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Theta series and number fields: theorems and experiments

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Abstract Let d and n be positive integers and let K be a totally real number field of discriminant d and degree n. We construct a theta series $\theta_K \in \mathcal{M}_{d,n}$, where $\mathcal{M}_{d,n}$ is a space of modular forms defined in terms of n and d. Moreover, if d is square free and n is at most 4 then θ_K is a complete invariant for K. We also investigate whether or not the collection of θ-series, associated to the set of isomorphism classes of quartic number fields of a fixed square free discriminant d, is a linearly independent subset of $\mathcal{M}_{d,4}$. This is known to be true if the degree of the number field is less than or equal to 3. We give computational and heuristic evidence suggesting that in degree 4 these theta series should be independent as well.

Keywords Quartic fields · Theta series

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1 Introduction and statement of results

Quadratic forms and number fields have been intertwined since the beginning of the development of algebraic number theory. An important method to create modular forms is the construction of θ-series associated to quadratic forms. In this paper, we investigate the relationship between modular forms and number fields. In [20], a particular modular form, a linear combination of two θ-series is shown to reveal a great deal of information about the quadratic forms of discriminant -23, the quadratic field $\mathbb{Q}(\sqrt{-23})$ and modular forms of level 23.

Perhaps the most extensively studied modular form is the Jacobi θ-function

$$\vartheta(z) = \sum_{n=-\infty}^{\infty} q^{n^2},$$

where $q = e^{\pi i z}$ for z in the complex upper half-plane. Let d be a squarefree positive integer. A slight generalization of Jacobi's form is the modular form $\vartheta_d(z) = \sum_{n=-\infty}^{\infty} q^{dn^2}$. Using the level, or by more elementary arguments, we see that the function $\vartheta_d(z)$ determines the value d. This could be interpreted in terms of number fields as giving a complete invariant for totally real number fields: Let K be a real quadratic number field and let d_K be the unique square free positive integer such that $K = \mathbb{Q}(\sqrt{d_K})$. In this context the assignment

$$K \mapsto \vartheta_{d_K}$$

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is a complete invariant. It is an invariant for K since the map is independent of the isomorphism class of K, and it is complete since it is injective. This simple construction leads one naturally to consider the following:
Question. In what ways can one associate a modular form to a number field so that one can recover meaningful information about the number field?

We aim, in part, to generalize the construction $K \mapsto \vartheta_{d_K}$ from degree $n = 2$ to arbitrary higher degree n, and to show, that for n up to 4 this generalization provides a complete invariant. A first natural invariant for number fields is the discriminant. Since the discriminant of a number field K is the determinant of the trace pairing, with respect to any integral basis, it is natural to consider the isometry class of the integral bilinear pairing

$$\begin{aligned} \mathrm{tr}_{K/\mathbb{Q}} : O_K \times O_K &\to \mathbb{Z} \\ (x, y) &\mapsto \mathrm{tr}_{K/\mathbb{Q}}(xy). \end{aligned}$$

as the first natural object refining the discriminant.

In [13] the second author associated a space of weight 1 modular forms to totally real cubic number fields. In particular, he showed that there is an injection from the set of isomorphism classes of totally real cubic number fields of fundamental discriminant d to the space of weight one modular forms of a prescribed level and character determined by d. Moreover, he showed that the forms in the image of this map are linearly independent. The modular forms that he constructs are θ-series derived from the map

$$\begin{aligned} t_K^0 : O_K^0 &\to \mathbb{Z} \\ x &\mapsto \tfrac{1}{2}\,\mathrm{tr}_{K/\mathbb{Q}}(x^2) \end{aligned}$$

where K is a number field and $O_K^0 \subset K$ is the set of integral elements with trace zero. This is shown to be a quadratic form over $\mathbb{Z}$ and the associated modular form he constructs is the theta series of the quadratic form. Combining other results of the second author [12,15] one concludes that these modular forms are invariants for number fields in the sense described above. In this paper, we study the same question but in greater generality. First for a degree n totally real number field K, of fundamental discriminant co-prime to n, we define a theta series θ_K that generalizes the definition of [13] to all n.

Theorem 1 *Let K be a totally real number field of degree n with discriminant d_K. Then if d_K is a fundamental discriminant and* $\gcd(n, d_K) = 1$*, the theta series*

$$\theta_K(z) := \sum_{\alpha \in \mathcal{O}_K^0} e^{\pi i \,\mathrm{Tr}_{K/\mathbb{Q}}(\alpha^2) z} \qquad (z \in \mathbb{H}),$$

satisfies

$$\theta_K \in M_{\frac{n-1}{2}}\left(\Gamma_0(2nd_K), \left(\frac{\delta_n n d_K}{\cdot}\right)\right),$$

where $\delta_n = \begin{cases} (-1)^{\frac{n-1}{2}} & \text{if } n \text{ is odd,} \\ \frac{1}{2} & \text{if } n \text{ is even.} \end{cases}$

Remark 1 By convention we define $\theta_{\mathbb{Q}}(z)$ to be the constant function 1.

Remark 2 We mention that there is a similar construction of theta series of weight 3/2 via maximal orders in quaternion algebras. More precisely, these theta series are constructed as sums

$$g_i(\tau) := \frac{1}{2} \sum_{b \in S_i^0} e^{2\pi i \mathbf{N}(b) z} \qquad (z \in \mathbb{H}),$$

where S_i^0 is a certain subgroup of trace zero elements in a maximal order R_i of the quaternion algebra, and $\mathbf{N}(b)$ is the reduced norm of b. The interested reader can consult for instance section 12 of [7].

Generalizing the results of [13] from degree 3 to degree 4 we show that there is an injection from the set of isomorphic classes of totally real quartic number fields of square free discriminant d to a prescribed space of weight 3/2 modular forms.

Theorem 2 *Let K and L be totally real number fields of degree $n \leq 4$ with discriminants $d_K = d_L = d$, where d is a fundamental discriminant. Then if $\gcd(n, d) = 1$, we have*

$$K \cong L \iff \theta_K = \theta_L,$$

i.e., the theta series θ_K completely determines the isomorphism class of the number field K.

At the moment we cannot show, as in the cases $n \leq 3$, that the θ-series in the image are linearly independent but we give both computational evidence and heuristic evidence that is true in the $n = 4$ case (see Section 5).

This paper is organized as follows. First, we provide necessary background and prove some basic theorems about the objects we are studying. Second, we prove the theorems stated above and we conclude with a summary of various computations and heuristics related to our expectation that the modular forms in the image of the map we define will be linearly independent.

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2 Background on quadratic spaces and modular forms

In this section we gather some standard basic results on quadratic spaces over $\mathbb{Z}$ and modular forms that will be needed later in the paper. We include this background in order to be as clear as possible as there are several slightly different ways to make some of these definitions. We start by recalling the definition of a quadratic form on a free $\mathbb{Z}$-module.

Definition 1 A quadratic form on a free $\mathbb{Z}$-module Λ is a function $\Phi : \Lambda \longrightarrow \mathbb{Z}$ such that

(i) $\Phi(m\lambda) = m^2\Phi(\lambda)$ for every $m \in \mathbb{Z}$ and every $\lambda \in \Lambda$;
(ii) The function $\phi : \Lambda \times \Lambda \longrightarrow \mathbb{Z}$ defined by

$$\phi(\alpha, \beta) := \Phi(\alpha + \beta) - \Phi(\alpha) - \Phi(\beta)$$

is a symmetric bilinear form on Λ.

The functions Φ and ϕ are said to be associated.

Definition 2 An ordered pair (Λ, Φ) with Λ a free $\mathbb{Z}$-module of finite rank and $\Phi : \Lambda \longrightarrow \mathbb{Z}$ a quadratic form on Λ will be called a quadratic space on $\mathbb{Z}$.

Now, if the free $\mathbb{Z}$-module Λ has finite rank n and $\mathcal{B} = \{\alpha_1, \ldots, \alpha_n\}$ is a $\mathbb{Z}$-basis for Λ, there is an associated symmetric matrix $A_{\Phi,\mathcal{B}} \in M_{n\times n}(\mathbb{Z})$ whose entries are given by

$$(A_{\Phi,\mathcal{B}})_{i,j} := \phi(\alpha_i, \alpha_j),$$

for $1 \leq i, j \leq n$. Clearly the matrix $A_{\Phi,\mathcal{B}}$ depends on the choice of $\mathbb{Z}$-basis for Λ. If we have two $\mathbb{Z}$-bases $\mathcal{B}$ and $\mathcal{C}$ for Λ and $P = P_{\mathcal{B}}^{\mathcal{C}} \in \mathrm{GL}_n(\mathbb{Z})$ is the change of basis matrix from $\mathcal{C}$ to $\mathcal{B}$, then

$$A_{\Phi,\mathcal{C}} = P^t A_{\Phi,\mathcal{B}} P.$$

The rank of the quadratic form Φ is the rank of the matrix $A_{\Phi,\mathcal{B}}$ with respect to any $\mathbb{Z}$-basis $\mathcal{B}$. Moreover, we define the discriminant of Φ to be $\Delta_\Phi = \mathrm{disc}(\Phi) := (-1)^{\frac{n(n-1)}{2}} \det(A_{\Phi,\mathcal{B}})$.

Observe that the matrix $A_{\Phi,\mathcal{B}}$ is what is called an even matrix, i.e., a matrix in $M_{n\times n}(\mathbb{Z})$ such that its diagonal entries lie in $2\mathbb{Z}$. This is because for any $x \in \Lambda$ we have $\phi(x, x) = \Phi(2x) - 2\Phi(x) = 2\Phi(x)$ and hence the diagonal entries of $A_{\Phi,\mathcal{B}}$ are

$$(A_{\Phi,\mathcal{B}})_{i,i} := \phi(\alpha_i, \alpha_i) = 2\Phi(\alpha_i) \in 2\mathbb{Z}.$$

Now, if $[x]_{\mathcal{B}} \in \mathbb{Z}^n$ denotes the vector of coordinates of an element $x \in \Lambda$ in the basis $\mathcal{B}$, then

$$\phi(x, y) = [x]_{\mathcal{B}}^t A_{\Phi,\mathcal{B}} [y]_{\mathcal{B}}.$$

Therefore, recalling that $\phi(x, x) = 2\Phi(x)$, we have

$$\Phi(x) = \frac{1}{2}[x]_{\mathcal{B}}^t A_{\Phi,\mathcal{B}} [x]_{\mathcal{B}}.$$

Thus, if $X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ we can associate a homogeneous quadratic polynomial

$$Q_{\Phi,\mathcal{B}}(x_1, \cdots, x_n) := \frac{1}{2} X^t A_{\Phi,\mathcal{B}} X = \frac{1}{2}[x_1, \ldots, x_n] A_{\Phi,\mathcal{B}} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \sum_{i=1}^{n} \frac{\phi(\alpha_i, \alpha_i)}{2} x_i^2 + \sum_{i<j} \phi(\alpha_i, \alpha_j) x_i x_j.$$

Moreover, in terms of this polynomial we have that if $[x]_{\mathcal{B}} = [x_1, \ldots, x_n]^t$, then

$$\Phi(x) = Q_{\Phi,\mathcal{B}}(x_1, \ldots, x_n). \tag{2.1}$$

The $\phi(\alpha_i, \alpha_j)$ are called the coefficients of the quadratic form.

A quadratic form Φ is said to be positive definite if the matrix $A_{\Phi,\mathcal{B}}$, thought of as a matrix in $M_{n\times n}(\mathbb{R})$, is positive definite.

Definition 3 If $\Phi_1 : \Lambda_1 \longrightarrow \mathbb{Z}$ and $\Phi_2 : \Lambda_2 \longrightarrow \mathbb{Z}$ are quadratic forms on $\mathbb{Z}$-modules Λ_1 and Λ_2, they are said to be isometric if there is a $\mathbb{Z}$-isomorphism $f : \Lambda_1 \longrightarrow \Lambda_2$ such that $\Phi_2(f(x)) = \Phi_1(x)$ for every $x \in \Lambda_1$. Moreover, the corresponding quadratic spaces (Λ_1, Φ_1) and (Λ_2, Φ_2) are said to be isomorphic if Φ_1 and Φ_2 are isometric and we write $(\Lambda_1, \Phi_1) \cong (\Lambda_2, \Phi_2)$ in this case.

2.1 Theta series

In this subsection we recall relevants facts about theta series. Again, like in the previous section, we do this in order to be explicit about what definitions and theorems we are using.

Let Λ be a free $\mathbb{Z}$-module of rank n and let $\Phi : \Lambda \longrightarrow \mathbb{Z}$ be a positive definite quadratic form on Λ. Then the theta series associated to Φ is the function

$$\theta_\Phi(z) := \sum_{\lambda \in \Lambda} q^{\Phi(\lambda)} = \sum_{\lambda \in \Lambda} e^{2\pi i \Phi(\lambda) z},$$

where as usual $q = e^{2\pi i z}$ for $z \in \mathbb{H}$. Requiring Φ to be positive definite guarantees that the series defining θ_Φ converges absolutely in $\mathbb{H}$ and moreover θ_Φ is a holomorphic function on $\mathbb{H}$ (see e.g. [14, chapter VI]). Now, choosing a $\mathbb{Z}$-basis $\mathcal{B}$ for Λ, equation (2.1) allows us to write

$$\theta_\Phi(z) = \sum_{x=(x_1,\ldots,x_n)\in\mathbb{Z}^n} q^{Q_{\Phi,\mathcal{B}}(x)} = \sum_{x=(x_1,\ldots,x_n)\in\mathbb{Z}^n} e^{2\pi i Q_{\Phi,\mathcal{B}}(x) z}.$$

In order to discuss the automorphy of the theta series (see Theorem 3 below), we need to introduce the notion of the level of a quadratic form.

To define the level of a quadratic form we first define the level of an even symmetric matrix. Thus, let $A \in M_{n\times n}(\mathbb{Z})$ be an even symmetric matrix (recall that even means that its entries are integers and that its diagonal entries lie in $2\mathbb{Z}$). Then, as is well known, we have the relation

$$A \operatorname{adj}(A) = \det(A) I_n,$$

where $\operatorname{adj}(A) \in M_{n\times n}(\mathbb{Z})$ is the adjugate matrix of A. If A is invertible, this relation implies that the product $\det(A)A^{-1} \in M_{n\times n}(\mathbb{Z})$ is also an even symmetric matrix (see e.g. the first lemma in Chapter VI of Ogg's book [14] and the subsequent discussion). Thus since this implies that $|\det(A)|A^{-1}$ is an even symmetric matrix in $M_{n\times n}(\mathbb{Z})$, we define the level of A to be the least positive integer N such that NA^{-1} is an even symmetric matrix in $M_{n\times n}(\mathbb{Z})$. Moreover, it can be seen that if $P \in \mathrm{GL}_n(\mathbb{Z})$ is an invertible matrix, then P^tAP also has level N. This allows us to define unambiguously the level of a quadratic form as follows.

Definition 4 The level of a quadratic form $\Phi : \Lambda \longrightarrow \mathbb{Z}$ is defined to be the level of any associated matrix $A_{\Phi,\mathcal{B}}$ corresponding to a $\mathbb{Z}$-basis $\mathcal{B}$ of Λ.

Now we recall the definition of the space of modular forms that we will consider, following [9] and [11]. First, we will require the quadratic residue symbol $\left(\frac{a}{b}\right)$, defined for $a, b \in \mathbb{Z}$ by the following properties.

1. If p is an odd prime, then $\left(\frac{a}{p}\right)$ is just the Legendre symbol.
2. If $a \in \mathbb{Z}$ is odd, then $\left(\frac{a}{2}\right) = (-1)^{(a^2-1)/8}$.
3. We have $\left(\frac{a}{-1}\right) = 1$ if $a \geq 0$ and $\left(\frac{a}{-1}\right) = -1$ if $a < 0$.
4. $\left(\frac{a}{b}\right) = 0$ if $\gcd(a, b) > 1$.
5. $\left(\frac{1}{0}\right) = 1$ and $\left(\frac{a}{0}\right) = 0$ for $a \neq 1$.
6. $\left(\frac{a}{bc}\right) = \left(\frac{a}{b}\right) \cdot \left(\frac{a}{c}\right)$ for every $a, b, c \in \mathbb{Z}$.

Second, if $d \in \mathbb{Z} \setminus \{0\}$, there are integers d_f and d_s such that we can write d uniquely as $d = d_f d_s^2$ with d_f square-free. Then we put

$$D_d := \begin{cases} d_f & \text{if } d_f \equiv 1 \pmod 4 \\ 4d_f & \text{if } d_f \equiv 2, 3 \pmod 4. \end{cases}$$

Note in particular that if d is not a perfect square, then D_d is just the discriminant of the quadratic number field $\mathbb{Q}(\sqrt{d})$. With this notation in place, we define the quadratic Dirichlet character $\chi_d(n) := \left(\frac{D_d}{n}\right)$. This character has conductor $|D_d|$.

Definition 5 Let $n \in \mathbb{Z}$ and $N \in \mathbb{Z}_{\geq 1}$. Moreover, assume that $4|N$ if n is odd. Let χ be a Dirichlet character modulo N. Then a holomorphic function $f : \mathbb{H} \longrightarrow \mathbb{C}$ is called a modular form of weight $n/2$, level N and character χ, if it is holomorphic at every cusp of $\Gamma_0(N)$ and if for every $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$ it transforms as

$$f(\gamma \cdot z) = \begin{cases} \chi(d)(cz+d)^{n/2} f(z) & \text{if } n \text{ is even,} \\ \chi(d)\chi_c(d)^n \varepsilon_d^{-n} (cz+d)^{n/2} f(z) & \text{if } n \text{ is odd,} \end{cases}$$

where

$$\varepsilon_d := \begin{cases} 1 & \text{if } d \equiv 1 \pmod 4, \\ i & \text{if } d \equiv 3 \pmod 4). \end{cases}$$

The $\mathbb{C}$-vector space of all such forms is denoted by $M_{n/2}(\Gamma_0(N), \chi)$.

The following result is a special case of a more general result of Shimura [17, Proposition 2.1] and it is stated in an equivalent way in [11, p. 400].

Theorem 3 *Let $\Phi : \Lambda \longrightarrow \mathbb{Z}$ be a quadratic form on a free $\mathbb{Z}$-module Λ of rank m. Let N_Φ be the level of the quadratic form Φ and let $A = A_{\Phi,\mathcal{B}}$ be the matrix of Φ with respect to some $\mathbb{Z}$-basis $\mathcal{B}$ of Λ. Define the integer d_Φ by*

$$d_\Phi := \begin{cases} \det(A) & \text{if } m \equiv 0 \pmod 4 \\ -\det(A) & \text{if } m \equiv 2 \pmod 4 \\ \dfrac{\det(A)}{2} & \text{if } m \equiv 1, 3 \pmod 4. \end{cases}$$

Then the theta function

$$\theta_\Phi(z) := \sum_{\lambda \in \Lambda} q^{\Phi(\lambda)} = \sum_{\lambda \in \Lambda} e^{2\pi i \Phi(\lambda) z}$$

is a modular form in the space $M_{\frac{m}{2}}(\Gamma_0(N_\Phi), \chi_{d_\Phi})$.

3 Trace zero forms on totally real number fields

Let K be a totally number field of degree n and let $\mathcal{O}_K$ be its ring of integers. Then the trace zero submodule of $\mathcal{O}_K$, defined by

$$\Lambda_K = \mathcal{O}_K^0 := \ker\left(\mathrm{Tr}_{K/\mathbb{Q}}\right) = \{\alpha \in \mathcal{O}_K \mid \mathrm{Tr}_{K/\mathbb{Q}}(\alpha) = 0\},$$

is a free $\mathbb{Z}$-module of rank $n-1$. This is because $\mathrm{Tr}_{K/\mathbb{Q}} : \mathcal{O}_K \longrightarrow \mathbb{Z}$ is a non-zero homomorphism and hence its image has rank 1.

The following proposition gives us a quadratic form constructed from the trace. We include a proof for the convenience of the reader.

Proposition 1 *The function $\Phi_K : \Lambda_K \longrightarrow \mathbb{Z}$ given by $\Phi_K(\alpha) := \frac{1}{2}\mathrm{Tr}_{K/\mathbb{Q}}(\alpha^2)$ is a positive definite quadratic form of rank $n-1$ and its associated bilinear function $\phi_K : \Lambda_K \times \Lambda_K \longrightarrow \mathbb{Z}$ satisfies $\phi_K(\alpha,\beta) = \mathrm{Tr}_{K/\mathbb{Q}}(\alpha\beta)$.*

Proof First we need to show that Φ_K takes values in $\mathbb{Z}$. Let $\alpha \in \Lambda_K$. Since $\Phi_K(\alpha)$ is rational it is enough to show that $\Phi_K(\alpha)$ is an algebraic integer. Now, let the embeddings $K \hookrightarrow \mathbb{R}$ be $\sigma_1, \ldots, \sigma_n$. Then for any $\alpha \in \Lambda_K$ we have

$$\begin{aligned}
\Phi_K(\alpha) &= \frac{1}{2}\mathrm{Tr}_{K/\mathbb{Q}}(\alpha^2) = \frac{1}{2}\sum_{i=1}^n \sigma_i(\alpha^2) = \frac{1}{2}\sum_{i=1}^n \sigma_i(\alpha)^2 \\
&= \frac{1}{2}\left(\left(\sum_{i=1}^n \sigma_i(\alpha)\right)^2 - 2\sum_{i<j}\sigma_i(\alpha)\sigma_j(\alpha)\right) \\
&= \frac{1}{2}\left(\mathrm{Tr}_{K/\mathbb{Q}}(\alpha)^2 - 2\sum_{i<j}\sigma_i(\alpha)\sigma_j(\alpha)\right) \\
&= -\sum_{i<j}\sigma_i(\alpha)\sigma_j(\alpha),
\end{aligned}$$

where we used that $\mathrm{Tr}_{K/\mathbb{Q}}(\alpha) = 0$. Since α is an algebraic integer, so are its conjugates $\sigma_k(\alpha)$ for all k, and in particular that implies that $\Phi_K(\alpha) = -\sum_{i<j}\sigma_i(\alpha)\sigma_j(\alpha)$ is an algebraic integer as we wanted to show.

Now, to see that Φ_K is a quadratic form, observe that the linearity of the trace implies that for any $m \in \mathbb{Z}$ and $\alpha \in \Lambda_K$ we have

$$\Phi_K(m\alpha) = \frac{1}{2}\mathrm{Tr}_{K/\mathbb{Q}}(m^2\alpha^2) = \frac{m^2}{2}\mathrm{Tr}_{K/\mathbb{Q}}(\alpha^2) = m^2\Phi_K(\alpha).$$

Also, the associated function $\phi_K : \Lambda_K \times \Lambda_K \longrightarrow \mathbb{Z}$, satisfies

$$\begin{aligned}
\phi_K(\alpha,\beta) = \Phi_K(\alpha+\beta) - \Phi(\alpha) - \Phi(\beta) &= \frac{1}{2}\left(\mathrm{Tr}_{K/\mathbb{Q}}((\alpha+\beta)^2) - \mathrm{Tr}_{K/\mathbb{Q}}(\alpha^2) - \mathrm{Tr}_{K/\mathbb{Q}}(\beta^2)\right) \\
&= \frac{1}{2}\left(\mathrm{Tr}_{K/\mathbb{Q}}(\alpha^2 + 2\alpha\beta + \beta^2) - \mathrm{Tr}_{K/\mathbb{Q}}(\alpha^2) - \mathrm{Tr}_{K/\mathbb{Q}}(\beta^2)\right) = \mathrm{Tr}_{K/\mathbb{Q}}(\alpha\beta).
\end{aligned}$$

Thus, since $\phi_K(\alpha,\beta) = \mathrm{Tr}_{K/\mathbb{Q}}(\alpha\beta)$, an easy calculation shows that this map is bilinear. Since K is totally real the values of Φ_K are sum of squares of real numbers, hence ϕ_K is a positive definite bilinear pairing.

Lemma 1 *Let K be a degree n number field of discriminant d. Suppose that m is a positive integer such that $\mathrm{Tr}_{K/\mathbb{Q}}(O_K) = m\mathbb{Z}$. Then the discriminant of the quadratic form Φ_K is equal to $(-1)^{\frac{(n-1)(n-2)}{2}}\dfrac{dn}{m^2}$. In particular, if d is not divisible by the nth power of any integer bigger than 1, then the discriminant of Φ_K is equal to $(-1)^{\frac{(n-1)(n-2)}{2}}dn$.*

Proof Since O_K^0 has $\mathbb{Z}$-rank equal to $n-1$ it is enough to show that the determinant d_0 of a Gram matrix of the bilinear pairing $\mathrm{Tr}_{K/\mathbb{Q}}(\cdot,\cdot)$ in any basis of O_K^0 is equal to $\frac{dn}{m^2}$. Since $\mathbb{Z}$ and O_K^0 are orthogonal with respect to the trace pairing, the determinant of the trace pairing over $\mathbb{Z} + O_K^0$ is equal to nd_0. On the other hand, since $\mathbb{Z} + O_K^0$ is a subgroup of O_K of full rank, the determinant is equal to $[O_K : \mathbb{Z} + O_K^0]^2 d$ and thus

$$d_0 = [O_K : \mathbb{Z} + O_K^0]^2 d/n.$$

Since $\mathbb{Z}+O_K^0$ is a subgroup of O_K that contains the Kernel of the trace map the group $O_K/(\mathbb{Z}+O_K^0)$ is isomorphic to $m\mathbb{Z}/n\mathbb{Z}$, in particular $[O_K:\mathbb{Z}+O_K^0]=\frac{n}{m}$ from which $d_0=\frac{nd}{m^2}$ follows. The final remark is due to the fact that m^n divides d.

Since the level of an integral quadratic form of discriminant Δ divides 2Δ we have that:

Corollary 1 *Let K be a degree n number field of discriminant d. Then, the level of the quadratic form Φ_K divides $2nd$.*

Remark 3 In the case of $n=3$ and d is fundamental it is not hard to observe, see [13], that the level is actually nd. This is not necessarily the case in higher degrees, and hence the best we can say is that the level divides $2nd$.

Now we prove Theorem 1. Recall that K is a totally real number field of degree n with discriminant d_K. Moreover, d_K is a fundamental discriminant and $\gcd(n,d_K)=1$.

Proof It follows from Lemma 1 that the form Φ_K has determinant nd_K, and by Corollary 1, its level is a divisor of $2nd_K$. By definition, the theta series associated to the form Φ_K is given by

$$\theta_K(z):=\sum_{\alpha\in\mathcal{O}_K^0} e^{\pi i\,\mathrm{Tr}_{K/\mathbb{Q}}(\alpha^2)z}.$$

Since Φ_K has rank $n-1$, thanks to Proposition 1, determinant nd_K and level dividing $2nd_K$, it follows from Theorem 3 that

$$\theta_K\in\mathcal{M}_{n,d_K}:=M_{\frac{n-1}{2}}\left(\Gamma_0(2nd_K),\left(\frac{\delta_n nd_K}{\cdot}\right)\right),$$

where $\delta_n=\begin{cases}(-1)^{\frac{n-1}{2}} & \text{if } n \text{ is odd,}\\ \frac{1}{2} & \text{if } n \text{ is even.}\end{cases}$

4 The case of number fields of small degree

As was shown in section 3, the quadratic form $\Phi_K=\frac{1}{2}\mathrm{Tr}_{K/\mathbb{Q}}$ on the trace zero module $\Lambda_K=\mathcal{O}_K^0$ has rank $m=n-1$. Since for any isomorphism $\sigma:K\to L$ between number fields, and for all $\alpha\in K$, we have that $\mathrm{Tr}_{K/\mathbb{Q}}(\alpha)=\mathrm{Tr}_{L/\mathbb{Q}}(\sigma(\alpha))$, the implication $K\cong L \implies \theta_K=\theta_L$ is trivial. We prove the reverse implication analyzing the cases $n=1,2,3,4$ individually.

4.1 $n=1$

There is nothing to prove here.

4.2 $n=2$

If K is the real quadratic field $\mathbb{Q}(\sqrt{d})$, where $d>1$ is a square free integer, then a calculation shows that $\mathcal{O}_K^0=\mathbb{Z}\cdot\sqrt{d}$ and therefore $\theta_K(z)=\vartheta_d(z)=1+2\sum_{n=1}^{\infty}e^{\pi i d n^2 z}$. Taking $z=i$, the result follows from the injectivity of the function $f(x)=\sum_{n=1}^{\infty}e^{-\pi x n^2}$ for $x>0$.

4.3 $n=3$

This was proved by the second author in [13, Theorem 3.7].

4.4 $n = 4$

Given a positive definite quadratic form $Q(x_1, \ldots, x_m) \in \mathbb{R}[x_1, \ldots, x_m]$ and a positive real number $t \in \mathbb{R}_{>0}$, we define the representation number $A(Q, t)$ by

$$A(Q,t) := \#\{x \in \mathbb{Z}^m \mid Q(x) = t\}.$$

Now, for the quadratic forms Φ_K and Φ_L, choose bases $\mathcal{B}_K$ and $\mathcal{B}_L$ and let $A_{\Phi_K,\mathcal{B}_K}$ and $A_{\Phi_L,\mathcal{B}_L}$ be the corresponding matrices in $M_{3\times 3}(\mathbb{Z})$. Moreover, let $Q_{K,\mathcal{B}_K}$ and $Q_{L,\mathcal{B}_L}$ be the corresponding quadratic polynomials as in Section 2. Then we have

$$\theta_K(z) = \sum_{\alpha \in \mathcal{O}_K^0} q^{\Phi_K(\alpha)} = \sum_{x \in \mathbb{Z}^3} q^{Q_{K,\mathcal{B}_K}(x)} = 1 + \sum_{n=1}^{\infty} A(Q_{K,\mathcal{B}_K}, n) q^n$$

and similarly for θ_L. Therefore, since $\theta_K = \theta_L$, the corresponding representation numbers are equal, i.e., we have

$$A(Q_{K,\mathcal{B}_K}, n) = A(Q_{L,\mathcal{B}_L}, n)$$

for every $n \geq 1$. Then since the forms $Q_{K,\mathcal{B}_K}$ and $Q_{L,\mathcal{B}_L}$ are positive definite and ternary, a result of Schiemann [16, Theorem 4.4] implies that there is a matrix $U \in \mathrm{GL}_3(\mathbb{Z})$ such that

$$Q_{L,\mathcal{B}_L}(Ux) = Q_{K,\mathcal{B}_K}(x). \tag{4.1}$$

Letting a basis $\mathcal{B}_L = \{\beta_1, \beta_2, \beta_3\}$ be given, we define the isomorphism of $\mathbb{Z}$-modules $T : \mathcal{O}_K^0 \longrightarrow \mathcal{O}_L^0$ by $T(\alpha) := [\beta_1, \beta_2, \beta_3] U [\alpha]_{\mathcal{B}_K}$, i.e., the coordinates of $T(\alpha)$ in the basis $\mathcal{B}_L$ are given by $[T(\alpha)]_{\mathcal{B}_L} = U[\alpha]_{\mathcal{B}_K}$ for every $\alpha \in \mathcal{O}_K^0$. Then, by equations (4.1) and (2.1), we see that $\Phi_L(T(\alpha)) = \Phi_K(\alpha)$ for every $\alpha \in \mathcal{O}_K^0$ and hence the quadratic forms Φ_K and Φ_L are isometric. Thus we have an isomorphism of quadratic spaces

$$(\mathcal{O}_K^0, \Phi_K) \cong (\mathcal{O}_L^0, \Phi_L).$$

Then a recent theorem of the second author and Rivera-Guaca [15, Theorem 2.12] implies that $K \cong L$, which completes the proof of the theorem.

5 Computations and heuristics for quartic number fields

Suppose we fix the discriminant $\mathrm{disc}(K)$ of the field K and ask a finer question about the theta series associated to the number fields of discriminant $\mathrm{disc}(K)$. In particular, it was proved in [13] that for cubic number fields these forms were linearly independent. We ask the same question here.

5.1 Computational evidence

The evidence we present below was computed in Sage [19]. The data related to the number fields was downloaded from the LMFDB [18] and from the database described in [8], which is due to Klüners and Malle. The LMFDB contains totally real quartic number fields up to discriminant 10^7 and the Klüners–Malle database contains totally real quartic number fields up to discriminant 10^{10}. We used both data sets to compare and verify results.

We restrict to those fields that satisfy our conditions (totally real, Galois group equal to S_4 [10]). We further restrict to those discriminants for which there is more than one field of that discriminant. This leave us with 1301494 fields to consider. It was shown computationally that for all the discriminants, the resulting theta series were linearly independent. The code and examples can be found at [1].

5.2 Two theta series represent the same prime

In [13] it was shown that if two integral binary quadratic forms of the same discriminant both represented the same prime p, then the two forms are equivalent. This was a key step in proving the linear independence of the set of quadratic forms associated to number fields of a fixed discriminant. We attempted to follow the same approach. But, as shown in Table 1 there are three quartic fields of discriminant 35537 satisfying our conditions but two of them represent the same prime.

Polynomial f	Quadratic form coefficients	Theta series θ_K
$x^4-2x^3-9x^2+5x+16$	$\begin{pmatrix} 95506 & 93618 & 261632 \\ * & 22943 & 128229 \\ * & * & 179181 \end{pmatrix}$	$1+2q^{23}+2q^{27}+O(q^{30})$
$x^4-x^3-8x^2-3x+4$	$\begin{pmatrix} 321 & 1038 & -505 \\ * & 851 & -861 \\ * & * & 245 \end{pmatrix}$	$1+2q^{21}+4q^{23}+O(q^{30})$
$x^4-2x^3-5x^2+5x+4$	$\begin{pmatrix} 527 & 3957 & 7078 \\ * & 7439 & 26613 \\ * & * & 23805 \end{pmatrix}$	$1+2q^{11}+2q^{26}+O(q^{30})$

Table 1 The field is $K=\mathbb{Q}[x]/\langle f(x)\rangle$, where $f(x)$ is the polynomial given in the first column. The symmetric matrix $A=(a_{ij})$ that is displayed in the second column contains the coefficients of $\Phi_K(x_1,x_2,x_3)=\sum_{i=1}^{3} a_{ii}x_i^2+2\sum_{j>i} a_{ij}x_ix_j$. Finally, the third column displays the first terms of the q-expansion of the theta series θ_K.

By computing the θ-series of the quadratic forms associated to K_1 and K_2 in Table 1 we see they both represent 23, rendering the approach in [13] not viable. This led us to consider a second approach we describe next.

5.3 Two theta series with the same minimum

If we knew that a set of quadratic forms of the same discriminant had different positive minima, then it would be easy to show that the set of forms were linearly independent. After the approach in [13] described in Section 5.2 was seen not to be viable, we considered this approach. As the following example shows, this approach was not viable either.

We remark that the notation is the same as in Table 1. We consider the two non-isomorphic quartic number fields with discriminant 4024049. Let K_1 have polynomial $x^4-x^3-37x^2+46x+20$ and K_2 have polynomial $x^4-2x^3-41x^2-49x+95$. The quadratic form associated to K_1 has coefficients

$$\begin{pmatrix} 10684425 & 71591860 & 106295143 \\ * & 119926773 & 356119635 \\ * & * & 264372149 \end{pmatrix}$$

and the quadratic form associated to K_2 has coefficients

$$\begin{pmatrix} 153730865 & 332210617 & 357281618 \\ * & 179475823 & 386040717 \\ * & * & 207587063 \end{pmatrix}.$$

If we proceed to compute the θ-series of the two quadratic forms we see that they both represent 43 and 43 is the smallest positive integer they both represent. The corresponding theta series have expansions

$$\theta_{K_1}(q)=1+2q^{43}+2q^{172}+O(q^{200}),$$
$$\theta_{K_2}(q)=1+2q^{43}+2q^{170}+2q^{172}+2q^{187}+O(q^{200}).$$

We had hoped that we could attempt to prove that theta series corresponding to fields of the same discriminant were linearly independent by showing that such forms had different positive minima, but as this example shows, this approach cannot yield a proof.

5.4 Successive minima

In [16] a condition for the equivalence of two ternary forms f and g is given in terms of representation numbers and successive minima. In particular, it is shown that if f and g have the same representation numbers up to some bound in terms of their successive minima, then the two forms are equivalent. It is possible that one could use these results to prove the linear independence of the theta series associated to number fields of a fixed discriminant. We leave that for future work.

5.5 Heuristic evidence

In order to have a chance for the set of θ-series to be linearly independent, we need that the number of quartic fields K of discriminant d_K that satisfy our conditions be less than the dimension of the corresponding space of modular forms of weight $\frac{3}{2}$. To that end we present an informal argument for why there are few θ-series of quartic number fields relative to the dimension of the corresponding space of weight $\frac{3}{2}$ modular forms.

Bounds for dimensions of spaces of weight $\frac{3}{2}$ modular forms

We want to calculate

$$\dim M_{\frac{3}{2}}(\Gamma_0(8d_K), \left(\tfrac{2d_K}{\cdot}\right)).$$

There are formulas in [5] that give the dimension of the space of forms that are of level $8d_K$. It suffices for our purposes to use those formulas. We recall them here.

If $p \mid 2d_K$, we define r_p to be the exponent of p in the prime factorization of $2d_k$ and we define s_p to be the exponent of p in the prime factorization of the conductor of $\left(\frac{2d_K}{\cdot}\right)$ (since the character is quadratic, we know that its conductor is $2d_K$). Recall that since d_K is odd, we know $2d_K$ is square-free and observe that $r_p = s_p = 1$. Now

$$\lambda(r_p, s_p, p) = \begin{cases} p^{r'} + p^{r'-1} & \text{if } 2s_p \le r_p = 2r' \\ 2p^{r'} & \text{if } 2s_p \le r_p = 2r' + 1 \\ 2p^{r_p - s_p} & \text{if } 2s_p > r_p. \end{cases}$$

In [5] it is shown that for $k \in \frac{1}{2} + \mathbb{Z}$, we know

$$\dim S_k(\Gamma_0(N), \chi) - \dim M_{2-k}(\Gamma_0(N), \chi) = \frac{k-1}{12} N \prod_{p \mid N} \left(1 + \tfrac{1}{p}\right) - \frac{\zeta}{2} \prod_{2 \ne p \mid N} \lambda(r_p, s_p, p).$$

Since $N = 8d_k$ and $r_2 = 3$, from a table in [5] we have $\zeta = 3$. Since $2s_p > r_p$ and $s_p = r_p = 1$ we get that $\lambda(r_p, s_p, p) = 2$. With this notation we have

$$\begin{aligned} \dim M_{\frac{3}{2}}(\Gamma_0(8d_K), \left(\tfrac{2d_K}{\cdot}\right)) &\ge \dim S_{\frac{3}{2}}(\Gamma_0(8d_K), \left(\tfrac{2d_K}{\cdot}\right)) \\ &= \dim M_{\frac{1}{2}}(\Gamma_0(8d_K), \left(\tfrac{2d_K}{\cdot}\right)) + \frac{d_K}{3} \prod_{p \mid 8d_K} \left(1 + \tfrac{1}{p}\right) - \frac{3}{2} \prod_{2 \ne p \mid N} 2 \\ &> \frac{d_K}{3} \prod_{p \mid 8d_K} \left(1 + \tfrac{1}{p}\right) - \frac{3}{2} \prod_{p \mid d_K} 2 \end{aligned}$$

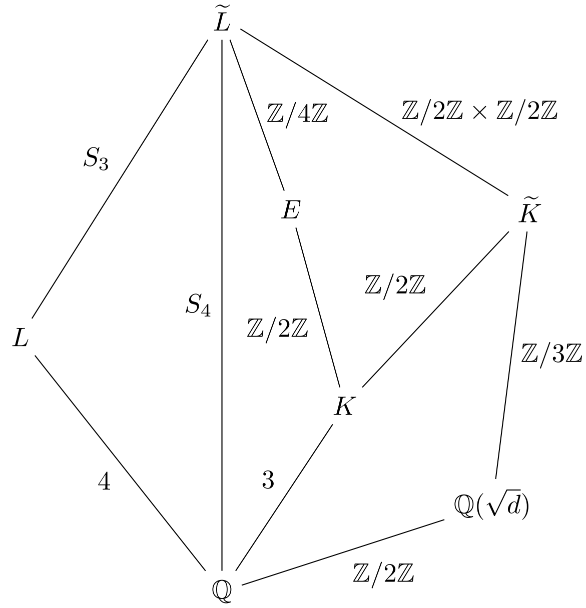
Since the average order of $\prod_{2 \ne p \mid N} 2 = 2^{\omega(d_K)-1}$ is $\log\log d_k$, we conclude that $\dim M_{\frac{3}{2}}(\Gamma_0(8d_K), \left(\frac{2d_K}{\cdot}\right))$ is roughly d_K. As we see next, this is far greater than the number of θ-series we construct.

The number of totally real quartic number fields

Let d be a positive square free discriminant and let $\mathcal{Q}_d$ be the set of isomorphism classes of totally real quartic fields of discriminant d. Let $\mathcal{F}_d := \{\theta_K : K \in \mathcal{Q}_d\}$. Now we estimate an upper bound of order $d^{0.62}$ for the number of elements in $\mathcal{F}_d$. In particular, $\#\mathcal{F}_d$ is much smaller than $O(d)$ which is a lower bound for the dimension of the space of modular forms in which the set $\mathcal{F}_d$ lives. Based on this, the computational evidence presented at the beginning of this section, and the analogous result for cubic fields (see [13]) we expect that the set $\mathcal{F}_d$ is linearly independent. More precisely:

Conjecture 1 Let d be a positive square free discriminant and let $\mathcal{Q}_d$ be the set of isomorphism classes of totally real quartic fields of discriminant d. Let $\mathcal{F}_d := \{\theta_K : K \in \mathcal{Q}_d\}$ where θ_K is the form defined in Theorem 1. Then, the set $\mathcal{F}_d$ is lineally independent.

For quartic fields of fundamental discriminant there is a classical duality between quartic fields and 2-torsion elements of their cubic resolvents. More precisely, if K is the cubic resolvent field of L then the field L corresponds to an index 2 subgroup of $Cl(K)$. In a similar fashion, the field K corresponds to an index 3 subgroup of its quadratic resolvent. Here we recall briefly how such correspondences occur; see [2] and [3, §3.1]. Let L be a quartic field of fundamental discriminant d. By a result of [10], if $\widetilde{L}$ is the Galois closure of L then $S_4 \cong \mathrm{Gal}(\widetilde{L}/\mathbb{Q})$. Hence, up to conjugation, $\widetilde{L}$ contains a unique cubic field K, called the resolvent of L. The Galois group $\mathrm{Gal}(\widetilde{L}/K)$ is isomorphic to the dihedral group of order 8; let E be the quadratic extension of K that corresponds, via Galois, to the unique cyclic subgroup of order 4. As it turns out, the extension E/K is unramified and thus, by class field theory, this corresponds to a index 2 subgroup of $Cl(K)$. Since d is square free this is also the discriminant of K, hence if $\widetilde{K}$ denotes the Galois closure of K then $S_3 \cong \mathrm{Gal}(\widetilde{K}/\mathbb{Q})$. Furthermore, the unique quadratic field inside $\widetilde{K}$, the quadratic resolvent of $\widetilde{K}$ is the quadratic field of discriminant d, i.e., $\mathbb{Q}(\sqrt{d})$. The Galois extension $\widetilde{K}/\mathbb{Q}(\sqrt{d})$ is unramified and hence it corresponds to an index 3 subgroup of $Cl(\mathbb{Q}(\sqrt{d}))$.



The above analysis is useful in bounding the number of quartic fields of fundamental discriminant equal to d. Given a finite abelian group G and a prime p it is not hard to see that the number of index p subgroups of G is given by

$$\frac{p^{\mathrm{rank}_p(G)} - 1}{p - 1}$$

where $\mathrm{rank_p}(G) = \dim_{\mathbb{F}_p}(G \otimes \mathbb{F}_p)$

Proposition 2 *Let d be a fundamental discriminant and let $n_4(d)$ be the number of isomorphism classes of quartic fields with discriminant d. Then, $n_4(d) \ll d^{0.62}$.*

Proof Let T_d be the number of cubic fields, up to isomorphism, of discriminant equal to d and let $\{K_1, ..., K_{T_d}\}$ be a set of representatives of such fields. The above analysis yields

$$n_4(d) = \sum_{i=1}^{T_d} \left(2^{\mathrm{rank}_2(Cl(K_i))} - 1\right) < \sum_{i=1}^{T_d} \left(2^{\mathrm{rank}_2(Cl(K_i))}\right) = \sum_{i=1}^{T_d} Cl(K_i)[2] \ll d^{0.28} T_d$$

where the last inequality is due to [4]. By the analysis on cubics above

$$T_d = \frac{3^{\mathrm{rank}_3(Cl(\mathbb{Q}(\sqrt{d})))} - 1}{2} = (Cl(\mathbb{Q}(\sqrt{d})[3] - 1)/2 \ll d^{1/3}$$

where the last inequality is due to [6]. The result follows since $0.28 + 1/3 < 0.62$.

As an immediate corollary of Theorem 2 we have:

Corollary 2 *Keeping with the notation of this section we have that* $\#\mathcal{F}_d \ll d^{0.62}$.

5.6 Higher degree fields

For all the totally real number fields of degrees 5, 6, 7 in the Klüners–Malle database [8], the set of associated theta series is linearly independent (in degree 5 there were 5870 fields where the number of associated theta series was greater than 1; in degree 6 there were 236 fields where the number of associated theta series was greater than 1; in degree 7 there were 16 fields where the associated number of associated theta series was greater than 1).

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