(\psi^1_{(0,\theta)}, \psi^1_{(0,0)}) = \sin(\theta/2)$.

Proof. Writing $a_\pm = a_1 \pm ia_2$ and $\sigma_\pm = \sigma_1 \pm i\sigma_2$, we get, for $a = a^*$:

$$[D_1, a] = \begin{pmatrix} \frac{1}{2}[\sigma_3, a] & [\sigma_-, a] \\ [\sigma_+, a] & -\frac{1}{2}[\sigma_3, a] \end{pmatrix} = \begin{pmatrix} 0 & a_+ & -a_+ & 0 \\ -a_- & 0 & 2a_3 & a_+ \\ a_- & -2a_3 & 0 & -a_+ \\ 0 & -a_- & a_- & 0 \end{pmatrix}.$$

The matrix $i[D_1, a]$ is hermitian, and its characteristic polynomial is easily seen to be $\det(\lambda - i[D_1, a]) = \lambda^2(\lambda^2 - 4|\vec{a}|^2)$, showing that its norm is $\|[D_1, a]\| = 2|\vec{a}|$.

The Cauchy–Schwarz inequality

$$\left|\omega_{\vec{x}}(a) - \omega_{\vec{y}}(a)\right| = \left|(\vec{x} - \vec{y}) \cdot \vec{a}\right| \leq |\vec{x} - \vec{y}|\,|\vec{a}|$$

is saturated when $\vec{a}$ is parallel to $\vec{x} - \vec{y}$. Thus $d_1(\omega_{\vec{x}}, \omega_{\vec{y}})$ is the supremum of $|\vec{x} - \vec{y}|\,|\vec{a}|$ over hermitian a with $\|[D_1, a]\| = 2|\vec{a}| \leq 1$. This establishes (4.4).

If $\vec{x} = (\sin\theta, 0, \cos\theta)$ and $\vec{y} = (0, 0, 1)$, then $|\vec{x} - \vec{y}|^2 = 2(1 - \cos\theta) = 4\sin^2(\theta/2)$, and thus $d_1(\omega_{\vec{x}}, \omega_{\vec{y}}) = \sin(\theta/2)$. $\qquad\square$

4.2 Distances between basis vectors

Similarly to Prop. 3.6 of [10], the distance between basis vectors can be exactly computed. For fixed $N = 2j$, and $m \in \{-j, \ldots, j\}$, the basic vector states are

$$\omega_m(a) := \langle j, m \mid a \mid j, m \rangle \,.$$

Proposition 4.4. *For any $m < n$ in $\{-j, \ldots, j\}$, the following distance formula holds:*

$$d_N(\omega_m, \omega_n) = \sum_{k=m+1}^{n} \frac{1}{\sqrt{(j+k)(j-k+1)}} \,. \tag{4.5}$$

Proof. If $a \in \mathcal{A}_N$, then

$$
\begin{aligned}
\omega_m(a) - \omega_n(a) &= \sum_{k=m+1}^{n} \langle j, k-1 \mid a \mid j, k-1 \rangle - \langle j, k \mid a \mid j, k \rangle \\
&= \sum_{k=m+1}^{n} \frac{1}{\sqrt{(j+k)(j-k+1)}} \langle j, k \mid [E, a] \mid j, k-1 \rangle .
\end{aligned}
$$

Using Lemma 4.1, we get the estimate

$$\left| \langle j, k \mid [E, a] \mid j, k-1 \rangle \right| \le \|[E, a]\| \le \|[D_N, a]\|$$

which shows that the left hand side of (4.5) is no greater than the right hand side. On the other hand, let $\hat{a}$ be the self-adjoint diagonal operator:

$$\hat{a} \mid j, m \rangle := -\left(\sum_{k=-j+1}^{m} \frac{1}{\sqrt{(j+k)(j-k+1)}} \right) \mid j, m \rangle. \tag{4.6}$$

The coefficients are chosen so that $[E, \hat{a}] \mid j, m \rangle = \mid j, m+1 \rangle$ for $m = -j, \ldots, j-1$. Notice that $\hat{a} \mid j, -j \rangle = 0$ and $[E, \hat{a}] \mid j, -j \rangle = 0$. Since $\hat{a} = \hat{a}^*$, Lemma 4.1 then shows that $\|[D_N, \hat{a}]\| = \|[E, \hat{a}]\| = 1$. Therefore,

$$d_N(\omega_m, \omega_n) \ge \omega_m(\hat{a}) - \omega_n(\hat{a}) = \sum_{k=m+1}^{n} \frac{1}{\sqrt{(j+k)(j-k+1)}} \,. \qquad \square$$

Note that the distance is additive on the chain of basic vector states:

$$d_N(\omega_m, \omega_n) = \sum_{k=m+1}^{n} d_N(\omega_{k-1}, \omega_k).$$

Corollary 4.5. *For any N, the distance between the north and south poles of the fuzzy sphere is:*

$$d_N(\psi_{(0,0)}^N, \psi_{(0,\pi)}^N) = \sum_{k=1}^{N} \frac{1}{\sqrt{k(N-k+1)}} \,. \tag{4.7}$$

Proof. By construction, the Bloch state vectors at the poles are basis vectors: $|0, 0\rangle_N = |j, -j\rangle$ and $|0, \pi\rangle_N = |j, j\rangle$. Therefore, $\psi_{(0,0)}^N = \omega_{-j}$ and $\psi_{(0,\pi)}^N = \omega_j$. From (4.5) we get (4.7), since the left hand side is just $d_N(\omega_{-j}, \omega_j)$. $\square$

4.3 An auxiliary distance

Let $\mathcal{B}_N \subset \mathcal{A}_N$ be the subalgebra of *diagonal* matrices. Note that if a is diagonal, then $\psi^N_{(\varphi,\theta)}(a) = \psi^N_{(0,\theta)}(a)$ for any φ. Define the distance

$$\rho_N(\theta) := \sup\{|\psi^N_{(0,\theta)}(a) - \psi^N_{(0,0)}(a)| : a = a^* \in \mathcal{B}_N, \ \|[D_N, a]\| \leq 1\}. \tag{4.8}$$

Proposition 4.6. *For any $0 \leq \theta \leq \pi$, $\rho_N(\theta)$ is given by:*

$$\rho_N(\theta) = \sum_{n=1}^{N} \binom{N}{n}(\sin\tfrac{\theta}{2})^{2n}(\cos\tfrac{\theta}{2})^{2(N-n)} \sum_{k=1}^{n} \frac{1}{\sqrt{k(N-k+1)}} . \tag{4.9}$$

Proof. Let $a = (\delta_{mn}c_m) \in \mathcal{B}_N$, with $c_m \in \mathbb{R}$. Then $\omega_m(a) = c_m$, which gives

$$\psi^N_{(0,0)}(a) - \psi^N_{(0,\theta)}(a) = \sum_{m=-j}^{j} \binom{2j}{j+m}(\sin\tfrac{\theta}{2})^{2(j+m)}(\cos\tfrac{\theta}{2})^{2(j-m)}(\omega_{-j}(a) - \omega_m(a)).$$

We also know that

$$\omega_{-j}(a) - \omega_m(a) \leq d_N(\omega_m, \omega_{-j}) = \sum_{m'=-j+1}^{m} \frac{1}{\sqrt{(j+m')(j-m'+1)}}$$

for all a with $\|[D_N, a]\| \leq 1$, with the supremum saturated on the diagonal element $\hat{a}$ given by (4.6). On substituting $n = j + m$ and $k = j + m'$, we arrive at (4.9). $\qquad\square$

Lemma 4.7. *The derivative $\rho'_N(\theta)$ of (4.9) satisfies $0 \leq \rho'_N(\theta) \leq 1$.*

Proof. From (4.3b) we deduce that $\psi^N_{(0,\theta)}([E,a]) = \frac{\partial}{\partial\theta}\psi^N_{(0,\theta)}(a)$ for all $a \in \mathcal{B}_N$. Using this relation and the equality $\rho_N(\theta) = \psi^N_{(0,\theta)}(\hat{a}) - \psi^N_{(0,0)}(\hat{a})$, with $\hat{a}$ the element in (4.6), we get:

$$\rho'_N(\theta) = \frac{\partial}{\partial\theta}\psi^N_{(0,\theta)}(\hat{a}) = \psi^N_{(0,\theta)}([E,\hat{a}]).$$

Since states are functionals with norm 1, it follows that

$$|\rho'_N(\theta)| = |\psi^N_{(0,\theta)}([E,\hat{a}])| \leq \psi^N_{(0,\theta)}(1) \, \|[E,\hat{a}]\| = 1.$$

On the other hand, since $L := [E, \hat{a}]$ is the ladder operator $|j, m\rangle \mapsto |j, m + 1\rangle$, we get

$$\rho'_N(\theta) = (0,\theta \mid L \mid 0,\theta)_N = \sum_{m=-j}^{j-1} \binom{2j}{j+m}^{\frac{1}{2}} \binom{2j}{j+m+1}^{\frac{1}{2}} (\sin\tfrac{\theta}{2})^{2j+2m+1}(\cos\tfrac{\theta}{2})^{2j-2m-1} \geq 0.$$

Actually, we see that $\rho'_N(\theta) > 0$ for $0 < \theta < \pi$. $\qquad\square$

The previous lemma has two consequences: $\rho_N(\theta)$ is strictly increasing on $0 \leq \theta \leq \pi$, for fixed N; and, for $0 < \theta \leq \pi$ the mean value theorem gives ϕ with $0 < \phi < \theta$ such that

$$\rho_N(\theta) = \rho_N(\theta) - \rho_N(0) = \theta \, \rho'_N(\phi) \leq \theta.$$

That is: $\rho_N(\theta)$ is no greater than the geodesic distance on the circle.

4.4 $SU(2)$-invariance of the distance

Lemma 4.8. *The distance function $d_N(\psi^N_{(\varphi,\theta)}, \psi^N_{(\varphi',\theta')})$ is $SU(2)$-invariant.*

Proof. Up to now, we have identified the element $a \in \mathcal{A}_N \simeq \mathrm{End}(V_j)$ with the operator $a \otimes 1_2$ acting on $\mathcal{H}_N = V_j \otimes V_{\frac{1}{2}}$. In this proof, we shall write explicitly $a \otimes 1_2$ to avoid ambiguities.

For any $g \in SU(2)$ and $a \in \mathcal{A}_N$, we write $a^g := \pi_j(g) a \pi_j(g)^*$. Since $\pi_{\frac{1}{2}}(g) \pi_{\frac{1}{2}}(g)^* = 1_2$ by unitarity of $\pi_{\frac{1}{2}}$, we get

$$a^g \otimes 1_2 = u(a \otimes 1_2)u^* \quad \text{where} \quad u := \pi_j(g) \otimes \pi_{\frac{1}{2}}(g).$$

Since D_N commutes with u, the operator $[D_N, a^g \otimes 1_2] = u[D_N, a \otimes 1_2]u^*$ has the same norm as $[D_N, a \otimes 1_2]$.

Given a state ω on $\mathcal{A}_N$ and $g \in SU(2)$, let $g_*\omega$ be the state defined by $g_*\omega(a) = \omega(a^g)$. For any pair of states ω, ω', we then obtain

$$
\begin{aligned}
d_N(g_*\omega, g_*\omega') &= \sup_{a \in \mathcal{A}_N} \{ |\omega(a^g) - \omega'(a^g)| : \|[D_N, a \otimes 1_2]\| \le 1 \} \\
&= \sup_{b \in \mathcal{A}_N} \{ |\omega(b) - \omega'(b)| : \|[D_N, b \otimes 1_2]\| \le 1 \} = d_N(\omega, \omega'),
\end{aligned}
$$

where we have put $b = a^g$ and used $\|[D_N, a^g \otimes 1_2]\| = \|[D_N, a \otimes 1_2]\|$. By construction, the action $\psi^N_{(\varphi,\theta)} \mapsto g_*\psi^N_{(\varphi,\theta)}$ corresponds to the usual rotation action of $SU(2)$ on $\mathbb{S}^2$. $\qquad\square$

4.5 Dependence on the dimension

We now show that the distance $d_N(\psi^N_{(\varphi,\theta)}, \psi^N_{(\varphi',\theta')})$ is non-decreasing with N. Using the fuzzy spinor basis (3.4), one defines injections $U_j^\pm : V_{j\pm\frac{1}{2}} \to V_j \otimes V_{\frac{1}{2}}$ by

$$U_j^+ |j + \tfrac{1}{2}, m + \tfrac{1}{2}\rangle := |j, m\rangle\!\rangle_+ , \qquad U_j^- |j - \tfrac{1}{2}, m + \tfrac{1}{2}\rangle := |j, m\rangle\!\rangle_- ,$$

using the same index sets as in (3.4), namely $m = -j - 1, \ldots, j$ for the range of U_j^+ and $m = -j, \ldots, j-1$ for the range of U_j^-. One easily checks that these $U_j^\pm$ are isometries, i.e., $(U_j^\pm)^* U_j^\pm = 1$, that intertwine the representations of $\mathfrak{su}(2)$. Also, $V_j \otimes V_{\frac{1}{2}}$ is the orthogonal direct sum of the ranges of U_j^+ and U_j^-.

Lemma 4.9. $U_j^+ |\varphi, \theta)_{N+1} = |\varphi, \theta)_N \otimes |\varphi, \theta)_1$ *for any* $(\varphi, \theta) \in \mathbb{S}^2$.

Proof. Note that $|\varphi, \theta)_1 = e^{-\frac{1}{2}\mathrm{i}\varphi} \sin \frac{\theta}{2} |\frac{1}{2}, -\frac{1}{2}\rangle + e^{\frac{1}{2}\mathrm{i}\varphi} \cos \frac{\theta}{2} |\frac{1}{2}, \frac{1}{2}\rangle$. The rest is an easy computation, using (4.2). $\qquad\square$

We define two injective linear maps

$$\eta_N^\pm : \mathcal{A}_N \to \mathcal{A}_{N\pm1}, \qquad \eta_N^\pm(a) := (U_j^\pm)^*(a \otimes 1_2)U_j^\pm .$$

They are unital and commute with the involution, but are neither surjective nor algebra morphisms, since $U_j^+(U_j^+)^* + U_j^-(U_j^-)^* = 1$. They are norm decreasing: the norm of $a \otimes 1_2$ on the range of $U_j^\pm$ is no greater than its norm on $V_j \otimes V_{\frac{1}{2}}$, which equals the norm of a on V_j.

Lemma 4.10. *For any $a \in \mathcal{A}_N$,*

$$\psi^{N+1}_{(\varphi,\theta)} \circ \eta^+_N(a) = \psi^N_{(\varphi,\theta)}(a), \tag{4.10}$$

and

$$\left\| [D_{N\pm1}, \eta^\pm_N(a)] \right\| \leq \left\| [D_N, a] \right\|. \tag{4.11}$$

Proof. The equality (4.10) follows from Lemma 4.9, because

$$(\varphi, \theta \mid \eta^+_N(a) \mid \varphi, \theta)_{N+1} = (\varphi, \theta \mid a \mid \varphi, \theta)_N \, (\varphi, \theta \mid \varphi, \theta)_1 = (\varphi, \theta \mid a \mid \varphi, \theta)_N \, .$$

Since $U^\pm_j$ intertwines representations of $\mathfrak{su}(2)$, i.e.,

$$U^\pm_j X = (X \otimes 1_2 + 1_2 \otimes X) U^\pm_j \quad \text{for all} \quad X \in \mathfrak{su}(2)$$

(the representation symbols are omitted), we conclude that

$$[X, \eta^\pm_N(a)] = (U^\pm_j)^* ([X, a] \otimes 1_2) U^\pm_j = \eta^\pm_N([X, a]).$$

In view of (4.1), therefore, $[D_{N\pm1}, \eta^\pm_N(a)] = \eta^\pm_N([D_N, a])$, where $[D_N, a] \in M_2(\mathcal{A}_N)$ and we extend $\eta^\pm_N$ from $\mathcal{A}_N$ to $M_2(\mathcal{A}_N)$ by applying it to each matrix entry. Since both $\eta^\pm_N$ are norm-decreasing maps, this proves (4.11). □

Proposition 4.11. *For any $N \geq 1$, the following majorization holds:*

$$d_{N+1}(\psi^{N+1}_{(\varphi,\theta)}, \psi^{N+1}_{(\varphi',\theta')}) \geq d_N(\psi^N_{(\varphi,\theta)}, \psi^N_{(\varphi',\theta')}).$$

Proof. We get directly:

$$
\begin{aligned}
d_{N+1}(\psi^{N+1}_{(\varphi,\theta)}, \psi^{N+1}_{(\varphi',\theta')}) &= \sup_{a \in \mathcal{A}_{N+1}} \left\{ \left| \psi^{N+1}_{(\varphi,\theta)}(a) - \psi^{N+1}_{(\varphi',\theta')}(a) \right| : \|[D_{N+1}, a]\| \leq 1 \right\} \\
&\geq \sup_{a \in \mathcal{A}_N} \left\{ \left| \psi^{N+1}_{(\varphi,\theta)} \circ \eta^+_N(a) - \psi^{N+1}_{(\varphi',\theta')} \circ \eta^+_N(a) \right| : \left\| [D_{N+1}, \eta^+_N(a)] \right\| \leq 1 \right\} \\
&= \sup_{a \in \mathcal{A}_N} \left\{ \left| \psi^N_{(\varphi,\theta)}(a) - \psi^N_{(\varphi',\theta')}(a) \right| : \left\| [D_{N+1}, \eta^+_N(a)] \right\| \leq 1 \right\} \\
&\geq \sup_{a \in \mathcal{A}_N} \left\{ \left| \psi^N_{(\varphi,\theta)}(a) - \psi^N_{(\varphi',\theta')}(a) \right| : \|[D_N, a]\| \leq 1 \right\} \\
&= d_N(\psi^N_{(\varphi,\theta)}, \psi^N_{(\varphi',\theta')}) \, .
\end{aligned}
$$

The first inequality follows since the supremum over the range of η^+_N in $\mathcal{A}_{N+1}$ is smaller than the supremum over the whole $\mathcal{A}_{N+1}$. In the next line (4.10) is used; and we get the second inequality from (4.11). □

Remark 4.12. The calculation in the proof of Prop. 4.11 can be adapted to establish that

$$\rho_{N+1}(\theta - \theta') \geq \rho_N(\theta - \theta'), \quad \text{for} \quad \theta, \theta' \in [0, \pi]. \tag{4.12}$$

For that, just restrict $a \in \mathcal{A}_N$ to (be self-adjoint and) lie in the diagonal subalgebra $\mathcal{B}_N$. The only thing to note that is that η^+_N maps $\mathcal{B}_N$ into a non-diagonal subalgebra of $\mathcal{A}_{N+1}$; but the notion of diagonal subalgebra is in any case basis-dependent. It is enough to replace $\mathcal{B}_{N+1}$ by a conjugate subalgebra that includes $\eta^+_N(\mathcal{B}_N)$, after conjugating $\mathcal{A}_{N+1}$ by a unitary operator commuting with the $SU(2)$ action via $\operatorname{ad} \pi_{j+\frac{1}{2}}$. This rotates the basis vectors in $V_{j+\frac{1}{2}}$, in such a way that the coherent states $\psi^{N+1}_{(\varphi,\theta)}$ are unchanged. Thus also, $\rho_{N+1}(\theta - \theta')$ is unchanged, and (4.12) holds.

4.6 Upper and lower bounds and the large N limit

Proposition 4.13. *The following inequalities hold, for all $(\varphi, \theta), (\varphi', \theta') \in \mathbb{S}^2$:*

$$\rho_N(\theta - \theta') \leq d_N(\psi^N_{(\varphi,\theta)}, \psi^N_{(\varphi',\theta')}) \leq d_{\text{geo}}((\varphi, \theta), (\varphi', \theta')), \tag{4.13}$$

where $\rho_N(\theta)$ is the auxiliary distance (4.8) and d_{geo} is the geodesic distance for the round metric of $\mathbb{S}^2$. In particular,

$$\rho_N(\theta) \leq d_N(\psi^N_{(0,\theta)}, \psi^N_{(0,0)}) \leq \theta. \tag{4.14}$$

Proof. Due to Lemma 4.8, the second inequality in (4.13) involves two $SU(2)$-invariant expressions. It is then enough to prove it when $(\varphi', \theta') = (0, \frac{\pi}{2})$ and $(\varphi, \theta) = (\varphi, \frac{\pi}{2})$. We thus need to prove that

$$d_N(\psi^N_{(\varphi,\frac{\pi}{2})}, \psi^N_{(0,\frac{\pi}{2})}) \leq |\varphi| \quad \text{for all} \quad -\pi < \varphi \leq \pi.$$

Integrating (4.3a), we find

$$\psi^N_{(\varphi,\frac{\pi}{2})}(a) - \psi^N_{(0,\frac{\pi}{2})}(a) = \mathrm{i} \int_0^\varphi \psi^N_{(\alpha,\frac{\pi}{2})}([H,a]) \, d\alpha,$$

and since $|\omega(A)| \leq \|A\|$ for any state ω and operator A, we obtain, using Lemma 4.1:

$$|\psi^N_{(\varphi,\frac{\pi}{2})}(a) - \psi^N_{(0,\frac{\pi}{2})}(a)| \leq \|[H,a]\| \left| \int_0^\varphi d\alpha \right| = |\varphi| \, \|[H,a]\| \leq |\varphi| \, \|[D_N,a]\|.$$

This proves the upper bound in (4.13). That of (4.14) follows from $d_{\text{geo}}((0,\theta),(0,0)) = \theta$.

A lower bound for the distance is given by the supremum over diagonal matrices:

$$d_N(\psi^N_{(\varphi,\theta)}, \psi^N_{(\varphi',\theta')}) \geq \sup_{a=a^*\in\mathcal{B}_N} \left\{ |\psi^N_{(\varphi,\theta)}(a) - \psi^N_{(\varphi',\theta')}(a)| : \|[D_N,a]\| \leq 1 \right\}.$$

Since $\psi^N_{(\varphi,\theta)}(a)$ is independent of φ for any diagonal a, we arrive at

$$d_N(\psi^N_{(\varphi,\theta)}, \psi^N_{(\varphi',\theta')}) \geq \sup_{a=a^*\in\mathcal{B}_N} \left\{ |\psi^N_{(0,\theta)}(a) - \psi^N_{(0,\theta')}(a)| : \|[D_N,a]\| \leq 1 \right\} = \rho_N(\theta - \theta'). \qquad \square$$

For $0 < \theta < \pi$, neither the upper nor the lower bound in (4.14) is sharp. On the other hand, $d_N(\psi^N_{(0,\pi)}, \psi^N_{(0,0)}) = \rho_N(\pi)$, since the formula (4.9) coincides with (4.7) when $\theta = \pi$. Thus the lower bound is sharp for $\theta = \pi$. In Figure 1a we show a plot of the upper bound (straight line) and lower bounds ρ_N for $N = 10, 30, 500$ (nondecreasing with N). It would seem that $\theta - \rho_N(\theta)$ has its maximum at $\theta = \pi$. Figure 1b plots $\theta - \rho_N(\theta)$ for $N = 5, 10, 20, 30$ (decreasing with N). This suggests how to prove our final result.

Proposition 4.14. *As $N \to \infty$, the sequence $\rho_N(\theta)$ is uniformly convergent to θ in $[0, \pi]$. Therefore,*

$$\lim_{N\to\infty} d_N(\psi^N_{(\varphi,\theta)}, \psi^N_{(\varphi',\theta')}) = d_{\text{geo}}((\varphi, \theta), (\varphi', \theta')).$$

<div style="text-align:center">

(a) Plot of θ and $\rho_N(\theta)$ for $N = 10, 30, 500$.   (b) Plot of $\theta - \rho_N(\theta)$ for $N = 5, 10, 20, 30$.
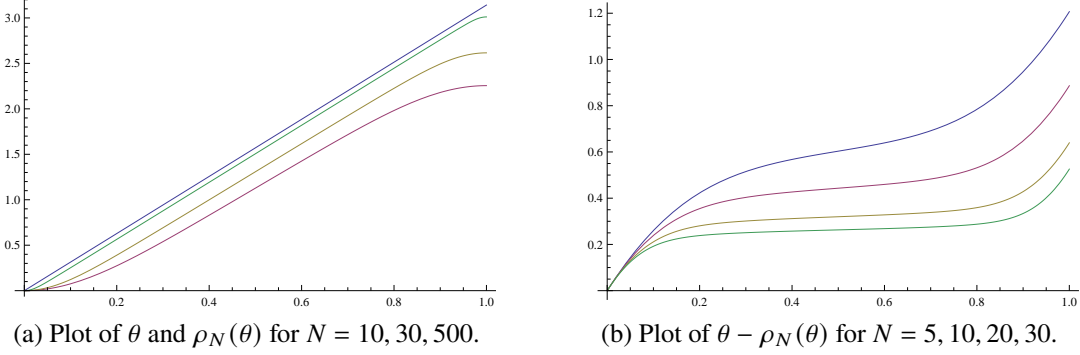
Figure 1: Comparison of $\rho_N(\theta)$ with θ. The abscissa is $x = \pi\theta$.

</div>

Proof. Let $f_N(\theta) := \theta - \rho_N(\theta)$. Clearly $f_N(0) = 0$, and $f'_N(\theta) \geq 0$ by Lemma 4.7. Hence $f_N(\theta)$ is a nondecreasing positive function of θ for each N, and

$$\|\theta - \rho_N(\theta)\|_\infty = \sup_{\theta \in [0,\pi]} f_N(\theta) \leq f_N(\pi) = \pi - \rho_N(\pi).$$

Therefore, the uniform convergence $\lim_{N\to\infty} \|\theta - \rho_N(\theta)\|_\infty = 0$ holds if and only if the diameter converges to π, i.e., $\lim_{N\to\infty} \rho_N(\pi) = \pi$.

The formula for $\rho_N(\pi)$ is given by (4.7). The sequence $\rho_N(\pi)$ is bounded, $\rho_N(\pi) \leq \pi$, and is nondecreasing by Remark 4.12. Hence it is convergent, and the limit can be computed using any subsequence. It is thus enough to prove that $\rho_N(\pi) \geq c_N$, where $c_N \to \pi$ as $N \to \infty$.

We consider the subsequence with odd N only. The function $(x(N - x + 1))^{-1/2}$ is positive for $1 \leq x \leq N$, symmetric about $x = \frac{1}{2}(N + 1)$, and monotonically decreasing for $1 \leq x \leq \frac{1}{2}(N + 1)$. Hence

$$\rho_N(\pi) = 2 \sum_{k=1}^{\frac{1}{2}(N-1)} \frac{1}{\sqrt{k(N - k + 1)}} + \frac{2}{N + 1} \geq 2 \int_1^{\frac{1}{2}(N+1)} \frac{dx}{\sqrt{x(N - x + 1)}} \, .$$

Substituting $x =: \frac{1}{2}(N + 1)(1 + \sin\xi)$, so that $d\xi = dx/\sqrt{x(N - x + 1)}$, we obtain

$$\rho_N(\pi) \geq 2 \arcsin \frac{N - 1}{N + 1} \, .$$

The right hand side converges monotonically to π as $N \to \infty$, thus $\lim_{N\to\infty} \rho_N(\pi) = \pi$ through odd N, and so, as noted above, through all N. (A slightly modified estimate gives $\lim_{N\to\infty} \rho_N(\pi) = \pi$ through even N, directly, without using Remark 4.12.) This proves the uniform convergence $\rho_N(\theta) \to \theta$.

The estimate (4.14) now shows that $d_N(\psi^N_{(0,\theta)}, \psi^N_{(0,0)})$ is uniformly convergent to θ; by $SU(2)$-invariance, $d_N(\psi^N_{(\varphi,\theta)}, \psi^N_{(\varphi',\theta')})$ converges to $d_{\text{geo}}((\varphi, \theta), (\varphi', \theta'))$ uniformly on $\mathbb{S}^2$. $\qquad\square$

Acknowledgments

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