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DYNAMIC ANALYSIS OF MODIFIED STRUCTURAL SYSTEMS

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<p>A set of methods is developed to determine the changes in frequencies and mode shapes of a structure resulting from modifications of the structure. These modifications can involve an increase in the number of degrees of freedom of the system as well as changes in already existing entries in the mass and stiffness matrices of the model of the structure. Indeed, the modification may consist entirely of a refinement of the structural model.</p> <p>In the procedures used, each mode and frequency is treated separately without any need to know the other modes and frequencies. An exception is the case of multiple or close frequencies, where the whole set of equal or close eigenvalues and their eigenvectors must be treated together. Calculations are carried out by a perturbation analysis, which can be carried as far as desired in an automatic fashion. Repeated use of the same coefficient matrix in the perturbation scheme leads to considerable economy of computational effort.</p> <p>Light, tuned systems attached to a structure can be treated advantageously by the procedure developed, as shown in examples.</p>			
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ABSTRACT

The subject of this study is the development of a set of methods for the calculation of the changes in the frequency of vibration and mode shape of a structure due to changes in the physical system. Such changes may consist of variations of the structural properties of a given system or even of the addition of new degrees of freedom.

One family of iterative methods that has been used effectively for improvement of modes and frequencies is based on an application of the Newton-Raphson technique to the equations defining modes and frequencies. The method is applicable to cases with multiple or close frequencies as well as to cases where they are widely separated. However, in their original statements, these methods are not capable of handling additional degrees of freedom arising from changes in the structural system. The current work extends these techniques to the cases of added degrees of freedom and, moreover, allows for considerable reduction in computational effort even when there are no additional degrees of freedom.

The original vibrating system is replaced by an augmented system, which may have more degrees of freedom than the original one. The augmentation is described by mass and stiffness matrices, and the original matrix expression of the system may have to be rewritten with suitable rows and columns of zeros to make the two sets of matrices conformable for addition. The augmentation mass and stiffness matrices are taken proportional to a single parameter which can vary from just

larger than zero to sizable positive values. A perturbation analysis is then performed on this augmented system.

The iterative character of the basic method for improving frequencies and modes is modified so that each step now corresponds to a calculation of the next order of perturbation. This modification eliminates the difficulty of dealing with very light additional masses and results in successive solutions of linear matrix equations all having the same coefficient matrix. Unlike in the original method, which requires a new triangularization at each step, the expensive repeated triangularizations are not needed in the perturbation scheme.

One important application of the perturbation method is to determine the modes and frequencies of a structure to which a light secondary system is added - say the piping in a power plant. In situations where a frequency of the added system is close to one or more frequencies of the original system, the so-called tuned case, the method is quite effective and general.

The additional system may also correspond to a large value of the parameter multiplying the augmentation matrices. Here the difficulties in adding degrees of freedom and separating initially close frequencies are resolved by the perturbation technique, and the larger values of the parameter are handled by applying the scheme to reduce the number of steps in the original iterative methods.

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1. INTRODUCTION

1.1 General Remarks

In structural engineering it is often desirable to be able to predict the consequences of modifying a structural system. Since dynamic analysis under earthquake, blast and wind loads often requires knowledge of the modes and frequencies of the structure, it is important to know how these dynamic properties of the structure are affected by modifications in the system. Such modifications may be the result of the inevitable difference between the analytical model and the system as actually constructed. Non-structural components may also contribute to such deviation from the analytical predictions of behavior.

Overall, the modifications that can be introduced to a given structural system may be viewed as consisting basically of two types. In the first instance, the modifications consist of simple variations in the magnitudes of the parameters of the system. In order to visualize this case, imagine that the structural system is discrete. The shear beam model of a rigid frame would represent a typical discrete system. If the masses i and j in Figure 1.1 are connected by an additional elastic connection then this will induce what may be called a local modification in the system. This could be the case of a frame with very rigid girders where some sort of connection with elastic properties is installed between two floors. Another possibility is that of rigidly attaching a new mass to the system. This could consist of a piece of equipment installed in the frame mentioned above. Thus the effect in the model is

that of an increase μ in the j th mass, as it is shown in Figure 1.2. This type of modifications may even be introduced in a system with the express purpose of changing the frequencies of the original unmodified structural system to avoid a resonance with an applied load.

The second kind of modification includes the attachment of spring-mass combinations to the structural system. This means that new coordinates are needed to describe the motion of these added masses. Furthermore, any of the added masses may have more than one point of attachment. Typical examples of this kind of modification are, the addition of piping systems into buildings as is the case in the coolant circulation systems of nuclear power plants, the sprinkler systems in vital buildings, ventilation systems, flexibly mounted engines, etc.

The prediction of the response of modified structural systems to the dynamic loadings mentioned above has attracted much attention in recent years. For the case when the modification consists of the addition of light attachments to the initial structural system, the problem has been approached as one of primary - secondary system. Under most circumstances, the response of the additional subsystem, if it is light, can be estimated accurately by imagining that the original system is not affected by the modification, or, in other words, that the old system and its new addition are decoupled. There are cases, however, referred to as tuned systems, where attempt of such decoupling leads to considerable errors. In these cases, the analysis is not as simple as in the detuned case.

The method proposed in this study permits accurate calculation of the modes and frequencies of a modified system even in the tuned case.

The method is also applicable to fairly heavy subsystems or to significant modifications of the original structure.

1.2 Object and Scope

The object of this study is the development of a set of computationally effective methods for the calculation of the changes in the frequencies of vibration and mode shapes of a structure due to changes in the physical system. Such changes may consist of variations of the structural properties of a given system or of the addition of new degrees of freedom.

In this work, only undamped linear systems are considered. Even for small damping, the consideration of damping in the combined structural systems can lead to non-classical modes.

As pointed out in Section 1.3, there are existing methods for finding and combining modal responses, for separated and close frequencies. The analysis in Chapters 2 and 3 and numerical results in Chapter 4 do not treat dynamic response problems, but concentrate on the problem of determining modes and frequencies.

Additional systems can consist of any configuration of additional masses and elastic elements attached to the main structure in any way. The magnitude of the elastic additions are characterized by a simple parameter. Extremely light additional systems corresponds to very small values of the parameter and heavier ones to larger values of the parameter.

From a mathematical standpoint, the problem is one of determining the eigenvalues and eigenvectors of a perturbed system of the form

$$K X = \lambda M X$$

where the modification can even involve new rows and columns of K and M .

1.3 Background

The problem of analyzing the dynamic response of modified structural systems has received considerable attention from researchers in several areas of the engineering profession. A brief review of some of the available literature is presented in this section.

The study of published work related to the topic of this investigation may be divided into two categories. The first one may be called the engineering approach and the second one the mathematical approach. The intention of the present study is to attempt to bridge this two approaches in order to solve the proposed problem.

In the engineering community the solution of modified structural systems has been addressed in many instances as the study of the dynamic behavior of the so-called primary-secondary systems. Several methods of solution have been proposed for these systems. They all pertain to the intended application: the response to blast and earthquake loads. Accordingly, the proposed solutions of the problem have been tailored to fit to the currently used approach in the respective field. That is, the solutions are based on the availability of the usual parameters for dynamic loadings: a description of the ground motion either as a time history of the displacements and accelerations, a ground response spectrum, or a shock response spectrum, together with some knowledge of the fixed base modal properties of the primary and secondary structures.

It is possible to categorize the different approaches to the dynamic response of primary-secondary systems in several ways. One of them would be to look at the methods employed for the solution of the combined system. Here, the approaches range from closed form solutions for the dynamic behavior of the combined system (usually by transforming to the frequency domain) to perturbation analyses of the resulting eigenvalue problem. Another is by the modeling of the combined system as either a continuous primary system with a discrete secondary system attached to it, or as a discrete combined system with all the different possibilities that it yields: SDOF-SDOF, MDOF-SDOF, MDOF-MDOF, and including either classical or nonclassical damping as well as the possibility of tuning.

One of the first solutions of the problem was to assume that the secondary system could be uncoupled from the primary system. In that sense, if the dynamic response of a primary-secondary system due to an earthquake was needed then the ground motion was used to find the response of the primary structure. Following that, the response of the secondary system was calculated based on the computation of its response to the motion of the floor to which it was attached. This gave rise to the floor response spectrum method. Several authors have done considerable work in this approach, among them Singh [58,59,60], Singh and Sharma [61], Singh and Wen [62], Gupta [19,20], Gupta and Tembulkar [21], Hadjian [23] and others [1,9,16,42,48]. The intention of this approach is to predict the response of the secondary system to a ground motion. This is done by predicting a floor response spectra for the secondary system or by estimating the maximum displacement and

acceleration of such a secondary system. To estimate the response of the secondary system an amplification factor is applied to the ground response spectra. The amplification factor is intended to take care of the response of the secondary system. The principal approximation in this group is the uncoupling of the two systems, primary and secondary.

The method has a limited range of applicability because uncoupling is only feasible when the ratio of the masses of the secondary system to the primary system is very small compared to unity. Another limitation is that uncoupling may only be used when the natural frequencies of both the primary and the secondary system are well separated and therefore the problem of close modes is avoided.

An improvement was provided by Ruzicka and Robinson [52] and Sackman and Kelly [54,55,56] when they included the possibility of interaction between primary and secondary systems. They showed that the interaction is critical when there is tuning between a frequency of the primary system and a frequency of the secondary system, even for the cases where their corresponding mass ratio is very small compared to unity. Sackman and Kelly and Kelly and Sackman [34] modeled the primary system as a continuous shear beam and the secondary system as a SDOF system connected to it at one point. They then provided a closed form solution to the problem. Their proposed method of solution is correct in every sense, however, it is a very limited result due to the constraints inherent to the model. For example, for tuning of very high modes, the mass of the secondary system has to be increasingly lighter in order to obtain a reasonable solution. Also it only considers one point of connection between the primary system and the secondary system. Other authors have

proposed closed form solutions to the problem of dynamic behavior of combined systems. The systems considered consist of beams or plates with spring-mass systems attached at discrete points. Dowell [12] studied the frequencies of a combined beam spring-mass system by using Lagrange's equations and by including the spring-mass system with the aid of a Lagrange multiplier. Later, Nicholson and Bergman [43] attacked the same problem by using separation of variables to obtain the frequencies and natural modes of the combined system in terms of the Green's functions for the vibrating beam or plate. Neither work gives a thorough consideration to the problem of tuning.

As mentioned above, the solution to the proposed problem has also been attempted by the use of perturbation methods. DerKiureghian et al. [10] and later Sackman et al. [51] used a perturbation approach to determine the dynamic properties of a MDOF-SDOF combined system. However, the SDOF secondary system considered has only one point of attachment to the primary system. In essence, they developed a mode superposition procedure for the response of very light equipment including the effect of closely spaced modes. They obtained the equipment response directly in terms of the dynamic properties of the primary structure, the dynamic properties of the secondary structure, and the design response spectrum for the primary structure (termed the ground response spectrum). The method of analysis employed consists essentially of a first order perturbation analysis of the combined system. A correcting scheme for the modes that are somewhat near the equipment frequency is then applied. However, no criteria are given to determine the closeness of the modes for which the correction is necessary. The

correction consists in the orthogonalization of the close mode with respect to the rest of the modes. This requires the knowledge of all the other modes. Later, Igusa and DerKiureghian [27,28] continued and expanded the scope of the work previously described. They considered the more general case of a MDOF-MDOF combined system where the secondary system may be multiply supported. The limitation of a very small added system helped them simplify the approach by allowing them to neglect the corresponding terms on the diagonal of the resulting stiffness matrix for the combined system. For tuned cases they used a frequency domain analysis of the approximate equations of motion for the combined system. Then, they applied a first order perturbation analysis to the determinant of the resulting complex frequency response matrix.

Other studies have been founded on a similar approach. Hernried and Sackman [25] and Ingham and Sackman [29] employed a first order perturbation analysis applied directly to the eigenproblem resulting from the equations of motion of the combined system.

Use of the perturbation methods for the solution of the eigenvalue problem is also found in the applied mathematics literature. Most of the effort has been dedicated to the solution of the standard form of the eigenvalue problem. See for example, Wilkinson [70, pp. 62-109]. Very little appears to have been written about the so-called generalized form of the eigenvalue problem, $Ax = \lambda Bx$. This is probably due to the fact that, for most applications, such generalized form may be converted to an equivalent standard form, $Cy = \lambda y$. However, this conversion requires that at least one of the matrices in the problem be positive definite. Several authors, [13,17,18] have treated the generalized eigenvalue

problems but have not taken on the case of semidefinite operators at $\mu=0$. As will be seen later, in this study both stiffness and mass matrices of the combined system are positive definite for all values of μ , but are semidefinite for $\mu=0$. Therefore, the transformation to the standard form is far from convenient, if it is, in fact, possible. Only a few references have been found that treat the generalized eigenvalue problem under these conditions. Among them, Fix and Heiberger [14] propose an algorithm for the calculation of a subset of the set of all eigenvalues on a generalized eigenvalue problem. Those eigenvalues correspond only to what will later be called the "old" modes. They do not study the "new" modes, because their interest is in examining the stability of the "old" modes. Stewart [61] presented a first order theory for the perturbation of generalized eigenvalue problem. It is based on the Rayleigh quotient and it essentially deals with the treatment of separate eigenvalues. The problem of predicting the behavior of close eigenvalues and its corresponding eigenvectors is not treated.

1.4 Outline of Method of Analysis

The method proposed, which has its origin in previous work by Robinson and Harris [49] and Lee and Robinson [36], aims at the investigation of a single mode or a set of modes with close frequencies. The effects of changes of parameters of the system on modes and frequencies were studied for separate frequencies in [49] and for close or equal frequencies in [36]. The approach was one of successive

approximations and did not contemplate the addition of degrees of freedom.

The present method involves two major modifications of the approaches of [49] and [36]. First, modifications of the structure that lead to additional degrees of freedom are considered. Secondly, the method is cast in the form of a perturbation calculation rather than one of successive approximations.

Since the added mass and the elastic connection with the primary structure usually correspond to additional terms in the mass and stiffness matrices that are very small in comparison to the original entries, a perturbation procedure is a natural approach. Both the added mass and stiffness are taken proportional to a single parameter μ , unlike the parametrization in Reference [55] where a constant spring stiffness is used.

One advantage of the present method lies in the fact that, for separate frequencies, a single mode may be considered without knowledge of the other modes. For close frequencies, all modes in the cluster must be considered at the same time, but no other modes need be known.

1.5 Notation

The symbols used in this study are defined in the text when they first appear. For convenient reference, the more important symbols are summarized here in alphabetical order. Some symbols are assigned more than one meaning; however, in the context of their use there are no ambiguities.

A, B = general linearized operators, may be matrices, differential or integral operators

X_i = i th eigenvector

λ_i = i th eigenvalue of the initial system

δ = increment operator

R = residual quantity

$K(\mu)$ = stiffness matrix for the augmented system

$M(\mu)$ = mass matrix for the augmented system

K = stiffness matrix for the initial system (bordered)

M = mass matrix for the initial system (bordered)

$K^{(1)}$ = augmentation stiffness matrix

$M^{(1)}$ = augmentation mass matrix

μ = perturbation parameter

$\lambda_i(\mu)$ = i th eigenvalue of the augmented system

λ_i = i th eigenvalue of the initial system

$\lambda_i^{(\alpha)}$ = α th coefficient of the expansion series for the i th eigenvalue of the augmented system

K_0 = stiffness matrix for the initial system

M_0 = mass matrix for the initial system

$X_i(\mu)$ = i th eigenvector of the augmented system

$X_i^{(\alpha)}$ = α th coefficient of the expansion series for the i th eigenvector

$\phi_i^{(0)}$ = is the eigenvector of order n corresponding to the i th eigenvalue of the initial system,

$\phi_i^{(k)}$ = for $k = 1, 2, \dots$ is a vector of order n

$\psi_i^{(k)}$ = is a vector of order m

s = number of close or multiple eigenvalues

Y_j = is a basis vector spanning the space R^*

c_{ij} = transformation coefficient

R = space spanned by the s close or multiple eigenvectors
corresponding to the pair (K,M)

R^* = s -dimensional space nearly orthogonal to the subspace
complementary to subspace R

β_{ij} = Lagrange multiplier

b_j = column corresponding to matrix of the β Lagrange multipliers

$[Y] = [Y_1 \ Y_2 \ \dots \ Y_s]$ matrix containing the s Y_i vectors

$\Delta b_j^{(k)}$ = incremental value for $b_j^{(k)}$

$\Delta Y_j^{(k)}$ = incremental value for $Y_j^{(k)}$

Λ = diagonal matrix containing the s dose or multiple eigenvalues
for the pair (K,M)

ϵ = perturbation parameter equal to $\mu^{1/2}$

δ_{ij} = Kronecker delta

l = number of close or multiple eigenvalues coming from the
initial system

2. DETUNED SYSTEMS

2.1 Introduction

In the previous chapter it was explained that the aim of this work is the presentation of a set of methods for the calculation of the modified and new eigenpairs that arise from the consideration of structural modifications to an initial system. The purpose of this chapter is to explain the proposed method in the simplest case of structural modification to a given system, the addition of a single-degree-of-freedom subsystem to a multiple-degree-of-freedom system. It is assumed in this chapter the eigenvalues of the resulting system are widely separated. The means of verifying that this is the case are explained in Section 2.4. In other words, the initial system does not have any clusters of eigenvalues and, furthermore, no clusters arise when the system is augmented. Such a system is said to be detuned.

2.2 The Robinson-Harris Method

The study of the detuned systems presented in this work is based partially on a generalization of the method proposed by Robinson and Harris in Reference [49]. Before proceeding with the analysis, a brief summary of the method is presented.

The Robinson-Harris Method consists of a procedure to improve approximations to eigenvalues and eigenvectors of a given system by an application of Newton's Method to the eigenvalue problem equations. The

method improves an eigenvalue and its corresponding eigenvector at the same time.

Essentially the method takes the generalized eigenvalue problem

$$AX = \lambda BX \quad (2.1)$$

which, in the present context, may be interpreted as taking A and B to be given operators that are in general dependent on the eigenvalue λ_i but not on the eigenvector X_i . Then an approximate solution to Equation (2.1) will yield:

$$(AX_i - \lambda_i BX_i)^{(j)} = R^{(j)} \quad (2.2)$$

where the superscript j indicates the j th approximation and R is a residual. Equation (2.1) is then expanded about the approximate solution represented by the j th iteration.

This yields,

$$(A\Delta X_i - \lambda_i B\Delta X_i)^{(j)} = (-\Delta A X_i + \Delta \lambda_i B X_i + \lambda_i \Delta B X_i - R)^{(j)} \quad (2.3)$$

or

$$\begin{aligned} (A\Delta X_i - \lambda_i B\Delta X_i)^{(j)} &= \Delta \lambda_i \left(-\frac{\partial A}{\partial \lambda} X_i + B X_i \right. \\ &\quad \left. + \lambda_i \frac{\partial B}{\partial \lambda_i} X_i \right)^{(j)} - R^{(j)} \end{aligned} \quad (2.4)$$

The unknowns in Equation (2.4) are ΔX_i and $\Delta \lambda_i$. Notice now that in order to solve Equation (2.4) uniquely, an extra condition is needed

since there are $n + 1$ unknowns and only n equations. The extra condition specified is

$$X_i^T B \Delta X_i = 0 \quad (2.5)$$

This side condition allows a unique solution for ΔX_i and $\Delta \lambda_i$ by eliminating the possibility of large changes in the eigenvector if the eigenvalue and approximate eigenvector are nearly correct. That is, any change of magnitude in the direction of the eigenvector is eliminated. An alternative statement of the condition is that $X_i^T B X_i$ is kept constant up to terms of first order.

Equations (2.4) and (2.5) together now form a system of $n + 1$ equations and $n + 1$ unknowns. That is

$$\left[\begin{array}{c|c} A - \lambda_i B & -B X_i \\ \hline -X_i^T B & 0 \end{array} \right] \begin{Bmatrix} \Delta X_i \\ \Delta \lambda_i \end{Bmatrix} = \begin{Bmatrix} -R \\ 0 \end{Bmatrix} \quad (2.6)$$

Provided that the matrix of coefficients is nonsingular, Equation (2.6) represents an iterative method that converges more rapidly than a second order process. See References [49] and [31].

A similar approach can now be used to perform a study of the dynamic properties of modified structural systems. However, there are two major differences in the approach. First, the method will not be used as a fixed-point iteration as Robinson and Harris suggest but rather as an extrapolation method to trace the variation of the eigenvalues and their corresponding eigenvectors. As mentioned above, this variation will be effected by a modification to the structural system. A second, and

perhaps more far-reaching modification, permits consideration of new degrees of freedom added to the system. That is, the Robinson-Harris method will be modified in order to handle an increase in the order of the operators. This increase then translates into an increase in the rank of both the stiffness and mass matrices for the case of a discrete model of a structure.

2.3 Statement of the Problem

When dealing with new degrees of freedom, it is necessary to define clearly an initial system and an augmented system.

Consider the free vibration of an undamped n-degree-of-freedom system like the one shown in Figure 2.1. The associated eigenvalue problem can be stated as

$$(K - \lambda_i M)\phi_i = 0 \quad i=1, \dots, n \quad (2.7)$$

where K is the stiffness matrix of the system,

M is the mass matrix of the system,

λ_i is the i th eigenvalue and

ϕ_i is the corresponding i th eigenvector.

A mass m^* is now connected elastically to this structural system at several points. For example, in Figure 2.2, a system is shown with the additional mass m^* attached to masses 2 and 3.

The question then becomes how to predict the solution to the new eigenvalue problem that arises when considering the augmented system. Naturally, one way of doing it is to simply solve the new eigenproblem by

the usual methods. This entails two difficulties. First of all, in large systems, it can be costly to recalculate everything if any slight changes are made in the new structure. That is, the added mass might be placed at different locations in the original structure during the process of design and the eigenproblem would have to be solved several times, one for each change in the configuration. Furthermore, if the magnitudes of the added mass and spring stiffnesses are small compared to those of the original structure, then accuracy problems can be present when the usual methods of solution are used. That is, there will be extremely small terms in the diagonal of both the mass and stiffness matrices of the system. For a complete reference on these methods, see Reference [70, pp. 62-109].

To go from the stiffness and mass matrices of the initial system to the augmented ones, the original matrices must be bordered with rows and columns of zeros. So-called augmentation stiffness and mass matrices are then added to the corresponding bordered matrices. These augmentation matrices contain all the information related to the interaction between the new degrees and the original components of the system. The case in which the additional springs are present but there is no added mass is a simple special case. Likewise, another special case arises when the modification consists of a mere addition of mass to an existing mass in the system.

The range of cases that can be treated is wide since no particular characteristics of the initial system are required. However, a fairly accurate solution of the eigenvalue problem of the initial system is expected to be available. Indeed, this solution actually constitutes the

zero-order approximation to the eigensolution of the augmented system, as will be explained in Section 2.4.

In order to examine the problem from a perturbation standpoint, augmentation matrices are taken proportional to a single positive parameter μ . For the case considered in this chapter, this parametric form can be introduced by setting

$$k^* = \mu k \quad (2.8a)$$

and

$$m^* = \mu m \quad (2.8b)$$

where the k and m are of the same order of magnitude as the spring stiffnesses and masses of the original system.

The eigenvalue problem for the new augmented system can be expressed as follows:

$$[K(\mu) - \lambda_i(\mu)M(\mu)]X_i(\mu) = 0 \quad i=1, \dots, n+1 \quad (2.9a)$$

where $K(\mu)$ is the stiffness matrix for the augmented system,

$$K(\mu) = K^{(0)} + \mu K^{(1)} \quad (2.9b)$$

$K^{(0)}$ is the bordered stiffness matrix of order $n + 1$,

$$K^{(0)} = \left[\begin{array}{c|c} K & 0 \\ \hline 0 & 0 \end{array} \right] \quad (2.9c)$$

$K^{(1)}$ is the augmentation stiffness matrix of order $n + 1$,

$$K^{(1)} = \left[\begin{array}{c|c} K_{11}^{(1)} & K_{12}^{(1)} \\ \hline K_{21}^{(1)} & K_{22}^{(1)} \end{array} \right] \quad (2.9d)$$

$M(\mu)$ is the mass matrix for the augmented system,

$$M(\mu) = M^{(0)} + \mu M^{(1)} \quad (2.9e)$$

$M^{(0)}$ is the bordered mass matrix of order $n + 1$,

$$M^{(0)} = \left[\begin{array}{c|c} M & 0 \\ \hline 0 & 0 \end{array} \right] \quad (2.9f)$$

$M^{(1)}$ is the augmentation mass matrix of order $n + 1$,

$$M^{(1)} = \left[\begin{array}{c|c} M_{11}^{(1)} & M_{12}^{(1)} \\ \hline M_{21}^{(1)} & M_{22}^{(1)} \end{array} \right] \quad (2.9g)$$

$\lambda_i(\mu)$ is the new i th eigenvalue,

$X_i(\mu)$ is the corresponding new i th eigenvector of order $n + 1$, and

μ is the proportionality parameter.

Notice that if the parameter μ is taken equal to zero, then the structure does not become an n -degree-of-freedom system, but rather one in which X_{n+1} is indeterminate. However, for the purpose of this study, μ will not be taken equal to zero. Instead, the limiting case when μ approaches zero through positive values will be studied. In fact, in practice, the most interesting case occurs when μ is small. From the

point of view of the applications, the smaller μ is, the more likely it is that the additional mass and springs will not be taken into consideration in the preliminary stages of design. Therefore, a method to estimate their effect in the structure is especially useful. It is important to notice also that as μ goes to zero, both the stiffness and the mass matrices for the augmented system become ill-conditioned. That is, their determinant becomes very small, almost zero. It is easy to see then why the common methods employed for the solution of the eigenvalue problem for a structural system fail. However, a perturbation analysis can be performed on the new system to eliminate this difficulty.

2.4 Perturbation Scheme. Old Modes.

After a mass is added, there are $n + 1$ degrees of freedom instead of n . It turns out that under certain restrictions, n of these have eigensolutions close to the original ones for small μ , and one of them is closely related to the characteristics of just the augmentation in stiffness and mass, see Reference [10]. This process is conceptually illustrated in Figure 2.4. The "old" modes, those close to the eigensolution of the original system, are considered in this section.

For convenience, the eigenvalues and corresponding eigenvectors are numbered so that the first n eigenvalues in the augmented system correspond to the eigenvalues of the initial system and then the $(n + 1)$ th. eigenvalue corresponds to the eigenvalue of the added system when considered separately. This section deals with the behavior of the modes of the initial structure. The study of the behavior of the new mode will be presented in the following section.

In the case of a detuned system, the eigenvalues are simple roots of an algebraic equation, that is, the roots of the characteristic equation.

For this reason, it is possible to state that the eigenvalues are single-valued analytic functions of μ in the neighborhood of $\mu = 0$. The same is therefore true of the components X_{ij} of the varied eigenvectors belonging to the eigenvalues λ_i . See Reference [8]. Thus, it is possible to use the Taylor series expansion about zero in order to express the eigenvalues of the augmented system. That is,

$$\lambda_i(\mu) = \lambda_i^{(0)} + \lambda_i^{(1)}\mu + \lambda_i^{(2)}\mu^2 + \dots \quad i=1, \dots, n \quad (2.10)$$

where $\lambda_i^{(0)}$ is the i th original eigenvalue of the initial system and $\lambda_i^{(k)}$ for $k = 1, 2, \dots$ are the k th coefficients of the expansion series for the i th eigenvalue.

Up to now, there has not been anything beyond the use of the perturbation method as explained in standard texts. The difference arises when it becomes necessary to treat the eigenvectors of the augmented system. Notice that these eigenvectors must now be of order $n + 1$. That is, they have to account for the displacement component corresponding to the new degree of freedom that has been added.

Note that the totality of eigenvectors of a given system is nothing other than a set of coordinates, a very particular one indeed, that must completely describe the motion of the system. Hence, for the augmented system, the eigenvectors must be of order $n + 1$ so that they define the location of every mass. In the literature, perturbations applied to a vibrating system never involve an increment in the number of degrees of

freedom of the system in question. Reference [70] contains a good summary on perturbation methods for the eigenvalue problem.

In the present case, the mere assumption that the constant term in the series expansion for the eigenvector corresponds to the initial zero-order solution, that is, when there is no perturbation, is obviously not enough. In view of this fact, the use of the following series expansion has been considered:

$$X_i(\mu) = X_i^{(0)} + X_i^{(1)}\mu + X_i^{(2)}\mu^2 + \dots \quad (2.11)$$

where $X_i^{(k)} = \begin{Bmatrix} \phi_i^{(k)} \\ -\frac{1}{\lambda_i^{(k)}} \\ \psi_i^{(k)} \end{Bmatrix}$ is the k th coefficient of the expansion series for

the i th eigenvector,

$\phi_i^{(0)}$ is the eigenvector of order n corresponding to the i th eigenvalue of the initial system,

$\phi_i^{(k)}$ for $k = 1, 2, \dots$ is an unknown vector of order n , and

$\psi_i^{(k)}$ is an unknown vector of order 1.

The physical interpretation of the vector $\psi_i^{(0)}$ will emerge from the analysis. At this point, Equations (2.10) and (2.11) can be substituted into the equation of the eigenvalue problem, Equation (2.9a),

$$\begin{aligned} [K^{(0)} + \mu K^{(1)}][X_i^{(0)} + X_i^{(1)}\mu + X_i^{(2)}\mu^2 + \dots] &= [\lambda_i^{(0)} + \lambda_i^{(1)}\mu + \lambda_i^{(2)}\mu^2 + \dots] \\ [M^{(0)} + \mu M^{(1)}][X_i^{(0)} + X_i^{(1)}\mu + X_i^{(2)}\mu^2 + \dots] & \end{aligned} \quad (2.12)$$

The indicated multiplications yield

$$\begin{aligned}
& [K^{(0)}X_i^{(0)} - \lambda_i^{(0)}M^{(0)}X_i^{(0)}] + [K^{(0)}X_i^{(1)} + K^{(1)}X_i^{(0)} - \lambda_i^{(0)}M^{(0)}X_i^{(1)} \\
& \quad - \lambda_i^{(0)}M^{(1)}X_i^{(0)} - \lambda_i^{(0)}M^{(0)}X_i^{(0)}] \mu + [K^{(0)}X_i^{(2)} + K^{(1)}X_i^{(1)} \\
& \quad + \lambda_i^{(0)}M^{(0)}X_i^{(2)} - \lambda_i^{(0)}M^{(1)}X_i^{(1)} - \lambda_i^{(1)}M^{(0)}X_i^{(1)} \\
& \quad - \lambda_i^{(1)}M^{(1)}X_i^{(0)} - \lambda_i^{(2)}M^{(0)}X_i^{(0)}] \mu^2 + \dots = 0 \tag{2.13}
\end{aligned}$$

Equations (2.13) represent the complete expansion about $\mu = 0$ of the eigenvalue problem. Because these equations must be satisfied for all values of μ and, moreover, the quantities $\lambda_i^{(k)}$ and the vectors $X_i^{(k)}$ are independent of μ , the coefficient of each power of μ must be equal to zero. Thus, the perturbation equations are,

$$[K^{(0)} - \lambda_i^{(0)}M^{(0)}]X_i^{(0)} = 0 \tag{2.14a}$$

$$\begin{aligned}
& [K^{(0)} - \lambda_i^{(0)}M^{(0)}]X_i^{(1)} + [K^{(1)} - \lambda_i^{(0)}M^{(1)}]X_i^{(0)} \\
& \quad - \lambda_i^{(1)}M^{(0)}X_i^{(0)} = 0 \tag{2.14b}
\end{aligned}$$

$$\begin{aligned}
& [K^{(0)} - \lambda_i^{(0)}M^{(0)}]X_i^{(2)} + [K^{(1)} - \lambda_i^{(0)}M^{(1)}]X_i^{(1)} - \lambda_i^{(1)}M^{(0)}X_i^{(1)} \\
& \quad - \lambda_i^{(1)}M^{(1)}X_i^{(0)} - \lambda_i^{(2)}M^{(0)}X_i^{(0)} = 0 \tag{2.14c}
\end{aligned}$$

...

This set of equations is now solved recursively. In the present case, a certain inconvenience seems to arise from the fact that $X_i^{(0)}$ is not known entirely. That is, the vector $\psi_i^{(0)}$ must be solved for.

If the system of Equations (2.14) is examined more closely, it can be seen that each one of these equations can be partitioned into two groups, one of order n and the other of order one. Thus, the first of Equations (2.14) gives,

$$\left[\left[\begin{array}{c|c} K & 0 \\ \hline 0 & 0 \end{array} \right] - \lambda_i^{(0)} \left[\begin{array}{c|c} M & 0 \\ \hline 0 & 0 \end{array} \right] \right] \begin{Bmatrix} \phi_i^{(0)} \\ \psi_i^{(0)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (2.15)$$

Equation (2.15) degenerates to two sets of equations

$$[K - \lambda_i^{(0)} M] \phi_i^{(0)} = 0 \quad (2.16a)$$

and

$$0 \cdot \psi_i^{(0)} = 0 \quad (2.16b)$$

It is seen here that the Equation (2.16a) is simply the statement of the eigenvalue problem for the initial system. Thus, the equation turns into an identity for each one of the old modes. Equation (2.16b) provides no information about $\psi_i^{(0)}$, which appears still to be arbitrary.

The results from Equation (2.16a) are consistent with the assumption made that the eigenvalues $\lambda_i^{(0)}$ of the initial system represent the zero-order approximation to the solution of the problem.

The second of Equations (2.14) is now subjected to the same partitioning procedure, yielding

$$\begin{aligned}
& \left[\left[\begin{array}{c|c} K & 0 \\ \hline 0 & 0 \end{array} \right] - \lambda_i^{(0)} \left[\begin{array}{c|c} M & 0 \\ \hline 0 & 0 \end{array} \right] \right] \begin{Bmatrix} \phi_i^{(1)} \\ \psi_i^{(1)} \end{Bmatrix} \\
& + \left[\left[\begin{array}{c|c} K_{11}^{(1)} & K_{12}^{(1)} \\ \hline K_{21}^{(1)} & K_{22}^{(1)} \end{array} \right] - \lambda_i^{(0)} \left[\begin{array}{c|c} M_{11}^{(1)} & M_{12}^{(1)} \\ \hline M_{21}^{(1)} & M_{22}^{(1)} \end{array} \right] \right] \begin{Bmatrix} \phi_i^{(0)} \\ \psi_i^{(0)} \end{Bmatrix} \\
& - \lambda_i^{(1)} \left[\begin{array}{c|c} M & 0 \\ \hline 0 & 0 \end{array} \right] \begin{Bmatrix} \phi_i^{(0)} \\ \psi_i^{(0)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{2.17}
\end{aligned}$$

Equation (2.17) can also be separated into two sets of equations,

$$\begin{aligned}
& [K - \lambda_i^{(0)} M] \phi_i^{(1)} + [K_{11}^{(1)} - \lambda_i^{(0)} M_{11}^{(1)}] \phi_i^{(0)} \\
& + [K_{12}^{(1)} - \lambda_i^{(0)} M_{12}^{(1)}] \psi_i^{(0)} - \lambda_i^{(1)} M \phi_i^{(0)} = 0 \tag{2.18}
\end{aligned}$$

and, the scalar equation

$$[K_{21}^{(1)} - \lambda_i^{(0)} M_{21}^{(1)}] \phi_i^{(0)} + [K_{22}^{(1)} - \lambda_i^{(0)} M_{22}^{(1)}] \psi_i^{(0)} = 0 \tag{2.19}$$

Notice that Equation (2.19) can be solved for $\psi_i^{(0)}$ provided that the scalar $[K_{22}^{(1)} - \lambda_i^{(0)} M_{22}^{(1)}]$ is not zero. $K_{22}^{(1)}$ and $M_{22}^{(1)}$ are the spring and mass respectively of the single-degree-of-freedom system being added to the initial configuration. For a detuned system,

$$\lambda_i \neq \frac{K_{22}^{(1)}}{M_{22}^{(1)}} \quad i=1, \dots, n \tag{2.20}$$

Thus, the solution of Equation (2.19) gives

$$\psi_i^{(0)} = - \frac{K_{21}^{(1)} - \lambda_i^{(0)} M_{21}^{(1)}}{K_{22}^{(1)} - \lambda_i^{(0)} M_{22}^{(1)}} \phi_i^{(0)} \quad (2.21)$$

It is seen from this equation that the term $\psi_i^{(0)}$ depends on $\phi_i^{(0)}$, the eigenvector corresponding to the i th mode of the initial system. Equation (2.21) states that, for the zero-order approximation, the response of the added mass is due to a forced vibration. That is, the motion of the added mass is due to the excitation of its supports. This becomes evident when one realizes that, in the special case of Figure 2.2, all but two terms of the row vector $K_{21}^{(1)}$ are zero. The two non-zero terms correspond to the points of support of the added mass. The result is neither surprising nor new; it is the basis of the so-called "floor response spectrum method." See References [1,59,60,61,62].

The perturbation process is now continued by substituting Equation (2.21) into Equation (2.18) and rearranging the resulting terms

$$[K - \lambda_i^{(0)} M] \phi_i^{(1)} - \lambda_i^{(1)} M \phi_i^{(0)} = P_i^{(1)} \quad (2.22)$$

where

$$P_i^{(1)} = - [K_{11}^{(1)} - \lambda_i^{(0)} M_{11}^{(1)}] \phi_i^{(0)} - [K_{12}^{(1)} - \lambda_i^{(0)} M_{12}^{(1)}] \psi_i^{(0)} \quad (2.23)$$

Notice that the unknowns in Equation (2.22) are the vector $\phi_i^{(1)}$ and the scalar $\lambda_i^{(1)}$. Hence, Equation (2.22) cannot be solved for the unknowns, since the vector $\phi_i^{(1)}$ is of order n . An extra condition is needed then in order to make this equation solvable.

The extra conditions that will be used are the condition of orthonormality of the eigenvectors with respect to the mass matrix. These are the same side conditions used by Robinson and Harris in their work. That is, the eigenvector is not allowed to have any changes in its magnitude. More specifically, only its direction is allowed to change.

Let us consider the orthonormality conditions for the augmented system. In general we have,

$$X_i^T(\mu)M(\mu)X_j(\mu) = \delta_{ij} \quad (2.24)$$

Substituting Equations (2.9e) and (2.11) into Equation (2.24) yields, for $i = j$

$$\begin{aligned} & [X_i^{(0)T} M^{(0)} X_i^{(0)}] + [2X_i^{(1)T} M^{(0)} X_i^{(0)} + X_i^{(0)T} M^{(1)} X_i^{(0)}] \mu \\ & + [X_i^{(1)T} M^{(0)} X_i^{(1)} + 2X_i^{(1)T} M^{(1)} X_i^{(0)} + 2X_i^{(2)T} M^{(0)} X_i^{(0)}] \mu^2 + \dots \quad (2.25) \end{aligned}$$

Equation (2.25) must be satisfied for all values of μ . Therefore, the coefficient of the zero power of μ in Equation (2.25) must be equal to one while the rest of the coefficients must be equal to zero. Thus,

$$X_i^{(0)T} M^{(0)} X_i^{(0)} = 1 \quad (2.26a)$$

$$2X_i^{(1)T} M^{(0)} X_i^{(0)} + X_i^{(0)T} M^{(1)} X_i^{(0)} = 0 \quad (2.26b)$$

$$X_i^{(1)T} M^{(0)} X_i^{(1)} + X_i^{(1)T} M^{(1)} X_i^{(0)} + 2X_i^{(2)T} M^{(0)} X_i^{(0)} = 0 \quad (2.26c)$$

A procedure similar to the one used for equations (2.14) can now be followed. That is, each of the Equations (2.26) can be partitioned and reduced to terms of initial system. For instance, Equation (2.26a) yields

$$\left\{ \begin{array}{c|c} \phi_i^{(0)T} & \psi_i^{(0)} \end{array} \right\} \left[\begin{array}{c|c} M & 0 \\ \hline 0 & 0 \end{array} \right] \left\{ \begin{array}{c} \phi_i^{(0)T} \\ \hline \psi_i^{(0)} \end{array} \right\} = 1 \quad (2.27)$$

which reduces to

$$\phi_i^{(0)T} M \phi_i^{(0)} = 1 \quad (2.28)$$

Equation (2.28) is obviously a statement of the normality condition for the "old" modes, and it is merely saying that the normality conditions for the augmented system is satisfied up to the zero-order approximation regardless of what the values of $\psi_i^{(0)}$ may be.

Also, Equation (2.26b) yields

$$\begin{aligned} & 2 \left[\begin{array}{c|c} \phi_i^{(1)T} & \psi_i^{(1)} \end{array} \right] \left[\begin{array}{c|c} M & 0 \\ \hline 0 & 0 \end{array} \right] \left\{ \begin{array}{c} \phi_i^{(0)T} \\ \hline \psi_i^{(0)} \end{array} \right\} \\ & + \left[\begin{array}{c|c} \phi_i^{(0)T} & \psi_i^{(0)} \end{array} \right] \left[\begin{array}{c|c} M_{11}^{(1)} & M_{12}^{(1)} \\ \hline M_{21}^{(1)} & M_{22}^{(1)} \end{array} \right] \left\{ \begin{array}{c} \phi_i^{(0)T} \\ \hline \psi_i^{(0)} \end{array} \right\} = 0 \end{aligned} \quad (2.29)$$

which reduces to

$$\begin{aligned}
2\phi_i^{(1)T} M_{\phi_i}^{(0)} + \phi_i^{(0)T} M_{11}^{(1)} \phi_i^{(0)} + \psi_i^{(0)T} M_{21}^{(1)} \phi_i^{(0)} \\
+ \phi_i^{(0)T} M_{12}^{(1)} \psi_i^{(0)} + \psi_i^{(0)T} M_{22}^{(1)} \psi_i^{(0)} = 0
\end{aligned} \tag{2.30}$$

A re-examination of Equation (2.22) shows that the additional imposition of Equation (2.30) permits unique solution for $\phi_i^{(1)}$ and $\lambda_i^{(1)}$.

That is,

$$\left[\begin{array}{c|c} K - \lambda_i^{(0)} M & -M\phi_i^{(0)} \\ \hline -\phi_i^{(0)T} M & 0 \end{array} \right] \begin{Bmatrix} \phi_i^{(1)} \\ \lambda_i^{(1)} \end{Bmatrix} = \begin{Bmatrix} P_i^{(1)} \\ Q_i^{(1)} \end{Bmatrix} \tag{2.31}$$

where

$$P_i^{(1)} = -[K_{11}^{(1)} - \lambda_i^{(0)} M_{11}^{(1)}] \phi_i^{(0)} - [K_{12}^{(1)} - \lambda_i^{(0)} M_{12}^{(1)}] \psi_i^{(0)}, \tag{2.23}$$

and

$$\begin{aligned}
Q_i^{(1)} = \frac{1}{2} [& \phi_i^{(0)T} M_{11}^{(1)} \phi_i^{(0)} + \psi_i^{(0)T} M_{21}^{(1)} \phi_i^{(0)} + \phi_i^{(0)T} M_{12}^{(1)} \psi_i^{(0)} \\
& + \psi_i^{(0)T} M_{22}^{(1)} \psi_i^{(0)}]
\end{aligned} \tag{2.32}$$

$$\psi_i^{(0)} = - \frac{K_{21}^{(1)} - \lambda_i^{(0)} M_{21}^{(1)}}{K_{22}^{(1)} - \lambda_i^{(0)} M_{22}^{(1)}} \phi_i^{(0)} \tag{2.21}$$

Indeed, Equation (2.31) constitutes a matrix equation of order $n + 1$. This equation is solvable for $\phi_i^{(1)}$ and $\lambda_i^{(1)}$. The matrix of coefficients is the same as the one obtained in Robinson and Harris [49],

where it is proved that this matrix of coefficients is non-singular if there are no multiple roots.

The solution of Equation (2.31) gives the values needed to calculate $\lambda(\mu)$ up to the first order perturbation. However, $X(\mu)$ is only known to the zero-order perturbation since the equations do not provide the value of $\psi_i^{(1)}$.

The second-order perturbation, Equations (2.14c), may now be analyzed by introducing the same type of partitioning used for the first-order perturbation Equations (2.14b). As in the previous case, Equations (2.14c) can be separated into two sets of equations,

$$\begin{aligned} & [K - \lambda_i^{(0)} M] \phi_i^{(2)} + [K_{11}^{(1)} - \lambda_i^{(0)} M_{11}^{(1)}] \phi_i^{(1)} \\ & + [K_{12}^{(1)} - \lambda_i^{(0)} M_{12}^{(1)}] \psi_i^{(1)} - \lambda_i^{(1)} M \phi_i^{(1)} \\ & - \lambda_i^{(1)} M_{11}^{(1)} \phi_i^{(0)} - \lambda_i^{(1)} M_{12}^{(1)} \psi_i^{(0)} - \lambda_i^{(2)} M \phi_i^{(0)} = 0 \end{aligned} \quad (2.33)$$

and

$$\begin{aligned} & [K_{21}^{(1)} - \lambda_i^{(0)} M_{21}^{(1)}] \phi_i^{(1)} + [K_{22}^{(1)} - \lambda_i^{(0)} M_{22}^{(1)}] \psi_i^{(1)} \\ & - \lambda_i^{(1)} M_{21}^{(1)} \phi_i^{(0)} - \lambda_i^{(1)} M_{22}^{(1)} \psi_i^{(0)} = 0 \end{aligned} \quad (2.34)$$

since $[K_{22}^{(1)} - \lambda_i^{(0)} M_{22}^{(1)}] \neq 0$, we have

$$\psi_i^{(1)} = \frac{-[K_{21}^{(1)} - \lambda_i^{(0)} M_{21}^{(1)}] \phi_i^{(1)} + \lambda_i^{(1)} M_{21}^{(1)} \phi_i^{(0)} + \lambda_i^{(1)} M_{22}^{(1)} \psi_i^{(0)}}{[K_{22}^{(1)} - \lambda_i^{(0)} M_{22}^{(1)}]} \quad (2.35)$$

Equation (2.35) can now be substituted into Equation (2.33) to yield

$$[K - \lambda_i^{(0)} M] \phi_i^{(2)} - \lambda_i^{(2)} M \phi_i^{(0)} = P_i^{(2)} \quad (2.36)$$

where

$$\begin{aligned} P_i^{(2)} = & -[K_{11}^{(1)} - \lambda_i^{(0)} M_{11}^{(1)}] \phi_i^{(1)} - [K_{12}^{(1)} - \lambda_i^{(0)} M_{12}^{(1)}] \psi_i^{(1)} \\ & + \lambda_i^{(1)} M_{\phi_i}^{(1)} + \lambda_i^{(1)} M_{11}^{(1)} \phi_i^{(0)} + \lambda_i^{(1)} M_{12}^{(1)} \psi_i^{(0)} \end{aligned} \quad (2.37)$$

Equation (2.36) is of the same form of Equation (2.22). Equation (2.26c) can be used as the extra condition needed for the solution of the Equations (2.36). But first notice that it can be reduced to

$$\begin{aligned} 2[\phi_i^{(0)T} M_{\phi_j}^{(2)} + \phi_i^{(0)T} M_{11}^{(1)} \phi_j^{(1)} + \psi_i^{(0)T} M_{21}^{(1)} \phi_j^{(1)} + \phi_i^{(0)T} M_{12}^{(1)} \psi_j^{(1)} \\ + \psi_i^{(0)T} M_{22}^{(1)} \psi_j^{(1)}] + \phi_i^{(1)T} M_{\phi_j}^{(1)} = 0 \end{aligned} \quad (2.38)$$

Now Equation (2.38) is used in conjunction with Equations (2.36) to yield an equation of the same type of Equations (2.31), that is

$$\left[\begin{array}{c|c} K - \lambda_i^{(0)} M & -M_{\phi_i}^{(0)} \\ \hline -\phi_i^{(0)T} M & 0 \end{array} \right] \left\{ \begin{array}{c} \phi_i^{(2)} \\ \lambda_i^{(2)} \end{array} \right\} = \left\{ \begin{array}{c} P_i^{(2)} \\ Q_i^{(2)} \end{array} \right\} \quad (2.39)$$

where

$$\begin{aligned} P_i^{(2)} = & -[K_{11}^{(1)} - \lambda_i^{(0)} M_{11}^{(1)}] \phi_i^{(1)} - [K_{12}^{(1)} - \lambda_i^{(0)} M_{12}^{(1)}] \psi_i^{(1)} \\ & + \lambda_i^{(1)} M_{\phi_i}^{(1)} + \lambda_i^{(1)} M_{11}^{(1)} \phi_i^{(0)} + \lambda_i^{(1)} M_{12}^{(1)} \psi_i^{(0)} \end{aligned} \quad (2.37)$$

and

$$\begin{aligned}
 Q_i^{(2)} = & \phi_i^{(0)T} M_{11}^{(1)} \phi_i^{(1)} + \psi_i^{(0)} M_{21}^{(1)} \phi_j^{(1)} + \phi_i^{(0)T} M_{12}^{(1)} \psi_i^{(1)} \\
 & + \psi_i^{(0)} M_{22}^{(1)} \psi_i^{(1)} + \frac{1}{2} \phi_i^{(1)T} M \phi_i^{(1)} \quad (2.40)
 \end{aligned}$$

The solution of Equation (2.39) gives the values of $\lambda_i^{(2)}$ and $\phi_i^{(2)}$. Hence, the eigenvalue $\lambda(\mu)$ can be calculated up to the second-order terms in μ . The eigenvector $X_i(\mu)$ can now also be approximated up to the first order by using Equation (2.35).

It is very important to point out the similarity of Equations (2.31) and (2.39). Both equations have the same matrix of coefficients. This means that in order to solve the two sets of equations numerically, the matrix would have to be triangularized only once. This represents considerable savings in computational effort. Furthermore, it is easy to see that perturbations of third or higher order will be of the same type as (2.31) and (2.39). That is,

$$\left[\begin{array}{c|c} K - \lambda_i^{(0)} M & -M \phi_i^{(0)} \\ \hline -\phi_i^{(0)T} M & 0 \end{array} \right] \left\{ \begin{array}{c} \phi_i^{(k)} \\ \lambda_i^{(k)} \end{array} \right\} = \left\{ \begin{array}{c} P_i^{(k)} \\ Q_i^{(k)} \end{array} \right\} \quad k=1, 2, \dots \quad (2.41)$$

where $P_i^{(k)}$ is a function of $\psi_i^{(k-1)}$ and $Q_i^{(k)}$ and $\psi_i^{(k-1)}$ are known scalars. Therefore, the eigenvalues and eigenvectors corresponding to the "old" modes of the initial system can be approximated to the desired order by using Equations (2.41) in conjunction with Equations (2.10) and (2.11).

2.5 Perturbation Scheme. New Mode.

This section deals with the behavior of the new mode that arises when a mass is added to the initial system. The same type of expansion series about $\mu = 0$ that was used for the "old" modes can now be used for the "new" mode. That is, Equations (2.10) and (2.11) can be used for $i = n + 1$. Hence,

$$\lambda_{n+1}(\mu) = \lambda_{n+1}^{(0)} + \mu \lambda_{n+1}^{(1)} + \mu^2 \lambda_{n+1}^{(2)} + \dots \quad (2.42)$$

where $\lambda_{n+1}^{(0)}$ is the eigenvalue of the added single-degree-of-freedom system

$\lambda_{n+1}^{(k)}$ for $k = 1, 2, \dots$ is the k th coefficient of the expansion series for the $(n+1)$ th eigenvalue,

and,

$$X_{n+1}^{(k)}(\mu) = X_{n+1}^{(0)} + \mu X_{n+1}^{(1)} + \mu^2 X_{n+1}^{(2)} + \dots \quad (2.43)$$

where $X_{n+1}^{(k)} = \begin{Bmatrix} \phi_{n+1}^{(k)} \\ \psi_{n+1}^{(k)} \end{Bmatrix}$ is the k th coefficient of the expansion series for the $(n+1)$ th eigenvector,

$\phi_{n+1}^{(k)}$ for $k = 0, 1, 2, \dots$ is an unknown vector of order n , and

$\psi_{n+1}^{(k)}$ for $k = 0, 1, 2, \dots$ is an unknown scalar.

Thus, the perturbation equations developed for the "old" modes continue to have validity for the "new" mode. However, some simplifications will arise as will be seen below.

The above equations are now examined in detail. For the new mode, Equation (2.16) is now

$$[K - \lambda_{n+1}^{(0)} M] \phi_{n+1}^{(0)} = 0 \quad (2.44)$$

It will be recalled that this equation is the zero-order perturbation equation for the augmented system. If $\lambda_{n+1}^{(0)}$ and $\phi_{n+1}^{(0)}$ are considered as unknowns, this equation also represents the eigenvalue problem equation for the initial system. The only way that the determinant $|K - \lambda_{n+1}^{(0)} M|$ can equal zero is if the quantity $\lambda_{n+1}^{(0)}$ is equal to any of the eigenvalues of the initial system. In that case we would have the so-called tuned problem, which will be discussed in detail in Chapter 3. In the case when $\lambda_{n+1}^{(0)}$ is not equal to any of the eigenvalues of the initial system, the determinant $|K - \lambda_{n+1}^{(0)} M|$ is different from zero. Hence, the only possible way in which Equation (2.44) can be satisfied is for the vector $\phi_{n+1}^{(0)}$ to be zero. This result implies that for the zero-order perturbation the "new" mode is completely uncoupled from the "old" modes.

The main simplification to the equations obtained in the previous section comes from the result that $\phi_{n+1}^{(0)}$ is zero. Under that condition, the first-order perturbation equations, i.e., Equations (2.18) and (2.19) become

$$[K - \lambda_{n+1}^{(0)} M] \phi_{n+1}^{(1)} + [K_{12}^{(1)} - \lambda_{n+1}^{(0)} M_{12}^{(1)}] \psi_{n+1}^{(0)} = 0 \quad (2.45)$$

$$[K_{22}^{(1)} - \lambda_{n+1}^{(0)} M_{22}^{(1)}] \psi_{n+1}^{(0)} = 0 \quad (2.46)$$

For a lumped mass system like the one shown in Figure 2.2, the first of the above equations can be written as

$$[K - \lambda_{n+1}^{(0)} M] \phi_{n+1}^{(1)} = -K_{12}^{(1)} \psi_{n+1}^{(0)} \quad (2.47)$$

The vector $\phi_{n+1}^{(1)}$ is nothing other than the motion of the original system caused by the dynamic reaction forces present when the system in Figure 2.3 is vibrating freely with an amplitude $\psi_{n+1}^{(0)}$. It is also evident here that such free vibration causes only a higher-order effect on the initial system. This fact corresponds to the assumptions of the floor response spectra method for very small added systems. See References [1,58,59,60,61,62].

The second equation, (2.46) is actually the eigenvalue problem for the system mentioned above, where the mass m^* is attached to a fixed base by the spring $\sum k_i^*$. The solution to this problem will yield the zero-order approximation for the $(n+1)$ th eigenvalue of the augmented system. However, it is also necessary to add an "orthonormality" condition in order to determine what $\psi_{n+1}^{(0)}$ can be. In other words, it is necessary to use an extra condition in order to solve this eigenvalue problem.

At this point, it is useful to consider Equation (2.24) again. This equation can be modified for the new mode in the following way by letting the size of the vector X_{n+1} depend on the parameter μ , i.e.

$$X_{n+1}^T(\mu) M(\mu) X_{n+1}(\mu) = \mu \quad (2.48)$$

The object of this modification is to avoid extremely large values of the entries in the "new" modal vector. Then Equations (2.26) will now become

$$X_{n+1}^{(0)T} M^{(0)} X_{n+1}^{(0)} = 0 \quad (2.49a)$$

$$2X_{n+1}^{(0)T} M^{(0)} X_{n+1}^{(1)} + X_{n+1}^{(0)T} M^{(1)} X_{n+1}^{(0)} = 1 \quad (2.49b)$$

$$2X_{n+1}^{(0)T} M^{(0)} X_{n+1}^{(2)} + 2X_{n+1}^{(0)T} M^{(1)} X_{n+1}^{(1)} + X_{n+1}^{(1)T} M^{(0)} X_{n+1}^{(1)} = 0 \quad (2.49c)$$

...

Equation (2.49a) is satisfied automatically because $\phi_{n+1}^{(0)}$ is zero.

Furthermore, Equation (2.49b) now reduces to

$$\psi_{n+1}^{(0)T} M_{22}^{(1)} \psi_{n+1}^{(0)} = 1 \quad (2.50)$$

It is interesting to remark now that Equation (2.50) represents the condition of orthonormality for a single-degree-of-freedom system like the one shown in Figure 2.3.

Combining Equations (2.46) and (2.50) it is possible to obtain a unique solution for the eigenvalue problem of the added system.

The second-order perturbation equations are now considered. Equations (2.33) and (2.34) become

$$\begin{aligned} [K - \lambda_{n+1}^{(0)} M] \phi_{n+1}^{(2)} + [K_{11}^{(1)} - \lambda_{n+1}^{(0)} M_{11}^{(1)}] \phi_{n+1}^{(1)} + [K_{12}^{(1)} \\ - \lambda_{n+1}^{(0)} M_{12}^{(1)}] \psi_{n+1}^{(1)} - \lambda_{n+1}^{(1)} M \phi_{n+1}^{(1)} - \lambda_{n+1}^{(1)} M_{12}^{(1)} \psi_{n+1}^{(0)} = 0 \end{aligned} \quad (2.51)$$

and

$$\begin{aligned} [K_{12}^{(1)} - \lambda_{n+1}^{(0)} M_{21}^{(1)}] \phi_{n+1}^{(1)} + [K_{22}^{(1)} - \lambda_{n+1}^{(0)} M_{22}^{(1)}] \psi_{n+1}^{(1)} \\ - \lambda_{n+1}^{(1)} M_{22}^{(1)} \psi_{n+1}^{(0)} = 0 \end{aligned} \quad (2.52)$$

Equation (2.52) has two unknowns, however, we can take this equation together with an appropriate side condition and solve for $\psi_{n+1}^{(1)}$ and $\lambda_{n+1}^{(1)}$. This side condition comes from the next higher order in the perturbation expansion for the orthogonality condition, i.e., Equation (2.49c). Such equation can be rewritten as

$$2\psi_{n+1}^{(0)} M_{21}^{(1)} \phi_{n+1}^{(1)} + 2\psi_{n+1}^{(0)} M_{22}^{(1)} \psi_{n+1}^{(1)} + \phi_{n+1}^{(1)} M \phi_{n+1}^{(1)} = 0 \quad (2.53)$$

The resulting system of equations is then

$$\left[\begin{array}{c|c} K_{22}^{(1)} - \lambda_{n+1}^{(0)} M_{22}^{(1)} & -M_{22}^{(1)} \psi_{n+1}^{(0)} \\ \hline -\psi_{n+1}^{(0)} M_{22}^{(1)} & 0 \end{array} \right] \begin{Bmatrix} \psi_{n+1}^{(1)} \\ \lambda_{n+1}^{(1)} \end{Bmatrix} = \begin{Bmatrix} P_{n+1}^{(1)} \\ Q_{n+1}^{(1)} \end{Bmatrix} \quad (2.54)$$

where

$$P_{n+1}^{(1)} = -[K_{21}^{(1)} - \lambda_{n+1}^{(0)} M_{21}^{(1)}] \phi_{n+1}^{(1)} \quad (2.55)$$

$$Q_{n+1}^{(1)} = \psi_{n+1}^{(0)} M_{21}^{(1)} \phi_{n+1}^{(1)} + \frac{1}{2} \phi_{n+1}^{(1)T} M \phi_{n+1}^{(1)} \quad (2.56)$$

The system of Equations (2.54) has a nonzero determinant just like the system (2.41). Hence, it is possible to solve Equations (2.54) for $\psi_{n+1}^{(1)}$ and $\lambda_{n+1}^{(1)}$. Substituting these values into Equations (2.51) will then yield the value for $\phi_{n+1}^{(2)}$. That is,

$$\begin{aligned} \phi_{n+1}^{(2)} = & -[K - \lambda_{n+1}^{(0)} M]^{-1} \{ [K_{11}^{(1)} - \lambda_{n+1}^{(0)} M_{11}^{(1)}] \phi_{n+1}^{(1)} + [K_{12}^{(1)} \\ & - \lambda_{n+1}^{(0)} M_{12}^{(1)}] \psi_{n+1}^{(1)} - \lambda_{n+1}^{(1)} M \phi_{n+1}^{(1)} - \lambda_{n+1}^{(1)} M_{12}^{(1)} \psi_{n+1}^{(0)} \} \end{aligned} \quad (2.57)$$

For higher orders of perturbation it is seen that equations (2.54) and (2.57) become

$$\left[\begin{array}{c|c} K_{22}^{(1)} - \lambda_{n+1}^{(0)} M_{22}^{(1)} & -M_{22}^{(1)} \psi_{n+1}^{(0)} \\ \hline -\psi_{n+1}^{(0)} M_{22}^{(1)} & 0 \end{array} \right] \begin{Bmatrix} \psi_{n+1}^{(k)} \\ \lambda_{n+1}^{(k)} \end{Bmatrix} = \begin{Bmatrix} P_{n+1}^{(k)} \\ Q_{n+1}^{(k)} \end{Bmatrix} \quad (2.58)$$

and

$$\phi_{n+1}^{(k+1)} = -[K - \lambda_{n+1}^{(0)} M]^{-1} R_{n+1}^{(k)} \quad (2.59)$$

where $R_{n+1}^{(k)}$ is a function of $\psi_{n+1}^{(k)}$ and $\lambda_{n+1}^{(k)}$ as well as $\phi_{n+1}^{(k)}$. Thus, the eigenvalue and the corresponding eigenvector for the "new" mode of the augmented system can be approximated to any desired order by using Equations (2.58) and (2.59).

2.6 Addition of Several Degrees of Freedom

The modeling of the secondary system as a single degree of freedom system is very limited when considering the dynamic behavior of flexible non-structural components [50]. Some examples of such components are piping systems, cable tray assemblages or perhaps internal elements of a large piece of equipment. A better model may then be obtained by adding of a system of several degrees of freedom to the initial system.

In this section, an extension is made to the case where an m degree-of-freedom system is added to an n degree-of-freedom system. The changes that will occur to the formulations presented in Sections 2.4 and 2.5 are small.

One of the most obvious changes will be in the augmentation matrices. They will now have to contain not only the information related to the interaction between the new masses and the original system but they will also describe any coupling within the added system. For the case of a lumped mass system, the augmentation stiffness matrix turns out to be equivalent to the stiffness matrix of a system that has rigid body modes. The number of rigid body modes is equal to the number of degrees of freedom of the initial system. For example, consider a combined structure composed of a four degree-of-freedom system to which a two degree-of-freedom system is added. Such a system is represented in Figure 2.5. For that system the augmentation stiffness matrix is equal to the stiffness matrix that would result from considering $k_i = 0$ for $i = 1, \dots, 4$. It is possible then to introduce a single parameter μ to vary the sizes of the augmentation matrices $K^{(1)}$ and $M^{(1)}$ in much the same way that was done in Section 2.3. Also, it may be remarked that the submatrices $K_{22}^{(1)}$ and $M_{22}^{(1)}$ will now be matrices of order $m \times m$ which will represent the structural properties of the added system when considered separately. In fact, for the example mentioned above, the matrices $K_{22}^{(1)}$ and $M_{22}^{(1)}$ would represent the structural characteristics of the system shown in Figure 2.6.

The results expressed in Sections 2.4 and 2.5 for the case of addition of a single-degree-of-freedom system to an initial one only requires a change from scalar notation to vector notation. Then Equation (2.41), which states the perturbation equation for order k , can be used without any modifications.

Another important result is given by a restatement of Equation (2.44), i.e.

$$[K - \lambda_j^{(0)} M] \phi_j^{(0)} = 0 \quad j = n+1, \dots, n+m \quad (2.60)$$

This equation implies that $\phi_i^{(0)}$ is equal to zero for each j . Thus the uncoupling between the "new" modes and the "old" ones for the zero-order perturbation is still present provided that there is no tuning. Also, the orthogonality condition presented in Section 2.5 can be used as stated without any difficulties.

For the previous reasons, it is possible to say that the Equations (2.58) and (2.59) can also be used without any inconveniences in the present case.

3. TUNED SYSTEMS

3.1 Introduction

The purpose of this chapter is to explain the proposed method as it pertains to the case where the structural modification introduced causes a more complicated result than in Chapter 2. In this chapter the eigenvalues of interest correspond to the addition of an m -degree-of-freedom system to an n -degree-of-freedom system where some of the eigenvalues are very close or equal.

The most general case of multiplicity of eigenvalues considered here is the one where the added system has a repeated eigenvalue of multiplicity q that is equal to a certain eigenvalue of multiplicity r in the initial system. The situation is then one of $q + r$ tuned modes. An example of the general system studied may be taken as an augmented system like the one shown in Figure 3.1. This system consists of an initial 2-degree-of-freedom to which a 2-degree-of-freedom subsystem has been added. The solution of this problem has been attempted before, but it has not been done by using the perturbation approach. In Section 3.2 a method proposed by Lee and Robinson in Reference [36] is explained. Then, in Section 3.3, the perturbation approach is presented and it is compared to the method discussed in Section 3.2.

3.2 The Lee-Robinson Method

The Robinson-Harris Method presented in the previous chapter was extended to equal eigenvalues (see Reference [49]) but was carried out in

an ad hoc method where the equations used lost their sparse character and suffered a deterioration of conditioning. Later, Lee and Robinson [36] addressed the problem of developing a better approximate method for close or repeated eigenvalues. However, their method is an iterative scheme, which is successful if the eigenvalues are equal or very close. As in the method of Chapter 2, very small masses call for a perturbation approach rather than an iterative one. The basic idea of the Lee-Robinson Method will be presented as a perturbation technique in Section 3.3. A brief summary of the original method is given here as a necessary introduction.

The Lee-Robinson Method finds all eigenvectors corresponding to multiple or close eigenvalues at the same time. The method finds the close eigenvalues for any cluster, together with the corresponding eigenvectors in a two-step procedure. The essence of the method consists first in finding the subspace spanned by the eigenvectors corresponding to the multiple or close eigenvalues. The subspace is found by minimizing a quadratic form subject to the constraint that the result is a new set that is also orthonormal with respect to the mass matrix M . After this step is completed for all the equal or close eigenvalues, the improved eigenvalues and eigenvectors are determined from the solution of a small eigenvalue problem.

The method considers the eigenvalue problem for the system:

$$KX_i = \lambda_i MX_i \quad i=1, \dots, n \quad (3.1)$$

where K and M are symmetric matrices of order n ,

M is positive definite,

X_i is an eigenvector, and

λ_i is an eigenvalue such that $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$

The set of all eigenvalues is then separated into two sets, one of them containing the s close or multiple eigenvalues. The s -dimensional subspace spanned by the X_j corresponding to close or multiple eigenvalues is denoted by R . A set of s vectors, Y_j , in the neighborhood of R is then selected so that they are orthonormal with respect to M . Figure 3.2 shows a schematic representation of the two frames X_i and Y_j , where $i=1,2,\dots,n$ and $j=1,2,\dots,s$. Each vector Y_j of the basis can be expanded as a series of the true eigenvectors X_i . Thus

$$Y_j = \sum_{i=1}^n c_{ij} X_i \quad j \in S = \{1, \dots, s\} \quad (3.2)$$

For close or equal eigenvalues, the coefficients fall into two groups of different order of magnitude,

$$\sum_{i \notin S} c_{ij}^2 \ll \sum_{i \in S} c_{ij}^2 \quad (3.3)$$

for sufficiently good approximations.

Equation (3.3) can be interpreted as the summation of the Euclidean norms of the projections of vectors of Y_j ($j \in S$) onto the two complementary subspaces. This means that the vectors Y_j are nearly orthogonal to the set of $n - s$ eigenvectors corresponding to the simple eigenvalues. However, a vector Y_j needs not be close to any one of the X_j ($j \in S$) in particular. A constrained stationary-value problem is set up to find the stationary values of

$$W = \sum_{j \in S} Y_j^T K Y_j \quad (3.4)$$

subject to

$$Y_i^T M Y_j = \delta_{ij} \quad (i, j \in S) \quad (3.5)$$

The fact that this stationary-value problem characterizes the subspace R of the eigenvectors X_j ($j \in S$) is proved in Reference [36]. Lagrange multipliers are introduced to solve the constrained stationary value problem. The Lagrangian resulting from Equations (3.4) and (3.5) can be written as

$$L = \sum_{i \in S} Y_i^T K Y_i - \sum_{i \in S} \sum_{j \in S} \beta_{ij} (Y_i^T M Y_j - \delta_{ij}) \quad (3.6)$$

where the β_{ij} 's form an $s \times s$ matrix of Lagrange multipliers.

The minimization yields

$$\frac{\partial L}{\partial Y_j} = 0; \quad K Y_j = \sum_{i \in S} \beta_{ij} M X_i \quad (j \in S) \quad (3.7)$$

$$\frac{\partial L}{\partial \beta_{ij}} = 0; \quad Y_i^T M Y_j = \delta_{ij} \quad (i, j \in S) \quad (3.8)$$

The first equation resembles an eigenvalue problem for matrices K and M . In matrix form, the equations are

$$K Y_j = M [Y] b_j \quad (j \in S) \quad (3.9)$$

where $b_j^T = (\beta_{1j}, \beta_{2j}, \dots, \beta_{sj})$, ($j \in S$), or

$$K[Y] = M[Y]B \quad (3.10)$$

where $B = (b_1, b_2, \dots, b_s)$, and

$$[Y]^T M[Y] = I_s \quad (3.11)$$

where $[Y] = [Y_1 \ Y_2 \ \dots \ Y_s]$.

Thus, the subspace R of the desired eigenvectors can be found by solving Equations (3.9) and (3.11).

The Newton-Raphson technique is then applied to the above equations. The initial values for B and $[Y]$ are assumed to be available. They are denoted by $B^{(0)}$ and $[Y^{(0)}]$. As mentioned above, it is assumed that the basis vectors forming $[Y^{(0)}]$ are in the neighborhood of the subspace of the eigenvectors.

$$[X] = [X_1 \ X_2 \ \dots \ X_s] \quad (3.12)$$

and that they satisfy the orthonormality condition with respect to M , that is

$$[Y^{(0)}]^T M[Y^{(0)}] = I_s \quad (3.13)$$

For the general k th iteration step, Lee and Robinson use

$$b_j^{(k+1)} = b_j^{(k)} + \Delta b_j^{(k)} \quad (3.14)$$

$$Y_j^{(k+1)} = Y_j^{(k)} + \Delta Y_j^{(k)} \quad (3.15)$$

where $\Delta b_j^{(k)}$ and $\Delta Y_j^{(k)}$ are unknown incremental values for $b_j^{(k)}$ and $Y_j^{(k)}$. By substituting $b_j^{(k+1)}$ and $Y_j^{(k+1)}$ into the above equations, and arrive at

$$\begin{aligned} K\Delta Y_j^{(k)} - M[Y_j^{(k)}] \Delta b_j^{(k)} &= -KY_j^{(k)} + M[Y_j^{(k)}] b_j^{(k)} \\ &+ M[\Delta Y_j^{(k)}] b_j^{(k)} \end{aligned} \quad (3.16)$$

and

$$[Y^{(k)}]^T M[Y^{(k)}] + 2[Y^{(k)}]^T M[\Delta Y^{(k)}] = I_s \quad (3.17)$$

The matrix B has zero off-diagonal terms for multiple roots. Thus, one starts with a certain matrix $B^{(0)}$ and a number of iterations later, when approaching convergence one should find that $B^{(k)}$ has either zero off-diagonal terms or that they are very small compared to the diagonal ones. For this reason the term $M[Y^{(k)}] b_j$ may be approximated by $\beta_{jj}^{(k)} M\Delta Y_j^{(k)}$, yielding

$$(K - \beta_{jj}^{(k)} M) \Delta Y_j^{(k)} - M[Y^{(k)}] \Delta b_j^{(k)} = -KY_j^{(k)} + M[Y^{(k)}] b_j^{(k)} \quad (3.18)$$

Equation (3.18) represents a set of n equations in $n + s$ unknowns that combined with the set of s equations in n unknowns in (3.17) yield a system of $n + s$ equations in $n + s$ unknowns.

Taking

$$[Y^{(k)}]^T M [Y^{(k)}] = I_s \quad (3.19)$$

Equation (3.17) becomes

$$[Y^{(k)}]^T M \Delta Y_j^{(k)} = 0 \quad (3.20)$$

Finally, combining Equations (3.18) and (3.20) yields

$$\begin{bmatrix} K - \beta_{jj}^{(k)} M & -M[Y^{(k)}] \\ \hline -[Y^{(k)}]^T M & 0 \end{bmatrix} \begin{Bmatrix} \Delta Y_j^{(k)} \\ \Delta b_j^{(k)} \end{Bmatrix} = \begin{Bmatrix} -r_j^{(k)} \\ 0 \end{Bmatrix} \quad (j \in S) \quad (3.21)$$

that is, a system of $n + s$ equations in $n + s$ unknowns.

When the roots are not exactly equal, the terms neglected in (3.18) are dropped as an approximation and the calculation is carried out as if the roots were multiple. When the roots are fairly far apart, the convergence of the procedure is adversely affected by this approximation. If the approximation of $M[Y^{(k)}]b_j$ by $\beta_{jj}^{(k)} M \Delta Y_j^{(k)}$ is not made, all s systems of the type (3.21) would be coupled by the extra terms. This would be a computationally unacceptable procedure. It should be noted that the closer the eigenvalues turn out to be the more rapid the convergence of the method will be.

A second step is needed to complete the problem. The matrix B of the Lagrangian coefficients has to be diagonalized (rotated) so that the Y vectors associated with B become the true eigenvectors of the subspace. This system of s eigenvectors and corresponding eigenvalues may be written as

$$K [X] = M [X] \Lambda \quad (3.22)$$

where $\Lambda = \text{diag} (\lambda_1, \lambda_2, \dots, \lambda_s)$.

If the rotation is expressed by a square matrix $[Z]$ of order s ,

$$[X] = [Y] [Z] \quad (3.23)$$

and (3.10)

$$K [Y] = M [Y] B \quad (3.10)$$

is taken into account, one has

$$K [Y] [Z] = M [Y] [Z] \Lambda = M [Y] B [Z] \quad (3.24)$$

Then, premultiplying by $[Y]^T$

$$[Y]^T M [Y] [Z] \Lambda = [Y]^T M [Y] B [Z] \quad (3.25)$$

and use of the orthonormality condition results in

$$B [Z] = [Z] \Lambda \quad (3.26)$$

Thus the second stage of the problem is reduced to the solution of an eigenvalue problem of order s .

3.3 Perturbation Scheme. Multiple Roots.

The method of solution that was presented in Chapter 2 breaks down for the case of systems that have either repeated or very close eigenvalues. It is for this reason that the approach to be followed in the solution of the problem has to be different. The work presented in this section uses the basic idea behind the method developed by Lee and Robinson [36] and explained in the previous section.

The Lee-Robinson Method is now implemented by means of series expansions for the Lagrangian coefficients β as well as for the Y vectors. Consider the eigenvalue problem for the augmented system represented by Equation (2.9a). In the detuned case, the solution for such a system is obtained one mode at a time. Moreover, the side condition is chosen as a normality condition for the old modes, thus Equations (2.26) are used. However, when considering a new mode, the scale could be left undetermined because normality is not an essential condition in the method. This approach leads to the side condition for the modes of the added system represented by Equation (2.48). By contrast, in the tuned case the solution must be obtained by dealing with the entire subspace of repeated modes. As explained in Section 3.2, the Lee-Robinson Method requires orthonormality among all the Y vectors. This normality condition forces the introduction of a $\mu^{-1/2}$ term in the expansion series for the Y vectors. In this way, the normality condition is satisfied for all the basis vectors Y regardless of whether they originate from the old eigenvectors or the new ones.

Taking $\epsilon = \mu^{1/2}$, the expansion utilized for the Lagrangian coefficients β is the following

$$\beta_{ij} = \beta_{ij}^{(0)} + \beta_{ij}^{(1)} \epsilon + \beta_{ij}^{(2)} \epsilon^2 + \dots \quad (i, j \in S) \quad (3.27)$$

and for the vector Y

$$Y_j = Y_j^{(-1)} \epsilon^{-1} + Y_j^{(0)} + Y_j^{(1)} \epsilon + Y_j^{(2)} \epsilon^2 + \dots \quad (j \in S) \quad (3.28)$$

Accordingly, it is convenient to choose the following formulation for the stiffness and mass matrices of the augmented system,

$$K(\epsilon) = K^{(0)} + \epsilon^2 K^{(2)} \quad (3.29)$$

and

$$M(\epsilon) = M^{(0)} + \epsilon^2 M^{(2)} \quad (3.30)$$

where the matrices $K^{(0)}$ and $M^{(0)}$ are the bordered stiffness and mass matrices of the initial system and $K^{(2)}$ and $M^{(2)}$ are the same augmentation matrices defined in the previous chapter.

Equations (3.27), (3.28), (3.29) and (3.30) can now be substituted into Equations (3.9)

$$\begin{aligned} & (K^{(0)} + \epsilon^2 K^{(2)}) (Y_j^{(-1)} \epsilon^{-1} + Y_j^{(0)} + Y_j^{(1)} \epsilon + Y_j^{(2)} \epsilon^2 + \dots) \\ & = (M^{(0)} + \epsilon^2 M^{(2)}) [Y_1 \ Y_2 \dots \ Y_s] b_j \quad (j \in S) \end{aligned} \quad (3.31)$$

where b_j represents the j th column of the matrix B of Lagrangian coefficients.

Equation (3.31) is now expanded for a certain $j = r$ such that $1 \leq r \leq s$. Rearranging the terms by increasing order of exponent in the parameter ϵ , Equation (3.31) becomes, for order $\alpha = -1$,

$$K^{(0)} Y_r^{(-1)} = \sum_{q=1}^s M^{(0)} Y_q^{(-1)} \beta_{qr}^{(0)} \quad (3.32)$$

for order $\alpha = 0$,

$$K^{(0)} Y_r^{(0)} = \sum_{q=1}^S M^{(0)} (Y_q^{(0)} \beta_{qr}^{(0)} + Y_q^{(-1)} \beta_{qr}^{(1)}) \quad (3.33)$$

for order $\alpha \geq 1$,

$$\begin{aligned} K^{(0)} Y_r^{(\alpha)} + K^{(2)} Y_r^{(\alpha-2)} &= M^{(0)} \sum_{q=1}^S \sum_{k=0}^{\alpha+1} Y_q^{(\alpha-k)} \beta_{qr}^{(k)} \\ &+ M^{(2)} \sum_{q=1}^S \sum_{k=0}^{\alpha-1} Y_q^{(\alpha-2-k)} \beta_{qr}^{(k)} \end{aligned} \quad (3.34)$$

In addition, Equations (3.28) and (3.30) can be substituted into equation (3.11)

$$\begin{bmatrix} Y_1^T \\ \vdots \\ Y_q^T \\ \vdots \\ Y_s^T \end{bmatrix} [M^{(0)} + \epsilon^2 M^{(2)}] [Y_1 \dots Y_q \dots Y_s] = I_s \quad (3.35)$$

for any q and r ,

$$Y_q^T [M^{(0)} + \epsilon^2 M^{(2)}] Y_r = \delta_{qr} \quad (3.36)$$

Substituting Equation (3.28) for $j = q$ and for $j = r$ into Equation (3.36) yields

$$\begin{aligned} (Y_q^{(-1)T} \epsilon^{-1} + Y_q^{(0)T} + Y_q^{(1)T} \epsilon + \dots) [M^{(0)} + \epsilon^2 M^{(2)}] \\ (Y_r^{(-1)} \epsilon^{-1} + Y_r^{(0)} + Y_r^{(1)} \epsilon + \dots) = \delta_{qr} \end{aligned} \quad (3.37)$$

performing the indicated multiplications and grouping by powers of ϵ ,

$$Y_q^{(-1)T} M^{(0)} Y_r^{(-1)} = 0 \quad (3.38)$$

$$Y_q^{(-1)T} M^{(0)} Y_r^{(0)} + Y_q^{(0)T} M^{(0)} Y_r^{(-1)} = 0 \quad (3.39)$$

$$\begin{aligned} Y_q^{(-1)T} M^{(0)} Y_r^{(1)} + Y_q^{(-1)T} M^{(2)} Y_r^{(-1)} + Y_q^{(0)T} M^{(0)} Y_r^{(0)} \\ + Y_q^{(1)T} M^{(0)} Y_r^{(-1)} = \delta_{qr} \end{aligned} \quad (3.40)$$

and for $\alpha \geq 1$

$$\sum_{k=0}^{\alpha+2} Y_q^{(\alpha-k+1)T} M^{(0)} Y_r^{(k-1)} + \sum_{k=0}^{\alpha} Y_q^{(\alpha-k-1)T} M^{(2)} Y_r^{(k-1)} = 0 \quad (3.41)$$

Equations (3.38), (3.39), (3.40) can be satisfied identically by making the proper choice of matrices $[Y^{(-1)}]$ and $[Y^{(0)}]$.

The choice is the following: ℓ of the s vectors Y will be of the form

$$Y_\ell^{(-1)} = \{0\}$$

and

$$Y_\ell^{(0)} = \left\{ \begin{array}{c} \phi_\ell^{(0)} \\ \hline 0 \end{array} \right\}$$

where $\phi_\ell^{(0)}$ corresponds to one of the eigenvectors of the repeated eigenvalue when considered for the initial system. The remaining $s-\ell$ vectors Y will be of the form

$$Y_{\ell-s}^{(-1)} = \left\{ \begin{array}{c} 0 \\ \psi_{\ell-s}^{(-1)} \end{array} \right\}$$

and

$$Y_{\ell-s}^{(0)} = \{0\}$$

where $\psi_{\ell-s}^{(-1)}$ corresponds to one of the eigenvectors of the repeated eigenvalue for the added system alone. Thus,

$$[Y^{(-1)}] = \left[\begin{array}{ccc|ccc} 0 & \dots & 0 & 0 & \dots & 0 \\ \hline 0 & \dots & 0 & \psi_{\ell+1}^{(-1)} & \dots & \psi_s^{(-1)} \end{array} \right] \quad (3.42)$$

and

$$[Y^{(0)}] = \left[\begin{array}{ccc|ccc} \phi_1^{(0)} & \dots & \phi_\ell^{(0)} & 0 & \dots & 0 \\ \hline 0 & \dots & 0 & 0 & \dots & 0 \end{array} \right] \quad (3.43)$$

Also $B^{(0)}$, the initial approximation to the matrix of Lagrangian coefficients, can be chosen as a diagonal matrix containing the repeated eigenvalues. That is,

$$B^{(0)} = \left[\begin{array}{cccccc} \lambda_1 & & & & & 0 \\ & \cdot & & & & \\ & & \lambda_\ell & & & \\ & 0 & & \lambda_{\ell+1} & & \\ & & & & \cdot & \lambda_s \end{array} \right] \quad (3.44)$$

Using the above representations, Equations (3.32) and (3.33) can be satisfied identically.

Rearranging Equation (3.34) yields,

$$\begin{aligned}
& (K^{(0)} - \beta_{rr}^{(0)} M^{(0)}) Y_r^{(\alpha)} - M^{(0)} \sum_{q=1}^s Y_q^{(0)} \beta_{qr}^{(\alpha)} = -K^{(2)} Y_r^{(\alpha-2)} \\
& + M^{(0)} \left[\sum_{\substack{k=1 \\ k \neq \alpha}}^{\alpha+1} Y_r^{(\alpha-k)} \beta_{rr}^{(k)} + \sum_{\substack{k=0 \\ k \neq \alpha}}^{\alpha+1} \sum_{\substack{q=1 \\ q \neq r}}^s Y_q^{(\alpha-k)} \beta_{qr}^{(k)} \right] \\
& + M^{(2)} \sum_{q=1}^s \sum_{k=0}^{\alpha-1} Y_q^{(\alpha-2-k)} \beta_{qr}^{(k)} \tag{3.48}
\end{aligned}$$

From Equation (3.41),

$$\begin{aligned}
& \sum_{\substack{k=0 \\ k \neq \alpha+1}}^{\alpha+2} Y_q^{(\alpha-k+1)} M^{(0)} Y_r^{(k-1)} + \sum_{\substack{k=0 \\ k \neq \alpha-1}}^{\alpha} Y_q^{(\alpha-k-1)} M^{(2)} Y_r^{(k-1)} \\
& + Y_q^{(0)} M^{(0)} Y_r^{(\alpha)} + Y_q^{(0)} M^{(2)} Y_r^{(\alpha-2)} = 0 \tag{3.49}
\end{aligned}$$

or

$$\begin{aligned}
Y_q^{(0)} M^{(0)} Y_r^{(\alpha)} + Y_q^{(0)} M^{(2)} Y_r^{(\alpha-2)} &= \sum_{\substack{k=0 \\ k \neq \alpha+1}}^{\alpha+2} Y_q^{(\alpha-k+1)} M^{(0)} Y_r^{(k-1)} \\
&+ \sum_{\substack{k=0 \\ k \neq \alpha-1}}^{\alpha} Y_q^{(\alpha-k-1)} M^{(2)} Y_r^{(k-1)} \tag{3.50}
\end{aligned}$$

Notice that the term $Y_q^{(0)} M^{(2)} Y_r^{(\alpha-2)}$ is always zero because of Equation (3.43).

Equation (3.50) represents a set of s independent equations. Thus, Equation (3.50) together with Equation (3.48) form a set of $n + s$ equations in $n + s$ unknowns,

$$\left[\begin{array}{c|ccc} [K^{(0)} - \beta_{rr}^{(0)} M^{(0)}] & -M^{(0)} [Y_1^{(0)} Y_2^{(0)} \dots Y_s^{(0)}] & & \\ \hline -Y_1^{(0)T} M^{(0)} & & 0 & \\ -Y_2^{(0)T} M^{(0)} & & 0 & \\ \vdots & & \vdots & \\ -Y_s^{(0)T} M^{(0)} & & 0 & \end{array} \right] \begin{pmatrix} Y_r^{(\alpha)} \\ \beta_{1r}^{(\alpha)} \\ \beta_{2r}^{(\alpha)} \\ \vdots \\ \beta_{sr}^{(\alpha)} \end{pmatrix} = \begin{pmatrix} P_r^{(\alpha)} \\ Q_{1r}^{(\alpha)} \\ Q_{2r}^{(\alpha)} \\ \vdots \\ Q_{sr}^{(\alpha)} \end{pmatrix} \quad (3.51)$$

where

$$P_r^{(\alpha)} = M^{(0)} \left[\sum_{\substack{k=1 \\ k \neq \alpha}}^{\alpha+1} Y_r^{(\alpha-k)} \beta_{rr}^{(k)} + \sum_{\substack{k=0 \\ k \neq \alpha}}^{\alpha+1} \sum_{\substack{q=1 \\ q \neq r}}^s Y_q^{(\alpha-k)} \beta_{qr}^{(k)} \right] - K^{(2)} Y_r^{(\alpha-2)} + M^{(2)} \sum_{q=1}^s \sum_{k=0}^{\alpha-1} Y_q^{(\alpha-2-k)} \beta_{qr}^{(k)} \quad (3.52)$$

$$Q_{qr}^{(\alpha)} = \sum_{\substack{k=0 \\ k \neq \alpha+1}}^{\alpha+2} Y_q^{(\alpha-k+1)} M^{(0)T} Y_r^{(k-1)} + \sum_{\substack{k=0 \\ k \neq \alpha-1}}^{\alpha} Y_q^{(\alpha-k-1)} M^{(2)T} Y_r^{(k-1)} \quad (3.53)$$

The coefficient matrix in Equation (3.51) can be further reduced. Notice that by using Equations (2.9c) and (2.9f),

$$(K - \beta_{rr}^{(0)} M^{(0)}) = \begin{bmatrix} K - \beta_{rr}^{(0)} M & 0 \\ 0 & 0 \end{bmatrix} \quad (3.54)$$

Substituting Equation (3.43) and Equation (3.54) into Equation (3.51) yields the following two equations,

$$\begin{bmatrix} K - \beta_{rr}^{(0)} M & -M[\phi_1^{(0)} \phi_2^{(0)} \dots \phi_\ell^{(0)}] \\ -\phi_1^{(0)T} M & 0 \\ -\phi_2^{(0)T} M & 0 \\ \vdots & \vdots \\ -\phi_\ell^{(0)T} M & 0 \end{bmatrix} \begin{Bmatrix} \phi_r^{(\alpha)} \\ \beta_{1r}^{(\alpha)} \\ \beta_{2r}^{(\alpha)} \\ \vdots \\ \beta_{\ell r}^{(\alpha)} \end{Bmatrix} = \begin{Bmatrix} P_r^{(\alpha)} \\ Q_{1r}^{(\alpha)} \\ Q_{2r}^{(\alpha)} \\ \vdots \\ Q_{\ell r}^{(\alpha)} \end{Bmatrix} \quad (3.55)$$

where

$$\begin{aligned} P_r^{(\alpha)} = & M \left[\sum_{\substack{k=1 \\ k \neq \alpha}}^{\alpha+1} \phi_r^{(\alpha-k)} \beta_{rr}^{(k)} + \sum_{\substack{k=0 \\ k \neq \alpha}}^{\alpha+1} \sum_{\substack{q=1 \\ q \neq r}}^s \phi_q^{(\alpha-k)} \beta_{qr}^{(k)} \right] - K_{11}^{(2)} \phi_r^{(\alpha-2)} \\ & - K_{12}^{(2)} \psi_r^{(\alpha-2)} + M_{11}^{(2)} \sum_{q=1}^s \sum_{k=0}^{\alpha-1} \phi_q^{(\alpha-2-k)} \beta_{qr}^{(k)} \\ & + M_{12}^{(2)} \sum_{q=1}^s \sum_{k=0}^{\alpha-1} \psi_q^{(\alpha-2-k)} \beta_{qr}^{(k)} \end{aligned} \quad (3.56)$$

$$Q_{qr}^{(\alpha)} = \sum_{\substack{k=0 \\ k \neq \alpha+1}}^{\alpha+2} \phi_q^{(\alpha-k+1)T} M_{\phi_r}^{(k-1)} + \sum_{\substack{k=0 \\ k \neq \alpha-1}}^{\alpha} \psi_q^{(\alpha-k-1)T} M_{22}^{(2)} \psi_r^{(k-1)} \quad (3.57)$$

and

$$\begin{aligned}
 & -K_{21}^{(2)} \phi_r^{(\alpha-2)} - K_{22}^{(2)} \psi_r^{(\alpha-2)} + M_{21}^{(2)} \sum_{q=1}^s \sum_{k=0}^{\alpha-1} \phi_q^{(\alpha-2-k)} \beta_{qr}^{(k)} \\
 & + M_{22}^{(2)} \sum_{q=1}^s \sum_{k=0}^{\alpha-1} \psi_q^{(\alpha-2-k)} \beta_{qr}^{(k)} = 0
 \end{aligned} \tag{3.58}$$

Equation (3.55) is of order $n + \ell$ and it is solvable because the matrix of coefficients is non-singular (see the Appendix of Reference [34]).

In solving the proposed perturbation problem, a definite sequence of calculations is required. Equation (3.55) is used first to find some of the parameters of the α th order perturbation and only then is Equation (3.58) used to complete the calculation.

A specific example is useful to illustrate the proposed method. Consider an augmented system like the one shown in Figure 3.1, consisting of an initial 2-degree-of-freedom system with two repeated eigenvalues to which a 2-degree-of-freedom subsystem has been added. Consider then the case when the added subsystem has also two repeated eigenvalues equal to those of the initial system; hence the combined system has only one eigenvalue of multiplicity four. For such a system, substituting Equation (3.42) into Equation (3.55) and taking $\alpha=1$ yields

$$\begin{bmatrix}
 K - \beta_{rr}^{(0)} M & -M\phi_1^{(0)} & -M\phi_2^{(0)} \\
 \hline
 -\phi_1^{(0)T} M & 0 & 0 \\
 -\phi_2^{(0)T} M & 0 & 0
 \end{bmatrix}
 \begin{Bmatrix}
 \phi_r^{(1)} \\
 \beta_{1r}^{(1)} \\
 \beta_{2r}^{(1)}
 \end{Bmatrix}
 =
 \begin{Bmatrix}
 -K_{12}^{(2)} \psi_r^{(-1)} \\
 0 \\
 0
 \end{Bmatrix} \tag{3.59}$$

and for $\alpha=2$, Equation (3.58) gives

$$-K_{21}^{(2)} \phi_r^{(0)} + M_{22}^{(2)} \sum_{q=3}^4 \psi_q^{(-1)} \beta_{qr}^{(1)} = 0 \quad (3.60)$$

Equations (3.59) and (3.60) can be solved for each mode r of the subspace R . Notice then that the solution must be carried out a definite sequence, first for the α th perturbation and then for the $(\alpha+1)$ th perturbation. Also the remaining $s-l$ orthonormality conditions that are not needed in Equation (3.55) are instead used to find the values for the ψ 's for the corresponding $(\alpha-1)$ th perturbation. Once the solution for each of the modes r is obtained, it is possible to continue to the next order by taking $\alpha=2$ in Equation (3.55),

$$\begin{bmatrix} K - \beta_{rr}^{(0)} M & -M\phi_1^{(0)} & -M\phi_2^{(0)} \\ \hline -\phi_1^{(0)T} M & 0 & 0 \\ -\phi_2^{(0)T} M & 0 & 0 \end{bmatrix} \begin{Bmatrix} \phi_r^{(2)} \\ \beta_{1r}^{(2)} \\ \beta_{2r}^{(2)} \end{Bmatrix} = \begin{Bmatrix} P_r^{(2)} \\ Q_{1r}^{(2)} \\ Q_{2r}^{(2)} \end{Bmatrix} \quad (3.61)$$

where

$$\begin{aligned} P_r^{(2)} &= M \sum_{q=1}^4 \phi_q^{(1)} \beta_{qr}^{(1)} - K_{11}^{(2)} \phi_r^{(0)} + M_{11}^{(2)} \sum_{q=1}^4 \phi_q^{(0)} \beta_{qr}^{(0)} \\ &+ M_{12}^{(2)} \sum_{q=1}^4 \psi_q^{(-1)} \beta_{qr}^{(1)} \end{aligned}$$

and

$$Q_{qr}^{(2)} = \phi_q^{(1)T} M \phi_r^{(1)} + \psi_q^{(1)T} M_{22}^{(2)} \psi_r^{(-1)} + \psi_q^{(-1)T} M_{22}^{(2)} \psi_r^{(1)}$$

and for $\alpha=3$ in Equation (3.58),

$$-K_{21}^{(2)} \phi_r^{(1)} - K_{22}^{(2)} \psi_r^{(1)} + M_{22}^{(2)} \sum_{q=1}^4 \sum_{k=0}^2 \psi_q^{(1-k)} \beta_{qr}^{(k)} = 0 \quad (3.62)$$

Higher order perturbations may be obtained in a similar way.

The above procedure can be simplified considerably by developing a computer program that can automatically generate the right-hand sides for Equations (3.55) and (3.58) for each order of perturbation. In Chapter 4, such a computer implementation is explained.

It should be noted that the perturbation calculations given here and in Chapter 2 are, at this point, just formal expansions. That the perturbation series converge for small enough values of the parameter ϵ (or μ in the detuned case) has not been demonstrated here.

It might be pointed out here that the perturbation procedure developed in this Chapter differs considerably from the one presented by DerKiureghian et al. [10]. In fact, the method employed in [28] and [29] might be more properly called a correction procedure. It does not give the entire first perturbation but does arrive at the same frequencies for the modes of tuned system obtained by the first perturbation in the method presented here. The procedure [10] cannot easily be extended to higher perturbations, especially in view of the neglect of part of the diagonal terms of the stiffness matrix.

4. NUMERICAL APPLICATIONS OF THE PERTURBATION PROCEDURES

4.1 Computer Implementation of the Perturbation Method

The difficult part of any perturbation problem is the determination of the successive right-hand sides of the sets of perturbation equations in terms of the results of perturbation solutions already carried out. In a program having the capability of calculating perturbations to a fairly high order, the number of terms on the right sides can become quite large for the higher-order perturbations.

A natural way to proceed would be to determine which terms occur on the right in exactly the same way as would be done in analytical calculation, and then to program the result. However, it has been found much more convenient to have the computer determine which terms occur and which do not. Then all that must be provided in the program is the general nonlinear operator that occurs in the problem. Simple logical calculations determine which terms in the sums of products of perturbation expansions arise for the stage of the perturbation under consideration. Those terms, and only those, are then calculated. Such a course of computation is not only more generally useful in perturbation calculations, but also requires much less programming than any specific case.

To carry out this method of calculation, the scalars, vectors, and matrices that contain the perturbation parameters are expressed as what will be termed "perturbation arrays." For instance, if six perturbations

are desired, an eigenvalue will be expressed in terms of six constants, the coefficients of the series

$$\lambda(\mu) = \sum_{i=0}^5 \lambda^{(i)} \mu^i \quad (4.1)$$

Similarly, eigenvectors will be expressed as an array of six vectors. It turns out that in the problems studied here, only two matrices are needed to form the perturbation array for K and M. The members of each array are marked by counters. In the example just given, the counters for the eigenvalues and eigenvectors will range from zero to five, and for the matrices from zero to one. (In the case of multiple eigenvalues, the counters for the matrices assume the values zero and two.) As will be shown, it is important to set all the unknown arrays equal to zero before the perturbation solution is started.

Detailed flow diagrams are given in the succeeding sections first for the case of separated eigenvalues and then for the repeated ones.

4.2 Separate Roots

The initial data necessary to run the program consist of the dynamic properties of each one of the separate structural systems that constitute the augmented system. That is, the stiffness and mass matrices corresponding to each system as well as their corresponding eigenvalues and eigenvectors. Thus, following the notation of Chapter 2, the starting values consist of $K^{(0)}$, $K^{(1)}$, $M^{(0)}$, $M^{(1)}$, $\phi_1^{(0)}$, ..., $\phi_n^{(0)}$, $\psi_1^{(0)}$, ..., $\psi_{n+m}^{(0)}$, and $\lambda_1^{(0)}$, ..., $\lambda_{n+m}^{(0)}$ and also

$$X_i^{(1)}, \dots, X_i^{(p)} = 0 \quad i=1, \dots, n+m$$

where

$$X_1^{(\alpha)} = \left\{ \begin{array}{c} \phi_1^{(\alpha)} \\ \psi_1^{(\alpha)} \end{array} \right\}$$

and

$$\lambda_i^{(1)}, \dots, \lambda_i^{(p)} = 0 \quad i=1, \dots, n+m$$

where p is the desired order of expansion.

The program assigns one counter to each of the terms involved in the expansion of Equation (2.9a) in the following way: $K(\mu)$ and $M(\mu)$ are assigned one counter, $X(\mu)$ is assigned another counter and finally $\lambda(\mu)$ is assigned a third counter.

Consider now how to evaluate a term in a product, say, λMX , that occurs in the solution. The sums of all possible counters for the three terms are found. Those for which the sum is a certain integer α , $\alpha \in \{1, \dots, p\}$, the order of perturbation being studied, are calculated and accumulated. The others, for which the sum of the counters is not equal to α , are not calculated. When all possibilities for the values of the counters have been exhausted, the contribution of the right-hand side for that product has been correctly calculated.

One point remains to be clarified. What happens to the term containing the unknown scalar or vector being calculated in the α th perturbation (where the other counters would usually be zero)? These products when calculated for the right-hand side are all zero because the values in question have not yet been substituted into the corresponding perturbation array. The initialization of the arrays to zero guarantees

that terms that are on the left-hand side of the perturbation equations do not also occur on the right. It is easier to perform a few unnecessary computations than to complicate the program by omitting those that really belong on the left. Since the forms of the left sides of the perturbation equations never change, it is only the perturbation number α that indicates which order of perturbation is being calculated and, therefore, where in the arrays the answers should be substituted once found. After this process is finished, the next perturbation can be started.

4.2.1 Old Modes

The procedure for the automatic calculation of the perturbation coefficients is different according to whether the modes of vibration of the augmented system belong to the initial structure, in which case they are called "old" modes, or to the added system, the "new" modes. This classification is the same as presented in Section 2.4 of Chapter 2 where the modes are separated into modes that are very close to the ones from the initial structure as opposed to those that are very close to the ones from the added system. The case of the "old" modes will be treated in this section. The flow diagram for the calculation of the right-hand side of the perturbation expansions represented by Equation (2.41) is presented in Figure 4.1. As the program advances progressively through every order of perturbation in the way explained above, every counter is varied from an initial value to a final value. An important point that must be mentioned is that the solution for each order of expansion must be accomplished in a staggered sequence. First one solves for $\psi_i^{(\alpha-1)}$, a

vector of order m , and then updating the current values, redo the calculation for the right-hand side of Equation (2.41). This time the result are the values $P_i^{(\alpha)}$ and $Q_i^{(\alpha)}$ that correspond to the proper right-hand side for Equation (2.41). More specifically, first the perturbation expansion of first order for the Equation (2.9a) is calculated. The result of this calculation is a vector of order $n+m$, whose lower portion (a vector of order m) is the right-hand side of Equation (2.19). Thus it is possible to solve Equation (2.19) for $\psi_i^{(0)}$. The resulting value can be stored in the corresponding member of the perturbation array for the eigenvector. Now, redo the calculation of the perturbation expansion of the first order for Equation (2.9a). Again, the result will be a vector of order $n+m$, however, this time its upper portion (a vector of order n) is selected and saved. Concurrently with the calculation performed for equation (2.9a) the expansion of Equation (2.24) for $i=j$, is also performed (see Figure 4.1). Now the above mentioned vector of order n that corresponds to the upper portion of the vector containing the expansion for the right-hand side of Equation (2.9a) together with the result of the expansion of Equation (2.24) can be assembled into an $n+1$ vector that corresponds to the right hand side of Equation (2.41) for a perturbation of first order, i.e., $P_i^{(1)}$ and $Q_i^{(1)}$. Finally, this new equation is solved and thus the calculation for the first-order perturbation is completed. This same procedure is repeated for each order of perturbation up until the desired perturbation number is reached.

It is important to mention at this point that the matrix of coefficients of Equation (2.41) is indefinite (i.e., not positive

definite). Thus, the algorithm to be employed for the solution of the equations must be special in order to avoid problems of stability. The algorithm proposed by Bunch and Parlett [6] and used in this work emphasizes the preservation of the symmetry of the matrix of coefficients and it represents a stable solution to the problem.

4.2.2 New Modes

In the case of the new modes the procedure is slightly different. However, the solution is still attained in staggered calculations. The difference comes from the fact that there is an initial step that must take place before the general algorithm is initiated.

The procedure for the calculation is performed as follows. The expansion of the perturbation equation for the first order is carried out and then the "top" part of the resulting vector is selected and saved. This is really the negative of the right-hand side of Equation (2.47), thus Equation (2.47) can be solved for $\phi_j^{(1)}$ and the result can be stored in the corresponding perturbation array. Once this first step is done, a somewhat different algorithm is followed for the higher orders. It is now necessary to perform the expansion for a second order. This time the "bottom" part of the resulting vector is taken and used in conjunction with the expansion of the orthonormality condition to build up the right-hand side of Equation (2.58) for $\alpha=2$. Then the solution for Equation (2.58) is found and the resulting values are stored in the corresponding perturbation arrays. The process can be summarized as follows: find $\psi_j^{(\alpha)}$, $\lambda_j^{(\alpha)}$ and then $\phi_j^{(\alpha+1)}$, at this point proceed to a next order. Notice that for the new modes, in contrast to the case of old

modes, the first-order perturbation yields only $\phi_j^{(1)}$ and then the second and higher order perturbations are needed to find $\phi_j^{(\alpha)}$, $\psi_j^{(\alpha-1)}$ and $\lambda_j^{(\alpha-1)}$. Thus, the expansion must be carried to one order higher for the new modes than for the old modes. Then the perturbation arrays will be complete to a certain predefined level for both the old and the new modes. This extra calculation does increase the number of calculations somewhat, but as explained above, it insures that the expansion for all the modes can be carried out to the same predetermined order.

4.3 Repeated Roots

The necessary data to run the program is the same as required for the case of separate roots. In this case however, the augmentation mass and stiffness matrices are proportional to the parameter squared, not to the first power. That is, the notation expressed in Equations (3.29) and (3.30) is followed.

The terms involved in the perturbation expansion are assigned one counter each. The equations involved this time are Equations (3.31). That is, for each one of the repeated modes, an expansion must be made, (a mode coming from either the "primary" or "secondary" system). Thus, the same subroutine for the calculation of the right hand side of the perturbation expansion presented for the case of separate roots can be used for the case of multiple roots. See Figure 4.1. The procedure used is the same, for each product of the terms, the sum of the counters for the terms is found and if the sum is equal to α , the corresponding product is taken into account. Inspection of Equations (3.32), (3.33) and (3.34) shows that sum of the counters can be performed in the same

way as for the separate roots. Also the term containing the unknown scalar or vector being calculated in the α th perturbation are again zero simply because they have not been substituted into the corresponding perturbation array. As it is seen, the mechanisms for the calculation of the right-hand side of Equation (3.55) for the α th perturbation are essentially the same as for the case of separate roots. The expressions in Equation (3.55) and (3.58) do not have to be programmed as such but are rather calculated internally in the machine, which simplifies the proposed perturbation approach. The calculation of the perturbation coefficients once the right-hand side is obtained is also performed in staggered fashion.

The two step procedure consists of first solving Equation (3.55) for α and then solving Equation (3.58) for $\alpha+1$ to complete the calculation of the Lagrange multipliers β for each mode. Notice that the solution for the order of perturbation α must be obtained for each one of the modes in the subspace R before the solution for the next perturbation ($\alpha+1$)th may be attempted. The values for $\psi^{(\alpha)}$ are obtained from the $s-l$ remaining equations for the orthogonality condition.

4.4 Some Numerical Examples

The proposed method was tested by using it to solve the eigenvalue problem for four different structural systems. The first two are systems whose eigenvalues are all separated and the last two have close eigenvalues.

The structural system considered in Example 1 is shown in Figure 4.2. The primary system consists of a two-degree-of-freedom system to

which a single-degree-of-freedom secondary system is added. The dynamic characteristics of the system are given in the figure. The initial frequencies for the augmented system, i.e., when $\mu=0$, are $\lambda_1=1/2$, $\lambda_2=2$, for the primary system and $\lambda_3=1$ for the secondary system. The values for all three eigenvalues were calculated using the proposed method for the interval $0.0 \leq \mu \leq 0.4$ and using five different orders of perturbation (from first order to fifth order). The results for this calculation are presented in Tables 1 and 2 and are shown in Figures 4.3, 4.4 and 4.5. Figure 4.6 represents log-log plots of the errors in eigenvalues versus the parameter μ for the results carried through the first perturbation. Figure 4.7 corresponds to the results through the fifth perturbation. The slope of the curves in Figure 4.6 is essentially 2.0; in Figure 4.7 the slopes are 6.0. This provides an interesting verification of the correctness of the perturbation approach, since the major part of the error in the first perturbation is of the order of the second perturbation. Similarly the errors in the fifth perturbation are terms of the order μ^6 .

Several comparisons are made to give some indication of the accuracy of the proposed method. One measure of error that will be shown to be convenient is what is here termed relative error. This is defined as

$$\% \text{ relative error} = \frac{\lambda_{\text{exact}}(\mu) - \lambda_{\text{approx}}(\mu)}{\lambda_{\text{exact}}(\mu) - \lambda(0)} * 100 \quad (4.2)$$

where $\lambda(0)$ is the initial eigenvalue for the system and $\lambda_{\text{exact}}(\mu)$ is the eigenvalue calculated using a standard eigenvalue solver package (IMSL).

Figure 4.8 shows the relative error for all of the eigenvalues of the augmented structure in Example 1. It is worth noticing that the error is very small for values of up to $\mu=0.25$, less than 10%, when the perturbation approximation is carried out up to the fifth order. The improvement in the use of higher order perturbations is represented in Figure 4.9, where the relative errors for first and fifth order expansion are given for the lowest eigenvalue. It is evident from Figure 4.3 that the perturbations are growing beyond the third one for the value $\mu=0.4$.

The second example consists of a combined system formed by a four-degree-of-freedom primary system and a two-degree-of-freedom secondary system. Figure 4.10 shows the structural properties of the system under consideration. Figures 4.11 through 4.16 show the values for all six eigenvalues as calculated by the proposed method for the interval $0.0 \leq \mu \leq 0.4$ and, again, using five different orders of perturbation (from first to fifth order). In each figure the proposed method is compared to the exact solution. Figures 4.17 and 4.18 show that the perturbation series used is appropriate, since the slopes in the graphs are 2 and 4 respectively.

The reason for choosing the relative error as a measure of accuracy can be illustrated in Figure 4.11. Here the change in the eigenvalue is a small percentage of the eigenvalue even for a large value of μ . However, if the question is whether the perturbation process is accurate or not, it is the closeness of the calculated deviation from $\lambda(0)$ to the actual deviation that is of significance. In this case, the first perturbation is in fact going in the wrong direction for $\mu \geq 0.05$. Note

that since the denominator can be zero here, the curve in Figure 4.19 is not very meaningful for $\mu < 0.1$.

This same example illustrates another interesting fact. It is well known (see Gantmacher [15]) that adding a single degree-of-freedom system to a spring-mass chain system will cause the old frequencies to shift outward from the new one. In the system of Figure 4.10, the augmentation is not simply the addition of a single degree of freedom system with a single point of attachment. Thus the resulting system is not a spring-mass chain system.

It is found here that the shift in frequency actually changes sign as μ increases. The theorem stated earlier in this paragraph really does require the limitation on type of primary and secondary system. It is also interesting to point out that even though the structural system under consideration has very well separated frequencies, the use of a first-order perturbation may indeed give erroneous answers. For higher frequencies, though, the first perturbation is all that seems to be needed (as seen in Figures 4.12 through 4.16).

The last two examples treat the case of structural systems that start out with repeated eigenvalues. Example 3 represents a structural system similar to the one used in Example 1. The frequency of the secondary system is tuned to one of the frequencies of the primary system. The properties of the resulting structural system are presented in Figure 4.21. Tables 5 through 7 present the results of the calculation of the eigenvalues of the combined system using a third order perturbation expansion. Figure 4.22 presents a comparison of the solution obtained using the proposed method with the solution obtained

using a standard eigenvalue solver (IMSL), and denoted as the exact solution. It may be noticed that the slope of the curves in Figure 4.23 reflect the order of the error between the two results.

For Example 3, the usefulness of higher order perturbations when the parameter is not very small can be illustrated by the following calculation. The values of $\lambda_2^{(1)}$ and $\lambda_2^{(2)}$ are 0.433014 and 0.20833. For $\mu=0.2$, the exact result for λ_2 is 0.739495. The results up to the first perturbation give a relative error for $\mu=0.2$ of about 19%. When the second perturbation is taken into account as well, the relative error falls to less than 1.8%.

Example 4 considers a combined system consisting in a primary system and two secondary systems. Both secondary systems are added to the primary system in the way shown in Figure 4.24. The frequencies of vibration of both secondary systems are tuned to a particular frequency of the primary system. Tables 8 through 11 present the results of the calculation of the eigenvalues of the combined system using a third order perturbation expansion. Figure 4.25 presents a comparison of the solution obtained using the proposed method with the exact solution. Also, similar to the case in Example 3, the slope of the curves in Figure 4.26 reflect the order of the error between the two results.

When the values of μ are so large that the perturbation process does not seem to converge, the results might most profitably be considered as an asymptotic representation. It is recommended that in this case the process terminate with the smallest change due to addition of the next perturbation, which will depend on the numerical value μ . The error is usually less than the first term neglected [32].

5. SUMMARY AND CONCLUSIONS

5.1 Summary

The subject of this study is the development of a set of methods for the calculation of the changes in the frequency of vibration and mode shape of a structure due to changes in the physical system. Such changes may consist of variations of the structural properties of a given system or even of the addition of new degrees of freedom.

One family of iterative methods that has been used effectively for improvement of modes and frequencies is based on an application of the Newton-Raphson technique to the equations defining modes and frequencies. The method is applicable to cases with multiple or close frequencies as well as to cases where they are widely separated. However, in their original statements, these methods are not capable of handling additional degrees of freedom arising from changes in the structural system. The current work extends these techniques to the cases of added degrees of freedom and, moreover, allows for considerable reduction in computational effort even when there are no additional degrees of freedom.

The original vibrating system is replaced by an augmented system, which may have more degrees of freedom than the original one. The augmentation is described by mass and stiffness matrices, and the original matrix expression of the system may have to be rewritten with suitable rows and columns of zeros to make the two sets of matrices conformable for addition. The augmentation mass and stiffness matrices

are taken proportional to a single parameter which can vary from just larger than zero to sizable positive values. A perturbation analysis is then performed on this augmented system.

The iterative character of the basic method for improving frequencies and modes is modified so that each step now corresponds to a calculation of the next order of perturbation. This modification eliminates the difficulty of dealing with very light additional masses and results in successive solutions of linear matrix equations all having the same coefficient matrix. Unlike in the original method, which requires a new triangularization at each step, the expensive repeated triangularizations are not needed in the perturbation scheme.

One important application of the perturbation method is to determine the modes and frequencies of a structure to which a light secondary system is added - say the piping in a power plant. In situations where a frequency of the added system is close to one or more frequencies of the original system, the so-called tuned case, the method is quite effective and general.

The additional system may also correspond to a large value of the parameter multiplying the augmentation matrices. Here the difficulties in adding degrees of freedom and separating initially close frequencies are resolved by the perturbation technique, and the larger values of the parameter are handled by applying the scheme to reduce the number of steps in the original iterative methods.

5.2 Conclusions

The method presented in this study has been shown to give the eigenvalues and eigenvectors for a general combined system. The method

covers cases of simple as well as multiple roots. The secondary system, tuned or untuned, may be attached to the primary system at several points. A considerable advantage of the method is that it can be applied to each eigenpair separately for the case of simple roots and to a small subspace in the case of close or multiple roots. It might be noted that although multiple roots do not occur in the primary system for most framed structures, they are clearly possible in slabs.

The mechanization of the perturbation process explained in Section 4.2 and 4.3 allows for the rapid and relatively inexpensive calculation of the approximate solution to the eigenvalue problem to any desired order of perturbation. Carrying out more than a simple perturbation is needed to predict the behavior at all accurately even for some systems with simple eigenvalues. As was seen in Section 4.4, it is important to be able to calculate more than just first order perturbations in order to find out how good the results really are. There seems to be no way to determine whether higher order perturbations are needed without calculating them.

5.3 Recommendations for Further Study

Several possible areas of further study to extend the proposed method may be suggested. The first is the extension of the mechanization of the perturbation process to all cases of repeated roots.

Another topic that deserves treatment is the examination of the convergence of the perturbation process at least for some nonzero μ . Actually, some progress has been made on this question that will be reported elsewhere.

Also, for large values of μ , the method presented might be considered an alternative to substructure methods [41]. In this case the identification of the primary and secondary system may be arbitrary. A further exploration of this topic in the context of perturbation methods may be useful.

A further possible area of improvement is the inclusion of several parameters in the formulation of the method. This would make possible the application of the proposed method to nonlinear systems approximated by bilinear ones. An important application of the use of several parameters is found in the optimization of design used in the aerospace industry. In fact, some partially successful attempts have been made in the past by authors using much less convenient perturbation techniques than the ones developed here [51].

TABLES

Table 4.1 Comparison of Exact Solution to Approximate Solution for First and Fifth Order Expansions for Example 1

μ	λ_1			% RELATIVE ERROR FOR 1ST ORDER EXPANSION	% RELATIVE ERROR FOR 5TH ORDER EXPANSION
	EXACT	1ST ORDER EXPANSION	5TH ORDER EXPANSION		
0.00	0.500000	0.500000	0.500000		
0.05	0.484442	0.483333	0.484442	7.128017	0.002203
0.10	0.470726	0.466667	0.470707	13.866731	0.065801
0.15	0.458438	0.450000	0.458242	20.300810	0.470840
0.20	0.447294	0.433333	0.446302	26.488300	1.883423
0.25	0.437093	0.416667	0.433641	32.471025	5.487993
0.30	0.427683	0.400000	0.418209	38.280286	13.100732
0.35	0.418948	0.383333	0.396844	43.940226	27.271649
0.40	0.410796	0.366667	0.364963	49.469922	51.379273

$$\% \text{ Relative Error} = \frac{\lambda_{\text{exact}}(\mu) - \lambda_{\text{approx}}(\mu)}{\lambda_{\text{exact}}(\mu) - \lambda(0)} \times 100$$

Table 4.2 Logarithm of the Difference Between the Exact and the Calculated Eigenvalues for Example 1

μ	λ_1	
	1ST ORDER EXPANSION	5TH ORDER EXPANSION
0.05	6.804336	14.886267
0.10	5.506734	10.857354
0.15	4.775067	8.538964
0.20	4.271497	6.91510
0.25	3.890922	5.668708
0.30	3.586933	4.659200
0.35	3.335002	3.811985
0.40	3.120633	3.08276

Table 4.3 Comparison of Exact Solution to Approximate Solution for First and Fifth Order Expansions for Example 2

μ	λ_1			% RELATIVE ERROR FOR 1ST ORDER EXPANSION	% RELATIVE ERROR FOR 5TH ORDER EXPANSION
	EXACT	1ST ORDER EXPANSION	5TH ORDER EXPANSION		
0.00	0.120615	0.120615	0.120615		
0.05	0.120581	0.120697	0.120581	342.075258	0.005736
0.10	0.120346	0.120780	0.120346	161.377258	0.044334
0.15	0.119952	0.120862	0.119953	137.300965	0.195226
0.20	0.119430	0.120944	0.119437	127.824097	0.586640
0.25	0.118806	0.121027	0.118831	122.780526	1.403669
0.30	0.118100	0.121109	0.118173	119.664577	2.892037
0.35	0.117329	0.121192	0.117505	117.559395	5.362183
0.40	0.116507	0.121274	0.116884	116.049957	9.193259

$$\% \text{ Relative Error} = \frac{\lambda_{\text{exact}}(\mu) - \lambda_{\text{approx}}(\mu)}{\lambda_{\text{exact}}(\mu) - \lambda(0)} \times 100$$

Table 4.4 Logarithm of the Difference Between the Exact and the Calculated Eigenvalues for Example 2

μ	λ_1	
	1ST ORDER EXPANSION	5TH ORDER EXPANSION
0.05	9.057932	20.053899
0.10	7.743861	15.943605
0.15	7.001945	13.557718
0.20	6.492667	11.876665
0.25	6.109783	10.581093
0.30	5.806080	9.528812
0.35	5.556449	8.644021
0.40	5.345957	7.881508

Table 4.5 Comparison of Exact Solution to Approximate Solution
for Third Order Expansion for Example 3

μ	λ_1		% RELATIVE ERROR FOR 3RD ORDER EXPANSION
	EXACT	3RD ORDER EXPANSION	
0.00	0.500000	0.500000	
0.04	0.419571	0.419578	0.011271
0.08	0.391891	0.391916	0.027657
0.12	0.371999	0.372051	0.047400
0.16	0.356141	0.356227	0.069624
0.20	0.342840	0.342970	0.093810
0.24	0.331338	0.331517	0.119603
0.28	0.321183	0.321417	0.146743
0.32	0.312081	0.312377	0.175026
0.36	0.303830	0.304192	0.204289
0.40	0.296281	0.296715	0.234399

$$\% \text{ Relative Error} = \frac{\lambda_{\text{exact}}(\mu) - \lambda_{\text{approx}}(\mu)}{\lambda_{\text{exact}}(\mu) - \lambda(0)} \times 100$$

Table 4.6 Comparison of Exact Solution to Approximate Solution for Third Order Expansion for Example 3

μ	λ_2		% RELATIVE ERROR FOR 3RD ORDER EXPANSION
	EXACT	3RD ORDER EXPANSION	
0.00	0.500000	0.500000	
0.04	0.597912	0.597921	0.012158
0.08	0.642212	0.642250	0.031290
0.12	0.678693	0.678782	0.055660
0.16	0.711110	0.711272	0.084394
0.20	0.740937	0.741196	0.116967
0.24	0.768934	0.769315	0.153019
0.28	0.795551	0.796082	0.192279
0.32	0.821082	0.821789	0.234536
0.36	0.845729	0.846641	0.279615
0.40	0.869639	0.870784	0.327373

$$\% \text{ Relative Error} = \frac{\lambda_{\text{exact}}(\mu) - \lambda_{\text{approx}}(\mu)}{\lambda_{\text{exact}}(\mu) - \lambda(0)} \times 100$$

Table 4.7 Logarithm of the Difference Between the Exact and the Calculated Eigenvalues for Example 3

μ	THIRD ORDER EXPANSION	
	λ_1	λ_2
0.04	11.879731	11.564324
0.08	10.610412	10.169843
0.12	9.870265	9.333078
0.16	9.350005	8.731212
0.20	8.950659	8.259938
0.24	8.627806	7.872068
0.28	8.357672	7.542177
0.32	8.126048	7.254975
0.36	7.923765	7.000539
0.40	7.744565	6.772064

Table 4.8 Comparison of Exact Solution to Approximate Solution for Third Order Expansion for λ_1 in Example 4

μ	λ_1		% RELATIVE ERROR FOR 3RD ORDER EXPANSION
	EXACT	3RD ORDER EXPANSION	
0.00	1.000000	1.000000	
0.04	0.953234	0.953405	0.496100
0.08	0.939907	0.940472	1.179991
0.12	0.930786	0.931871	1.906097
0.16	0.923572	0.925239	2.597862
0.20	0.917433	0.919694	3.215379
0.24	0.911976	0.914809	3.739264
0.28	0.906989	0.910350	4.162050
0.32	0.902346	0.906174	4.483105
0.36	0.897968	0.902193	4.705580
0.40	0.893804	0.898346	4.834583

$$\% \text{ Relative Error} = \frac{\lambda_{\text{exact}}(\mu) - \lambda_{\text{approx}}(\mu)}{\lambda_{\text{exact}}(\mu) - \lambda(0)} \times 100$$

Table 4.9 Comparison of Exact Solution to Approximate Solution
for Third Order Expansion for λ_2 in Example 4

μ	λ_2		% RELATIVE ERROR FOR 3RD ORDER EXPANSION
	EXACT	3RD ORDER EXPANSION	
0.00	1.000000	1.000000	
0.04	1.011224	1.011067	1.465051
0.08	1.023101	1.022589	2.267886
0.12	1.035932	1.034966	2.726921
0.16	1.049552	1.048109	2.942715
0.20	1.063827	1.061937	2.984927
0.24	1.078649	1.076381	2.901761
0.28	1.093930	1.091382	2.726815
0.32	1.109597	1.106887	2.483715
0.36	1.125592	1.122854	2.189223
0.40	1.141865	1.139242	1.855318

$$\% \text{ Relative Error} = \frac{\lambda_{\text{exact}}(\mu) - \lambda_{\text{approx}}(\mu)}{\lambda_{\text{exact}}(\mu) - \lambda(0)} \times 100$$

Table 4.10 Comparison of Exact Solution for Approximate Solution for Third Order Expansion for λ_3 in Example 4

μ	λ_3		% RELATIVE ERROR FOR 3RD ORDER EXPANSION
	EXACT	3RD ORDER EXPANSION	
0.00	1.000000	1.000000	
0.04	1.073803	1.074026	0.371692
0.08	1.111246	1.112105	0.878972
0.12	1.143092	1.144994	1.468598
0.16	1.171800	1.175150	2.117290
0.20	1.198337	1.203534	2.812983
0.24	1.223204	1.230641	3.548219
0.28	1.246701	1.256767	4.317842
0.32	1.269030	1.282104	5.117962
0.36	1.290332	1.306785	5.945436
0.40	1.310713	1.330911	6.797560

$$\% \text{ Relative Error} = \frac{\lambda_{\text{exact}}(\mu) - \lambda_{\text{approx}}(\mu)}{\lambda_{\text{exact}}(\mu) - \lambda(0)} \times 100$$

Table 4.11 Logarithm of the Difference Between the Exact
and the Calculated Eigenvalues for Example 4

μ	THIRD ORDER EXPANSION		
	λ_1	λ_2	λ_3
0.04	8.672675	8.758666	8.404680
0.08	7.479644	7.576127	7.060441
0.12	6.825559	6.942160	6.264892
0.16	6.396684	6.540728	5.698842
0.20	6.092065	6.271052	5.259674
0.24	5.866370	6.089005	4.901197
0.28	5.695501	5.972596	4.598639
0.32	5.565292	5.910943	4.337168
0.36	5.466870	5.900338	4.107205
0.40	5.394481	5.943533	3.902203

FIGURES

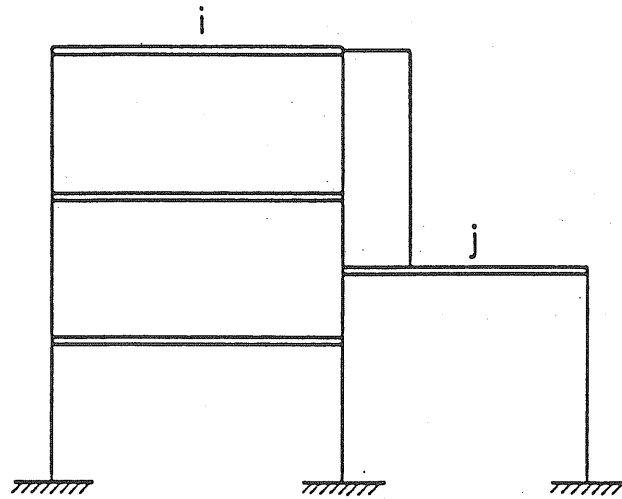


Fig. 1.1 Local Modification to Structural System Produced by Elastic Connector Between Masses i and j

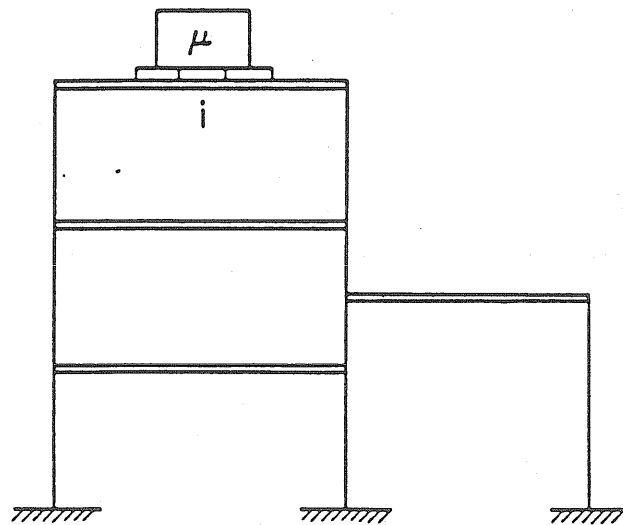


Fig. 1.2 Local Modification to Structural System Produced by Addition of Mass μ at Level i

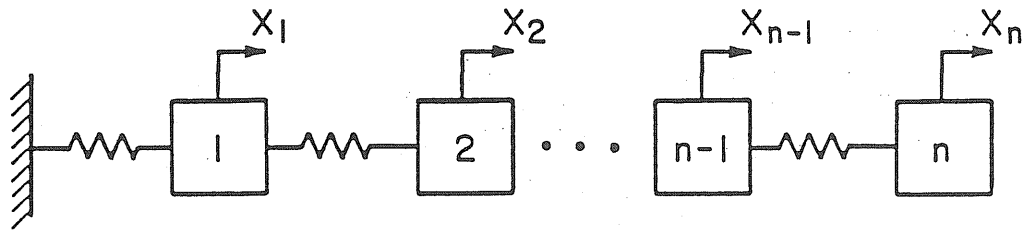


Fig. 2.1 Undamped n Degree-of-Freedom System

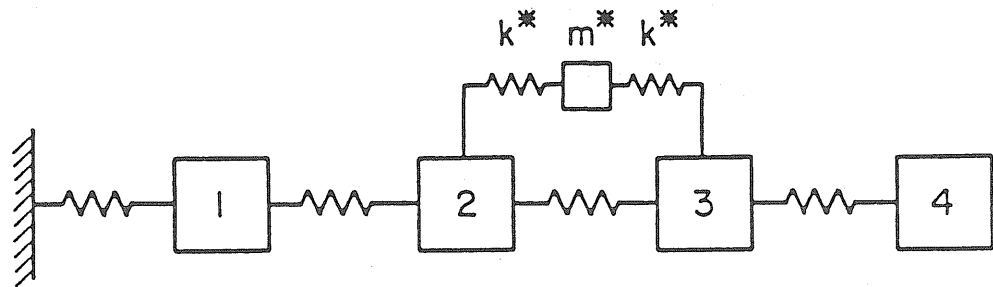


Fig. 2.2 Augmented Structural System Consisting of a Four-Degree-of-Freedom Initial System and a Single-Degree-of-Freedom Added System

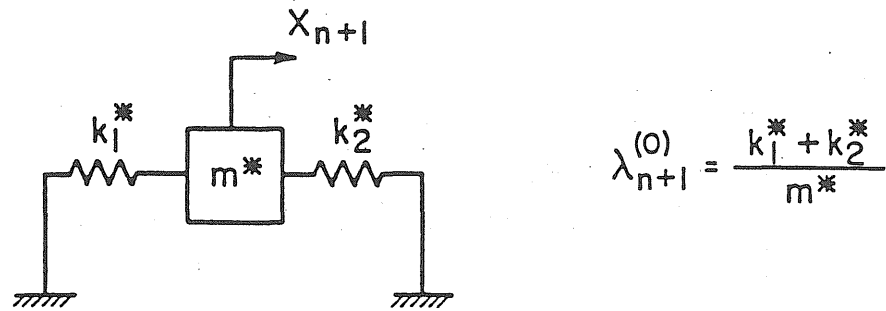


Fig. 2.3 Fixed-Base Model of Single-Degree-of-Freedom Added System

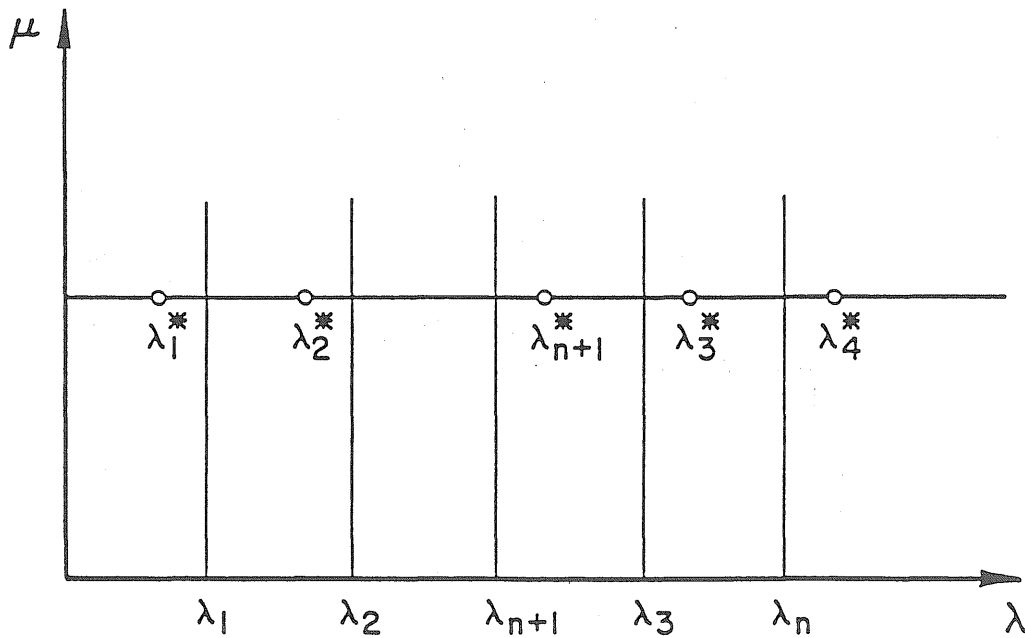


Fig. 2.4 Schematic Representation of the Variation of the Eigenvalues

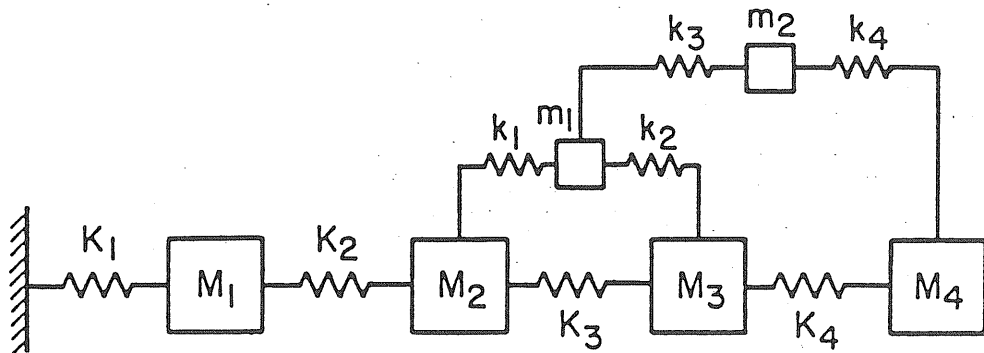


Fig. 2.5 Augmented Structural System Consisting of a Four-Degree-of-Freedom Initial System and a Two-Degree-of-Freedom Added System

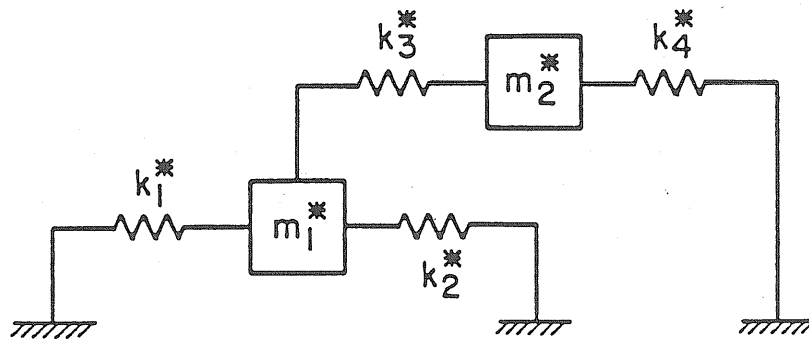


Fig. 2.6 Fixed-Based Model of Two-Degree-of-Freedom Added System

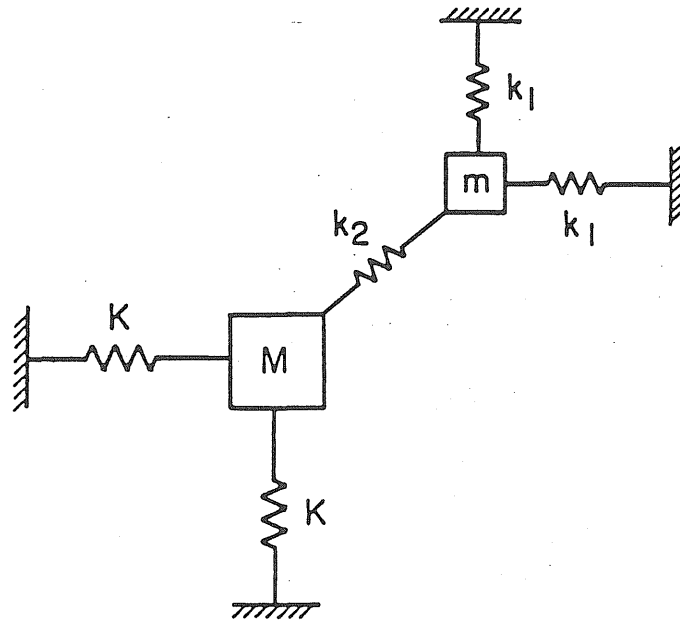


Fig. 3.1 Four-Degree-of-Freedom Tuned Structural System

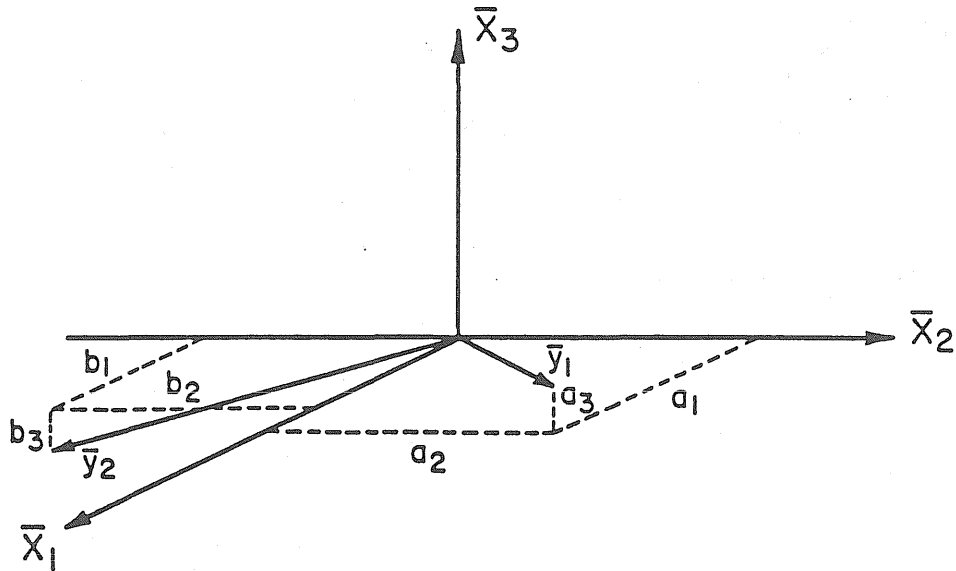


Fig. 3.2 Schematic Representation of Frames of Reference X_i and Y_j

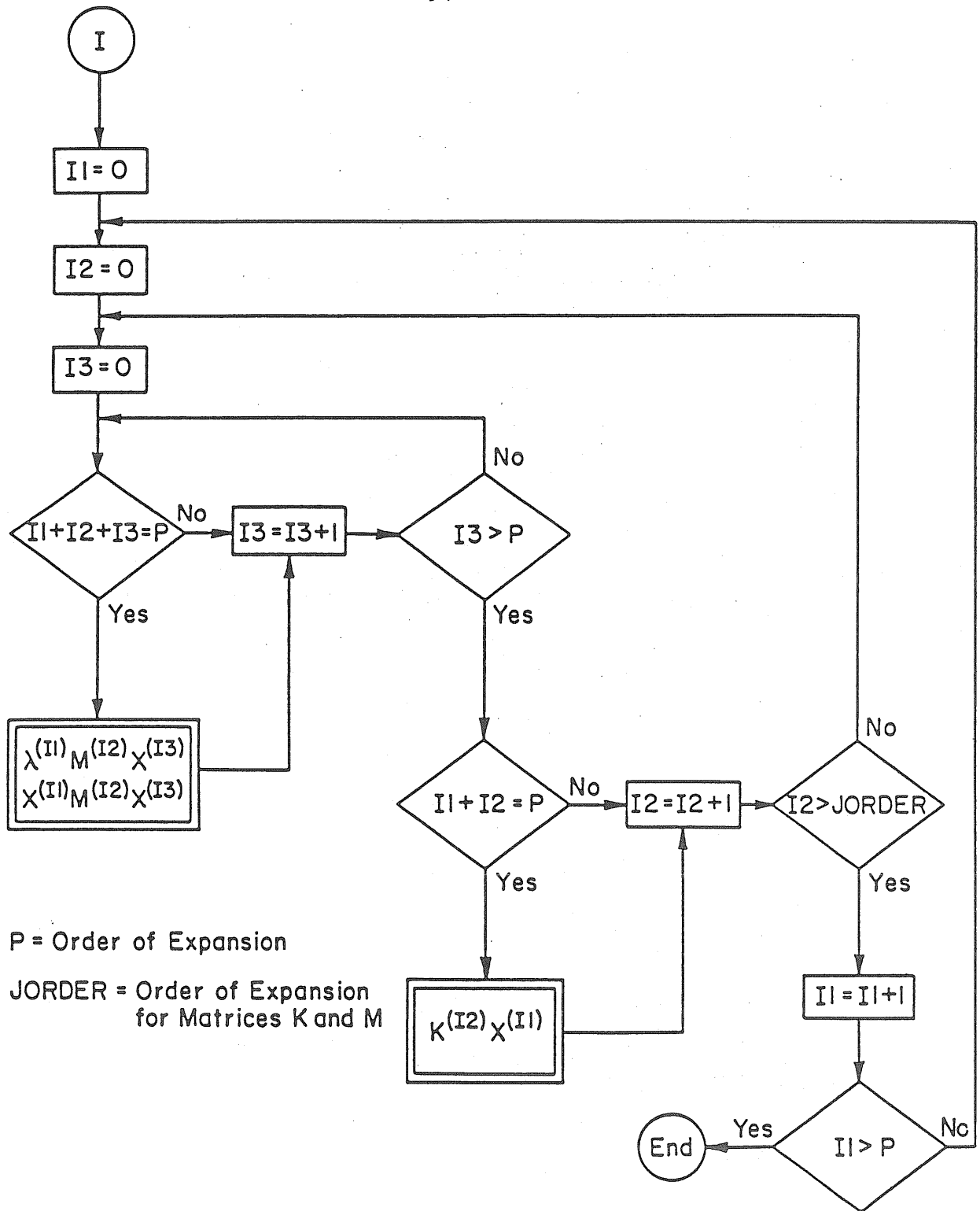


Fig. 4.1 Flow Diagram for the Calculation of the Right-Hand-Side of the Perturbation Equation

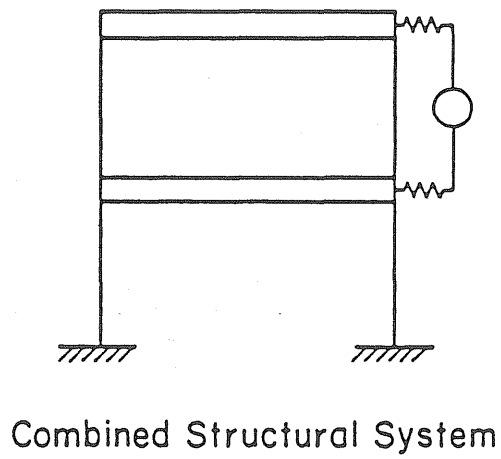
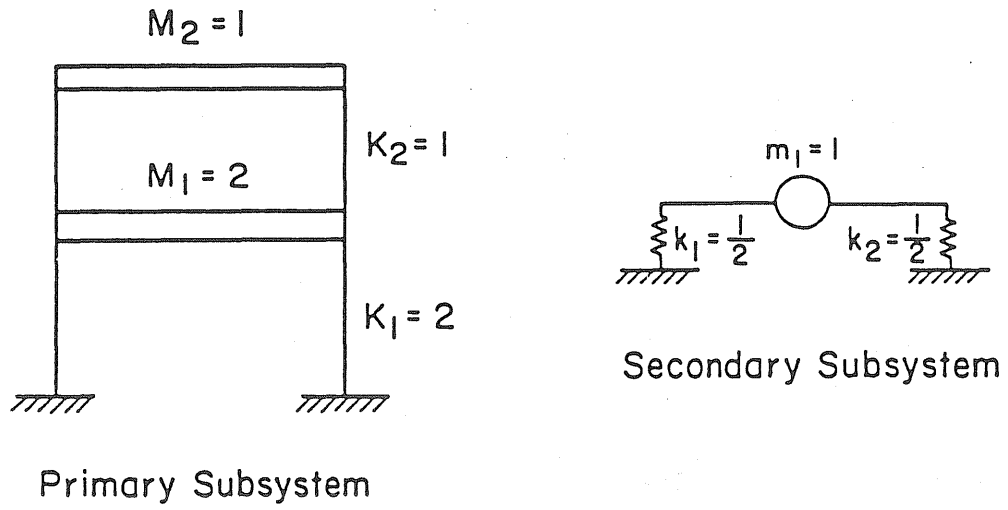


Fig. 4.2 Combined Structural System for Example 1

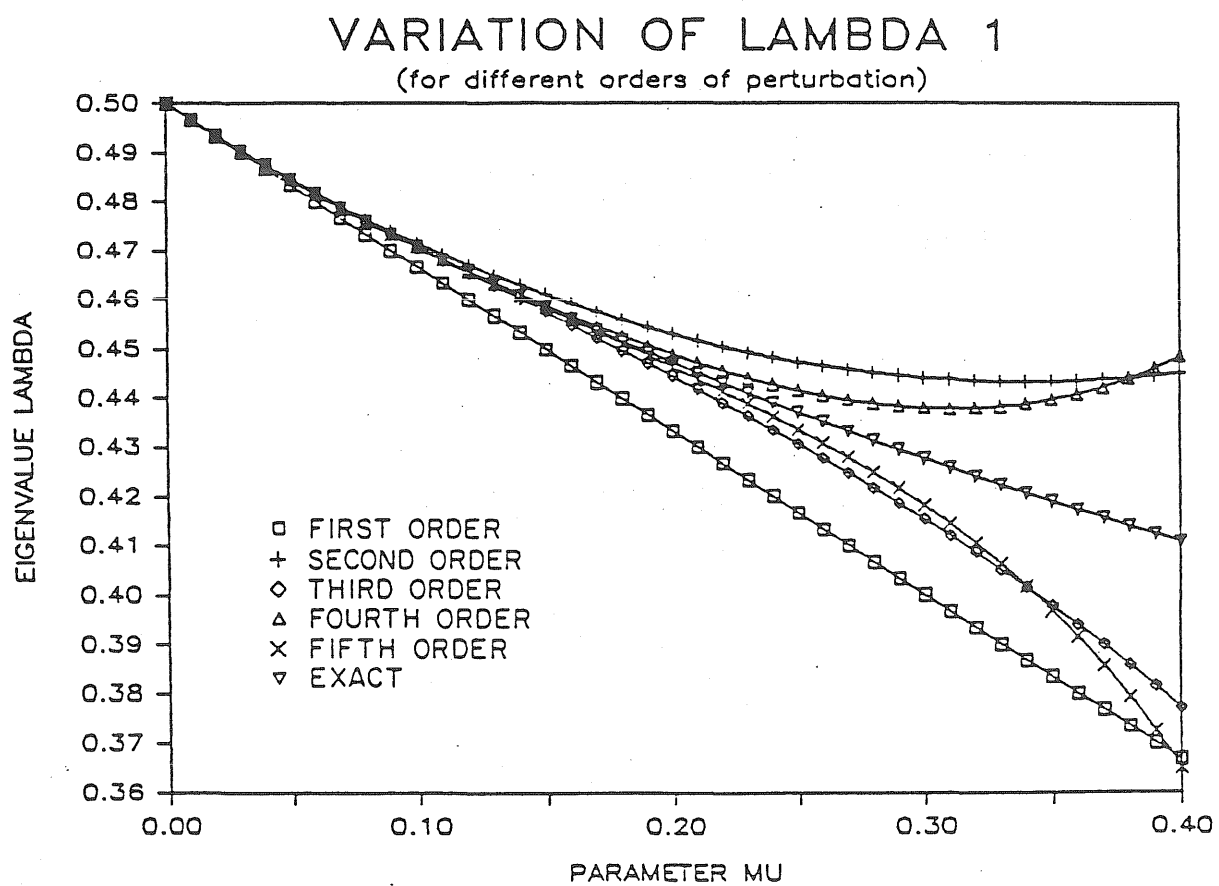


Fig. 4.3 Variation of λ_1 for Different Orders of Perturbation

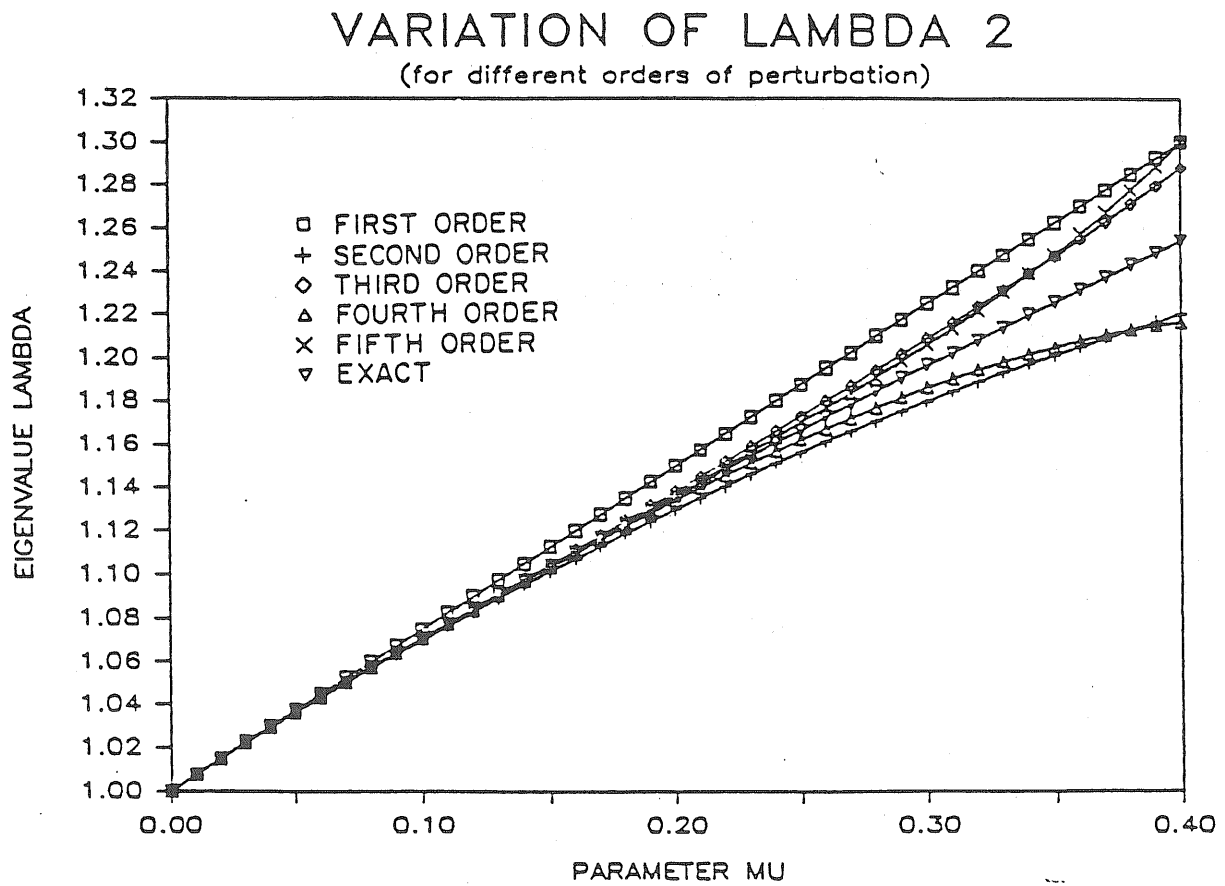


Fig. 4.4 Variation of λ_2 for Different Orders of Perturbation

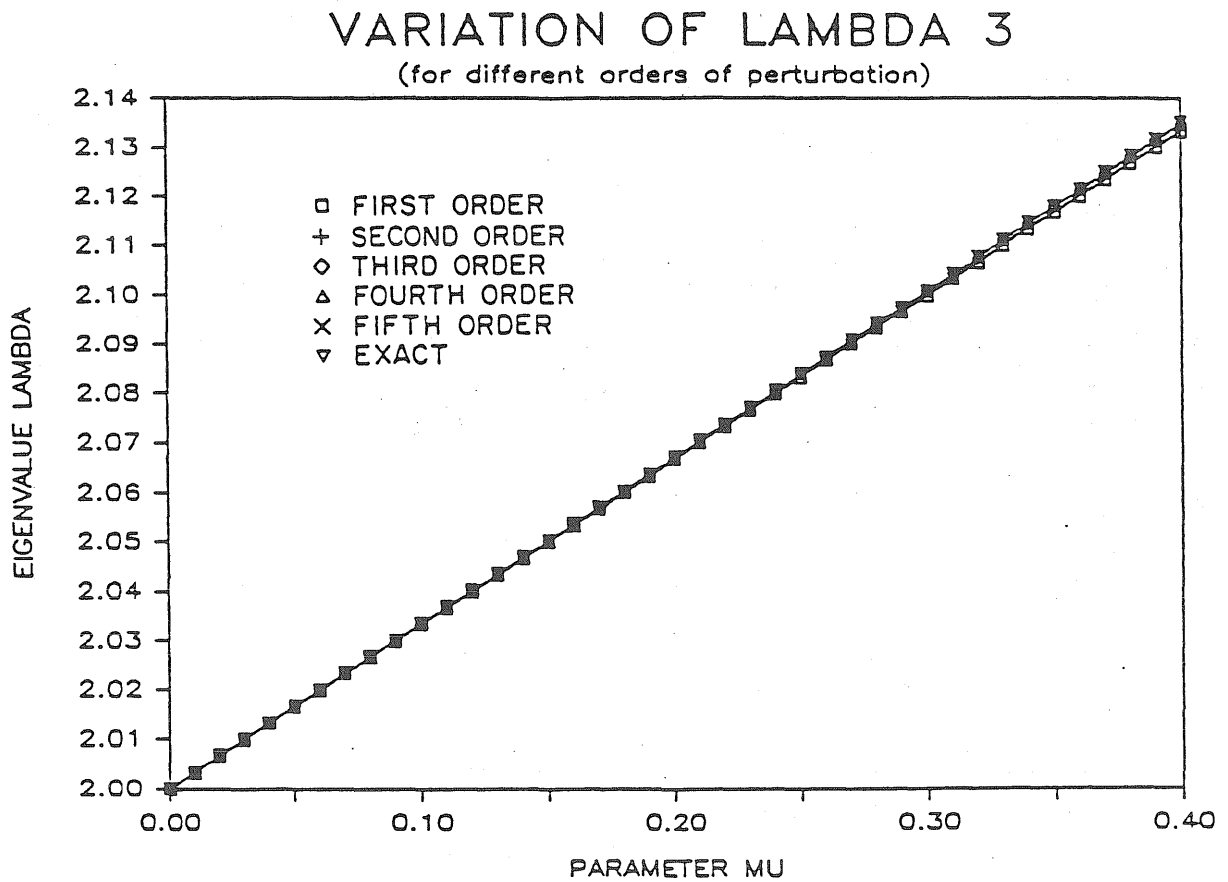


Fig. 4.5 Variation of λ_3 for Different Orders of Perturbation

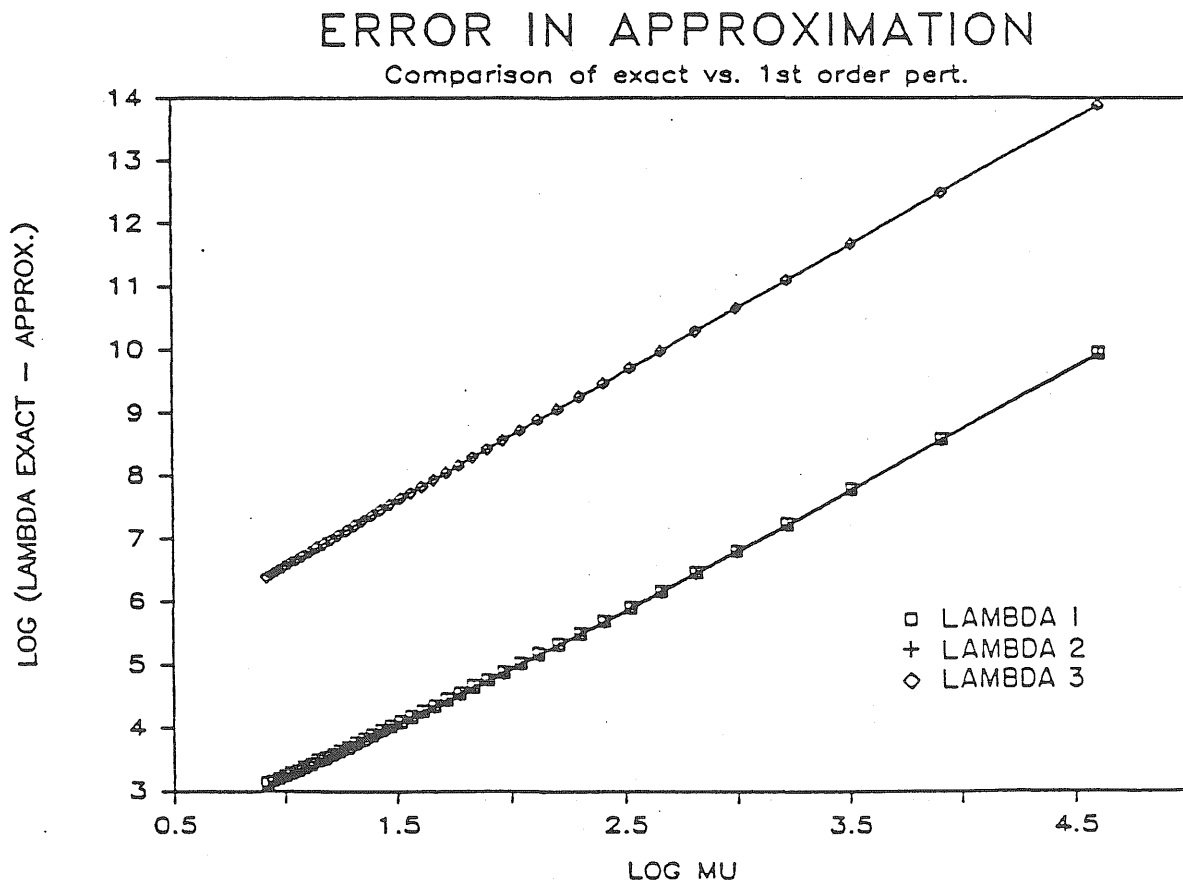


Fig. 4.6 Comparison of the Error in Approximation for the Exact Solution and the First Order Perturbation

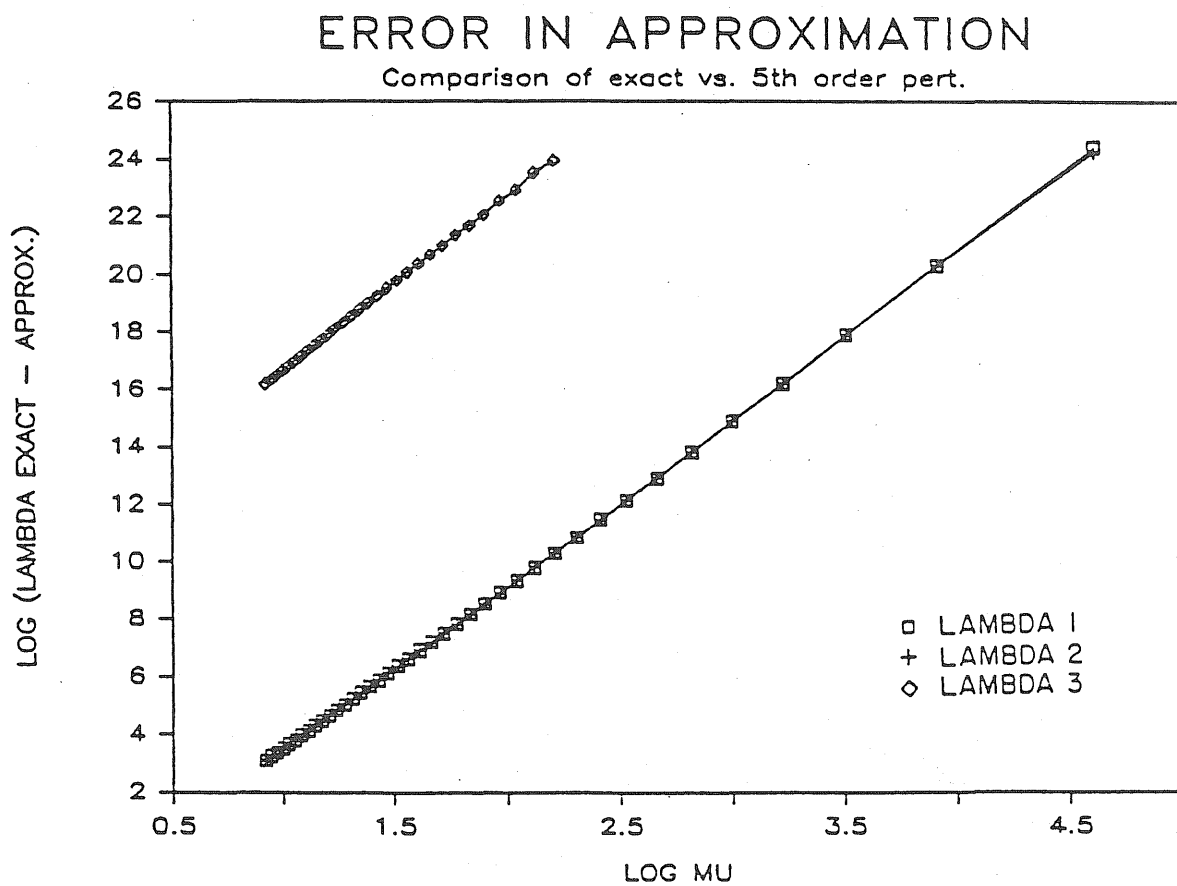


Fig. 4.7 Comparison of the Error in Approximation for the Exact Solution and the Fifth Order Perturbation

RELATIVE ERROR FOR DIFFERENT LAMBDA S (For 5th order perturbation.)

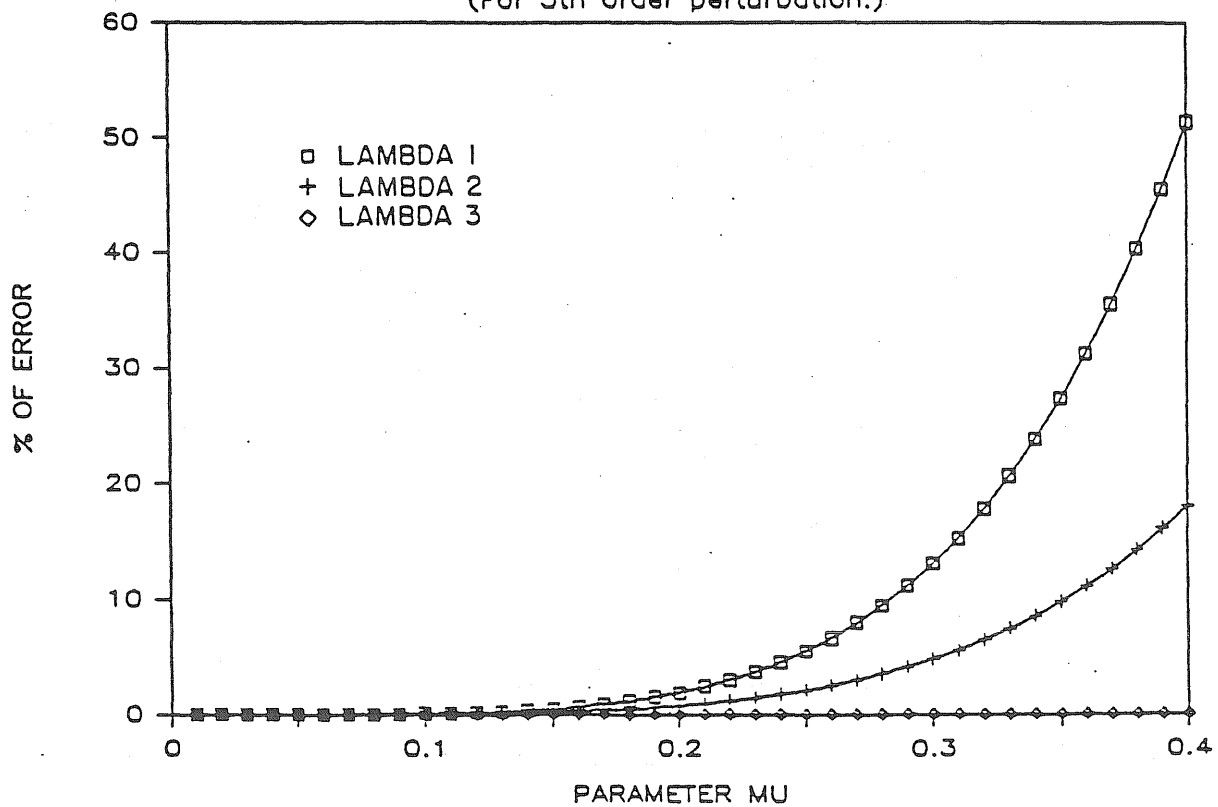


Fig. 4.8 Relative Error for Different λ 's
for Fifth Order Perturbation

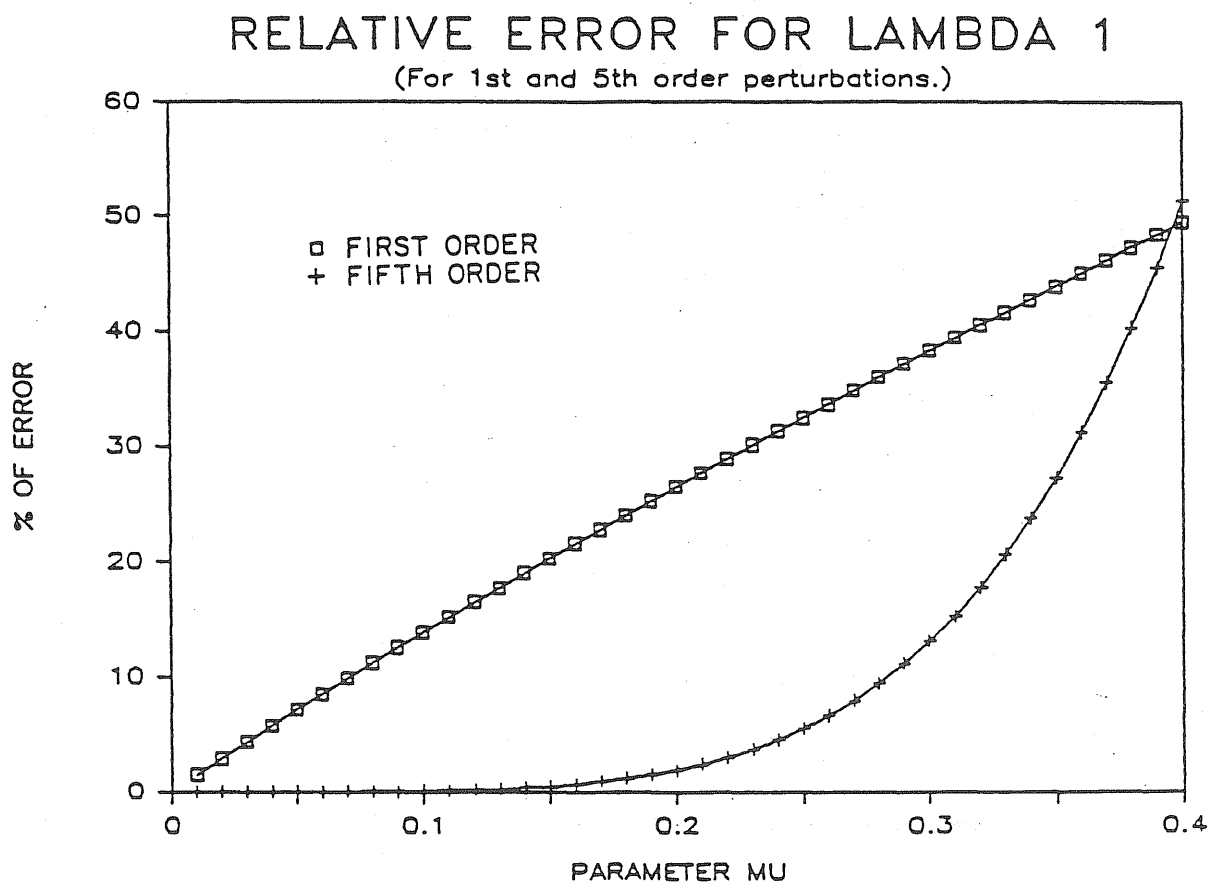
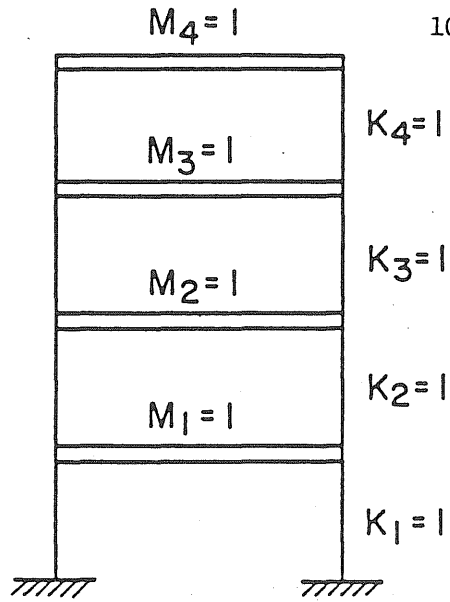
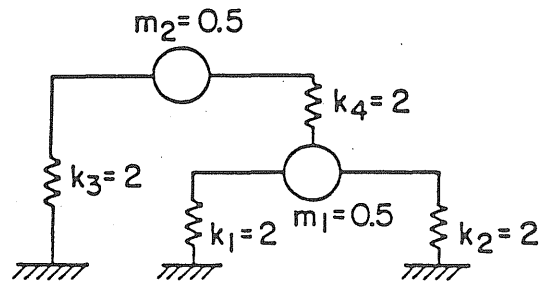


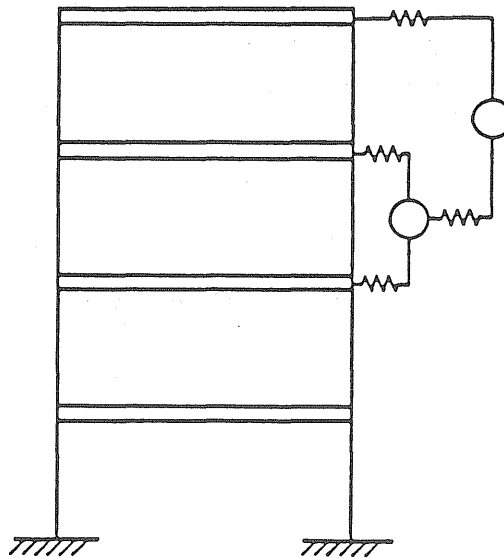
Fig. 4.9 Relative Error for Different λ_1 for First and Fifth Order Perturbations



Primary Subsystem



Secondary Subsystems



Combined Structural System

Fig. 4.10 Combined Structural System for Example 2

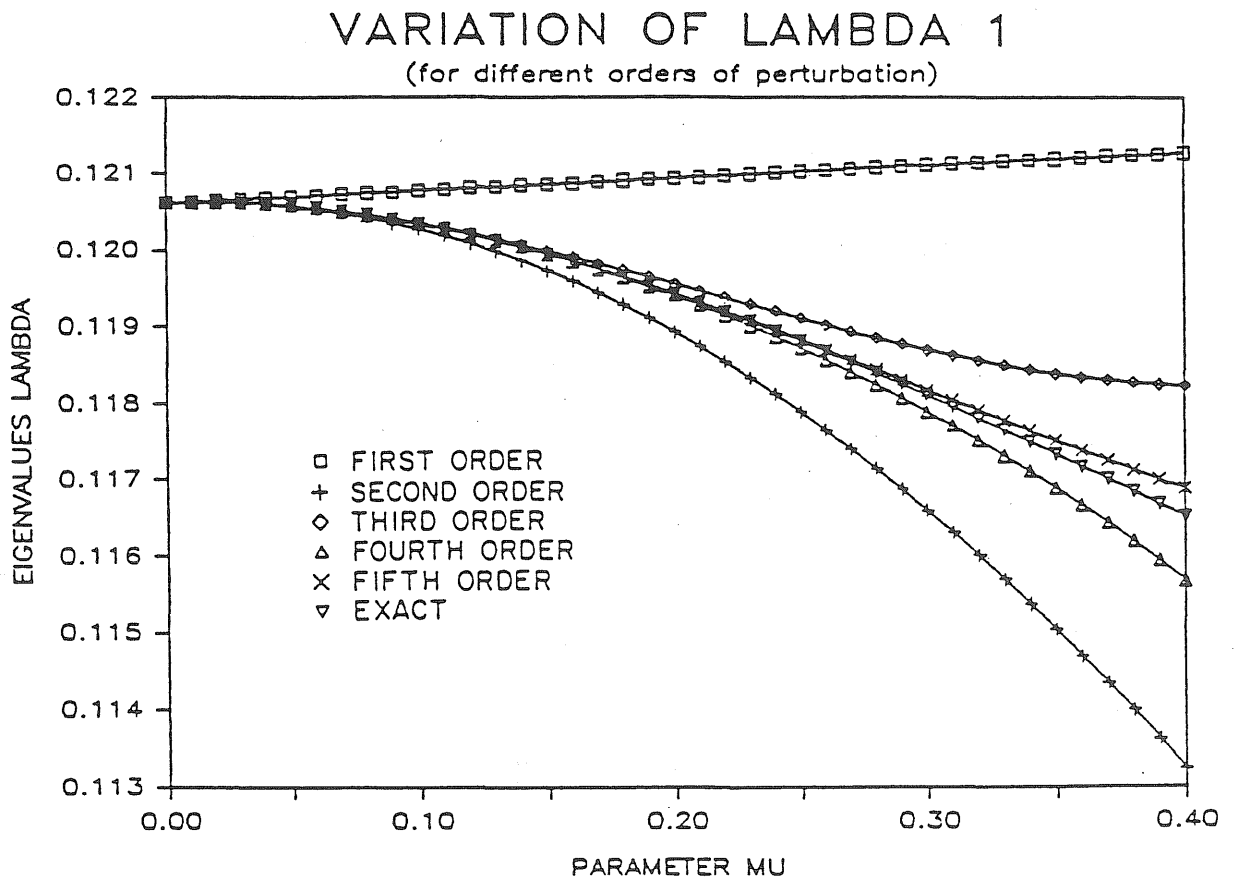


Fig. 4.11 Variation of λ_1 for Different Orders of Perturbation

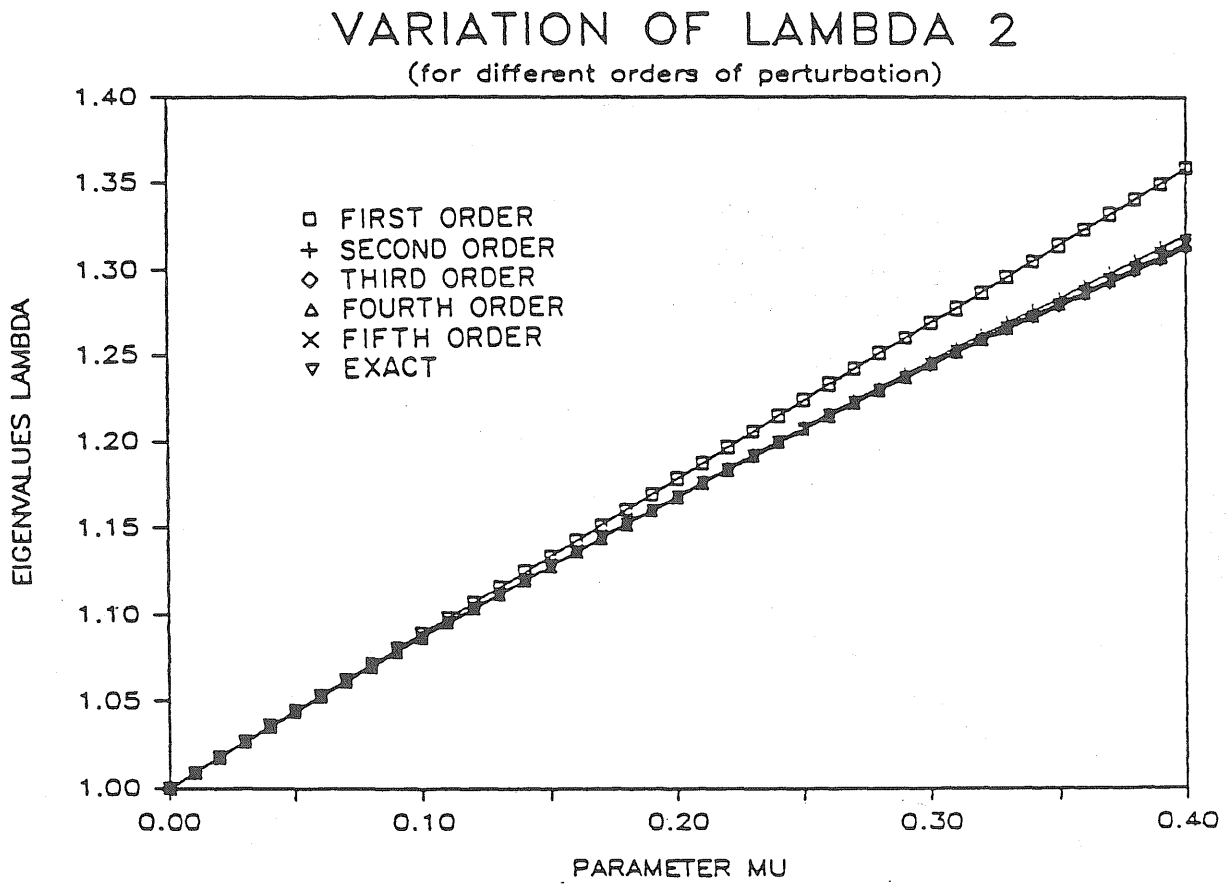


Fig. 4.12 Variation of λ_2 for Different Orders of Perturbation

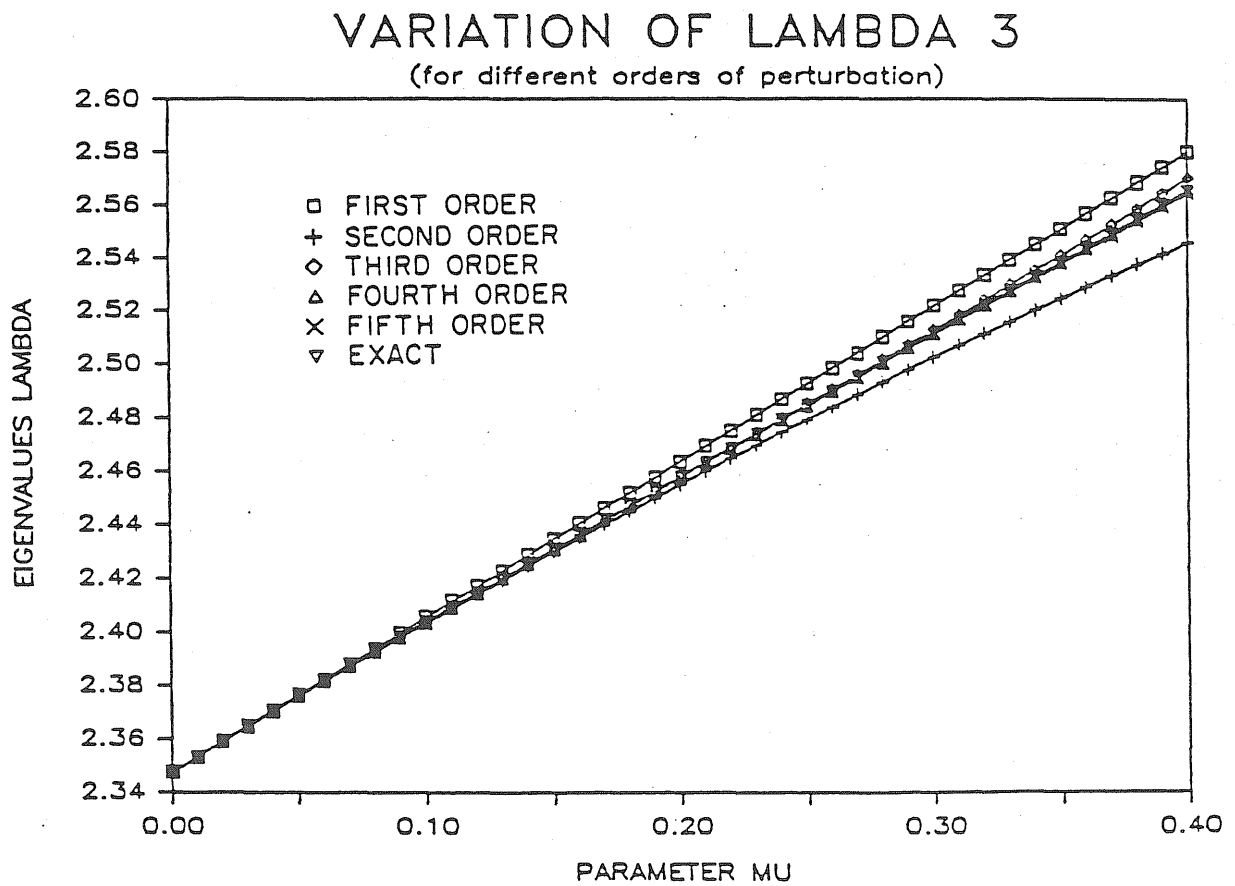


Fig. 4.13 Variation of λ_3 for Different Orders of Perturbation

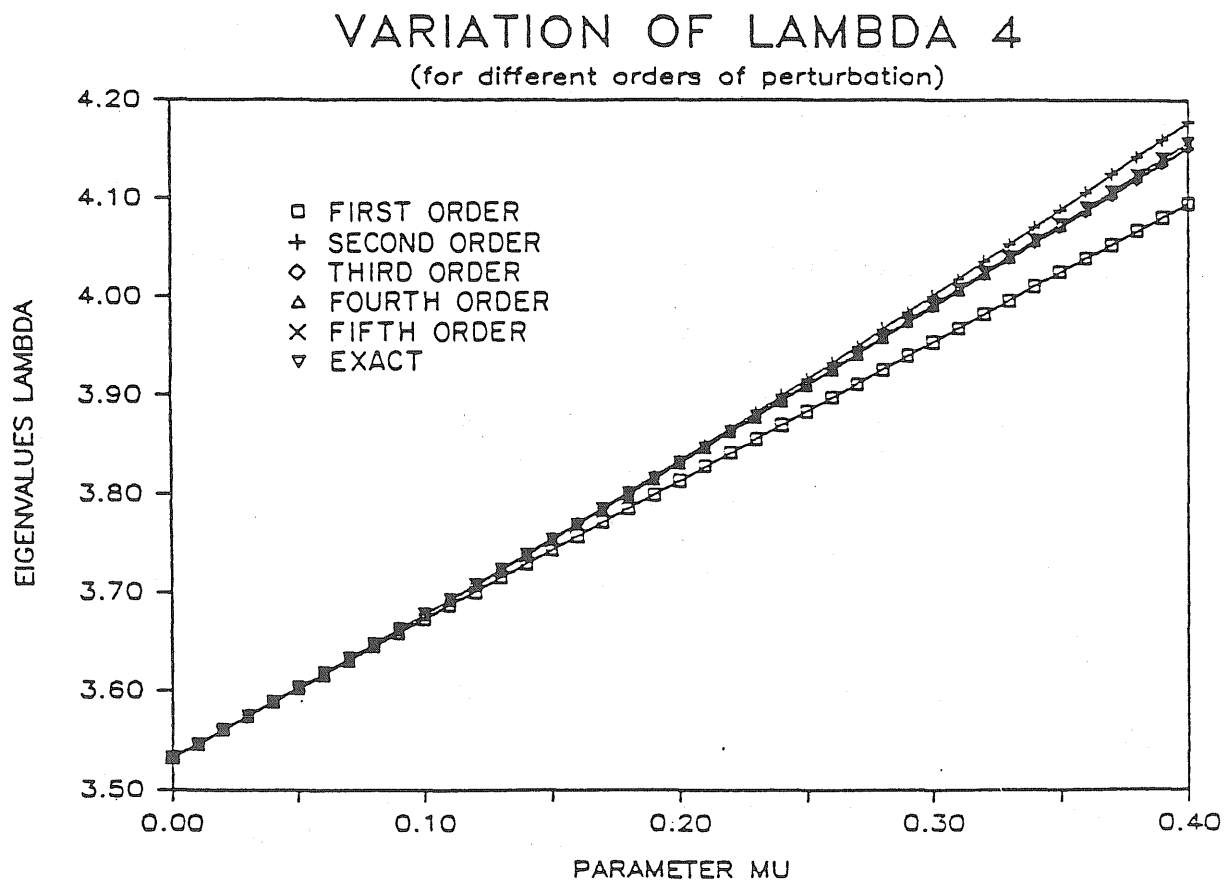


Fig. 4.14 Variation of λ_4 for Different Orders of Perturbation

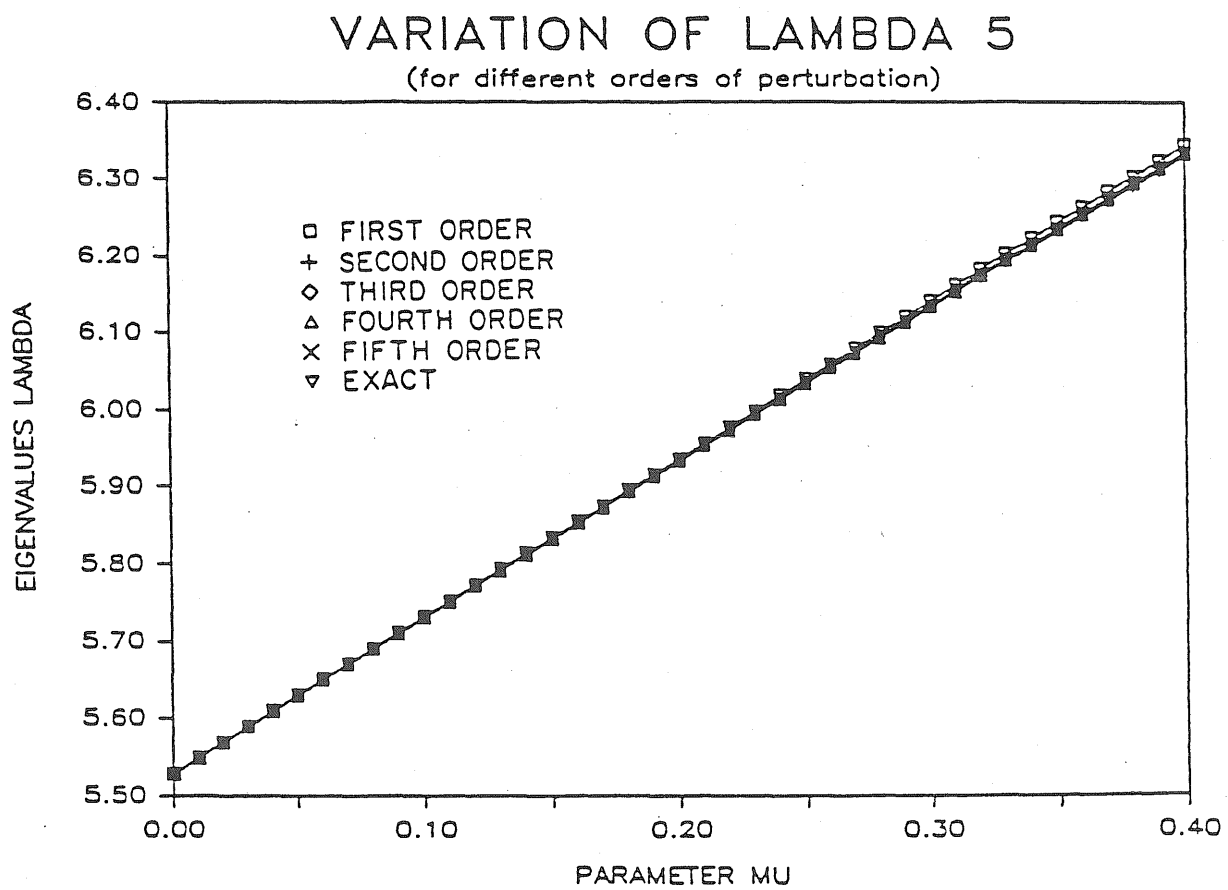


Fig. 4.15 Variation of λ_5 for Different Orders of Perturbation

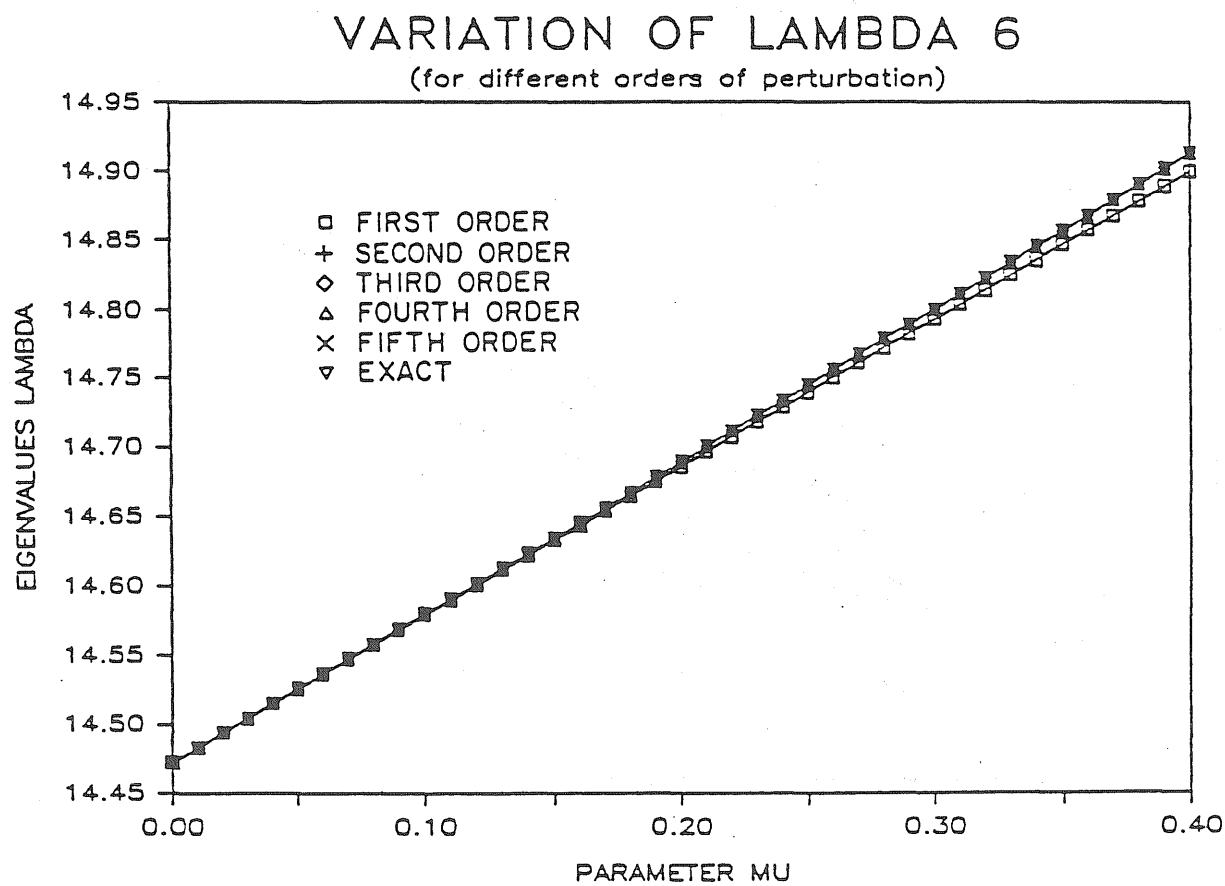


Fig. 4.16 Variation of λ_6 for Different Orders of Perturbation

ERROR IN APPROXIMATION

Comparison of exact vs. 1st order pert.

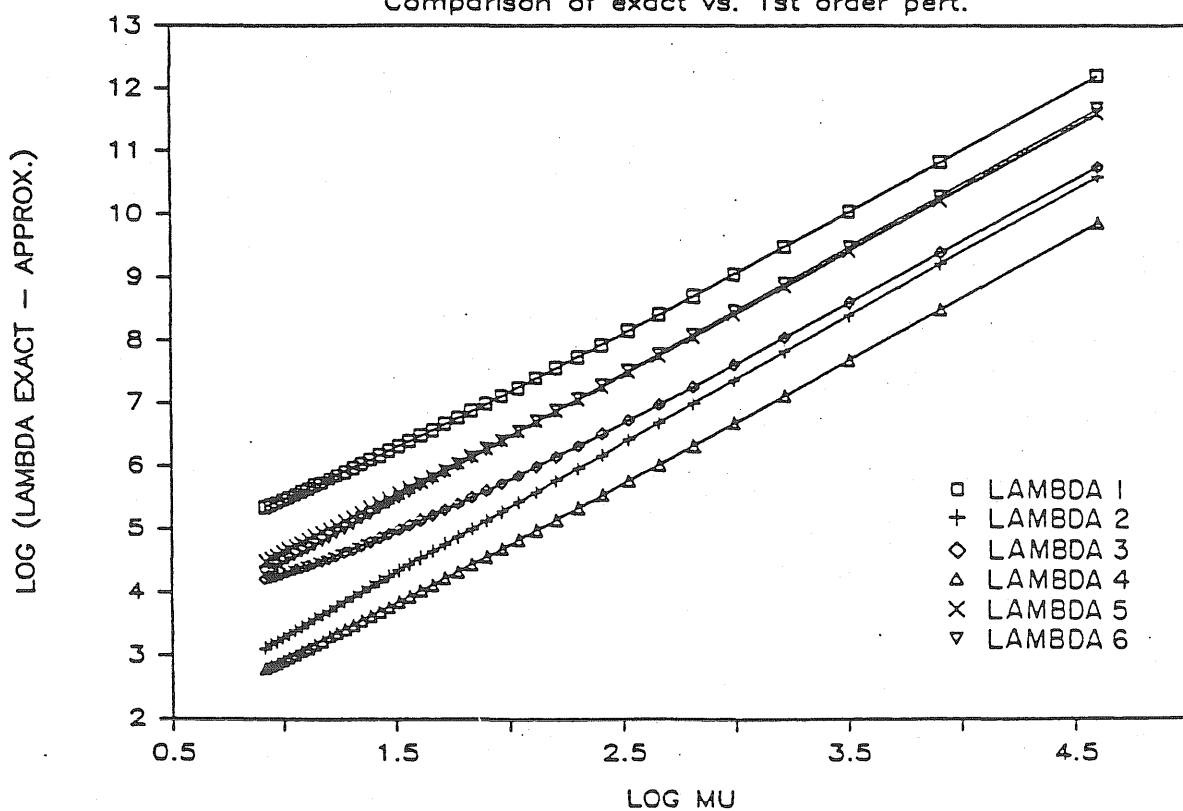


Fig. 4.17 Comparison of the Error in Approximation Between the Exact Solution and the First Order Perturbation Solution

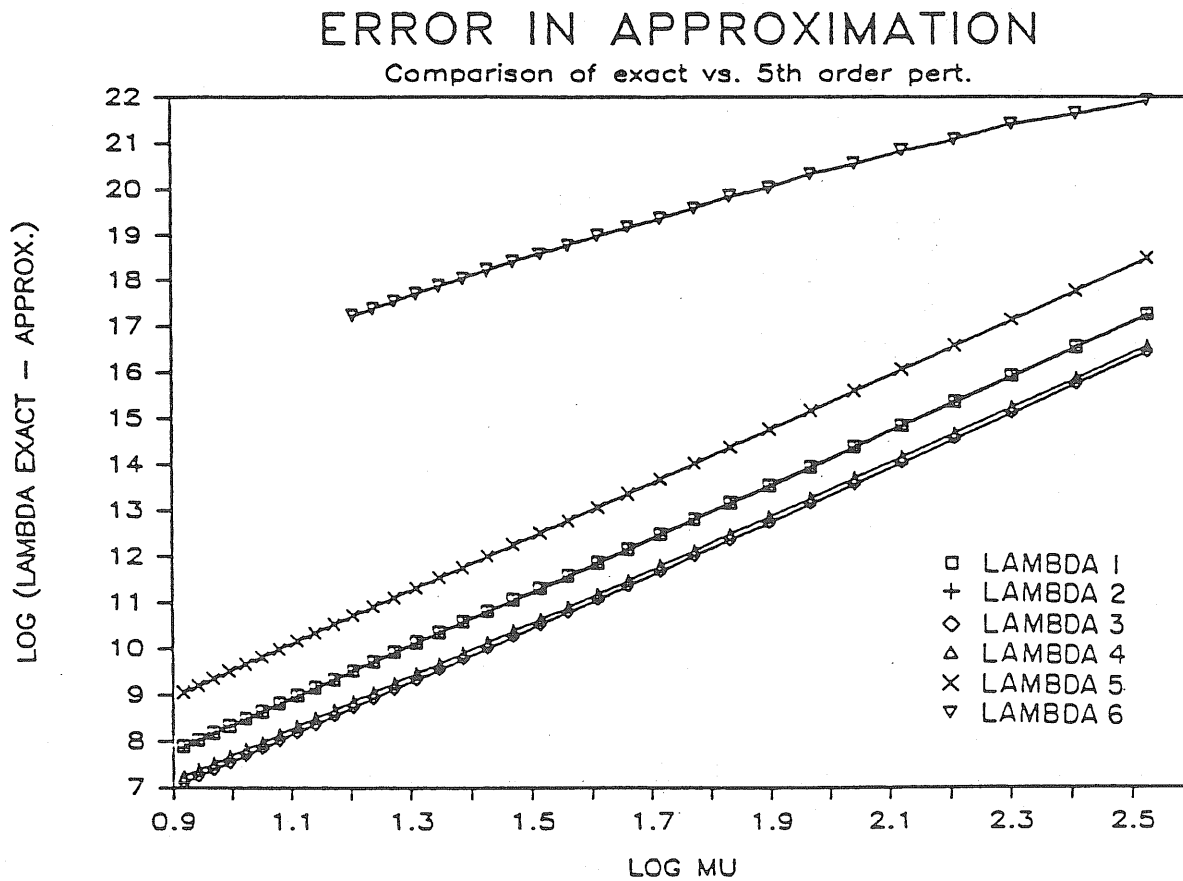


Fig. 4.18 Comparison of the Error in Approximation Between the Exact Solution and the Fifth Order Perturbation Solution

RELATIVE ERROR FOR DIFFERENT LAMBDA'S (For 5th order of perturbation.)

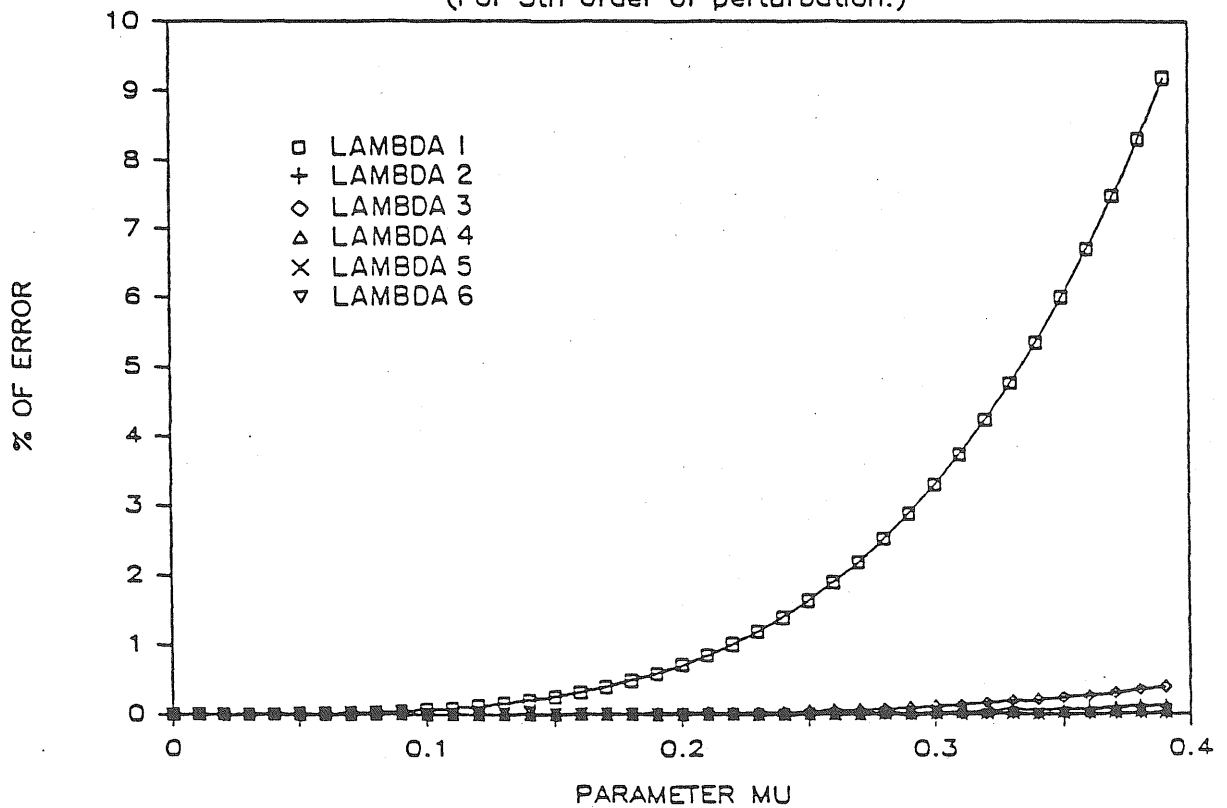


Fig. 4.19 Relative Error for Different λ 's
for Fifth Order of Perturbation

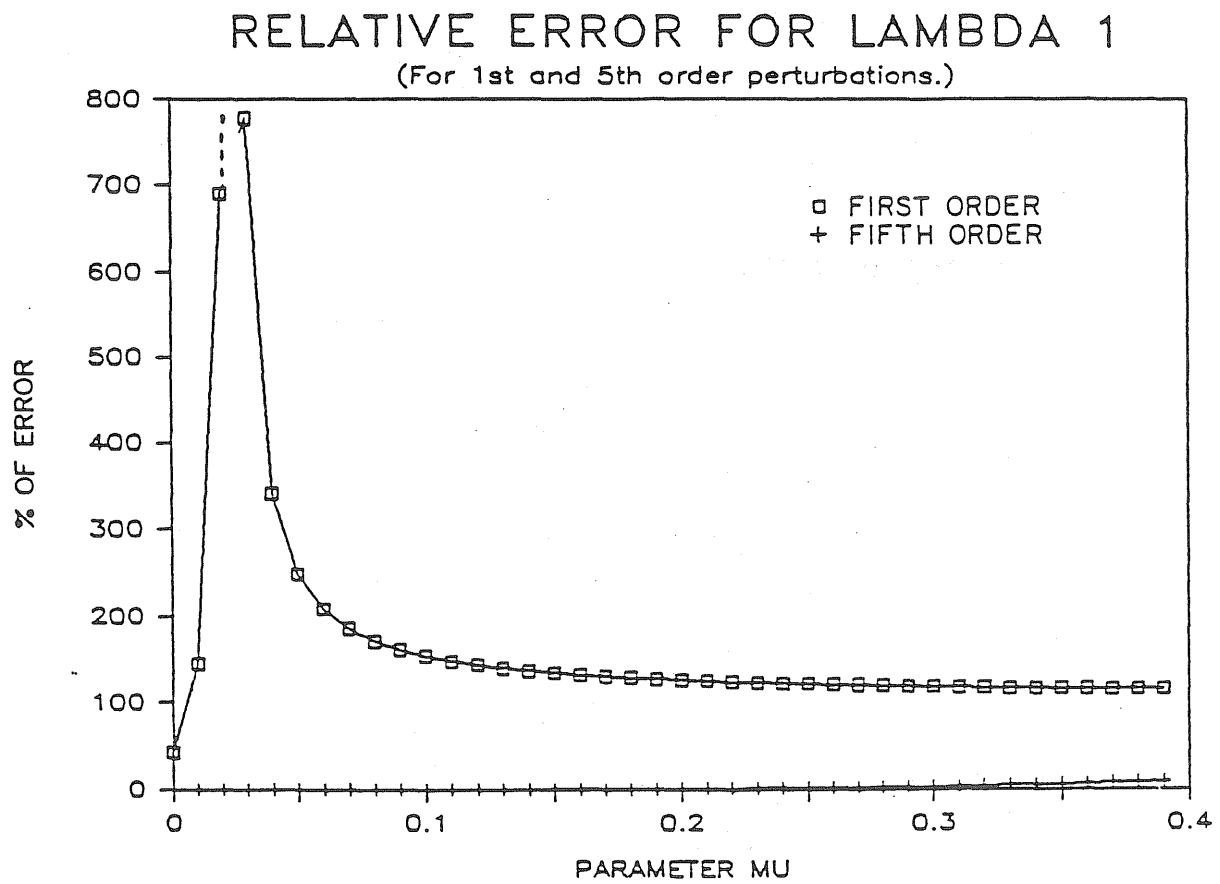
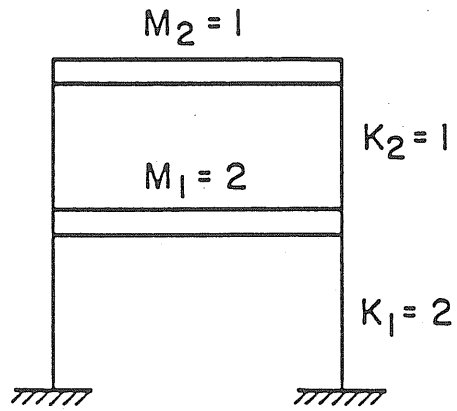
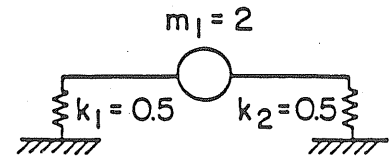


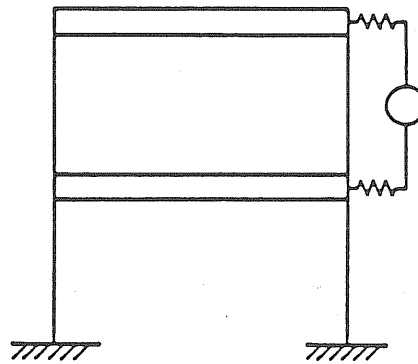
Fig. 4.20 Relative Error for λ_1 for First and Fifth Order Perturbations



Primary Subsystem



Secondary Subsystem



Combined Structural System

Fig. 4.21 Combined Structural System for Example 3

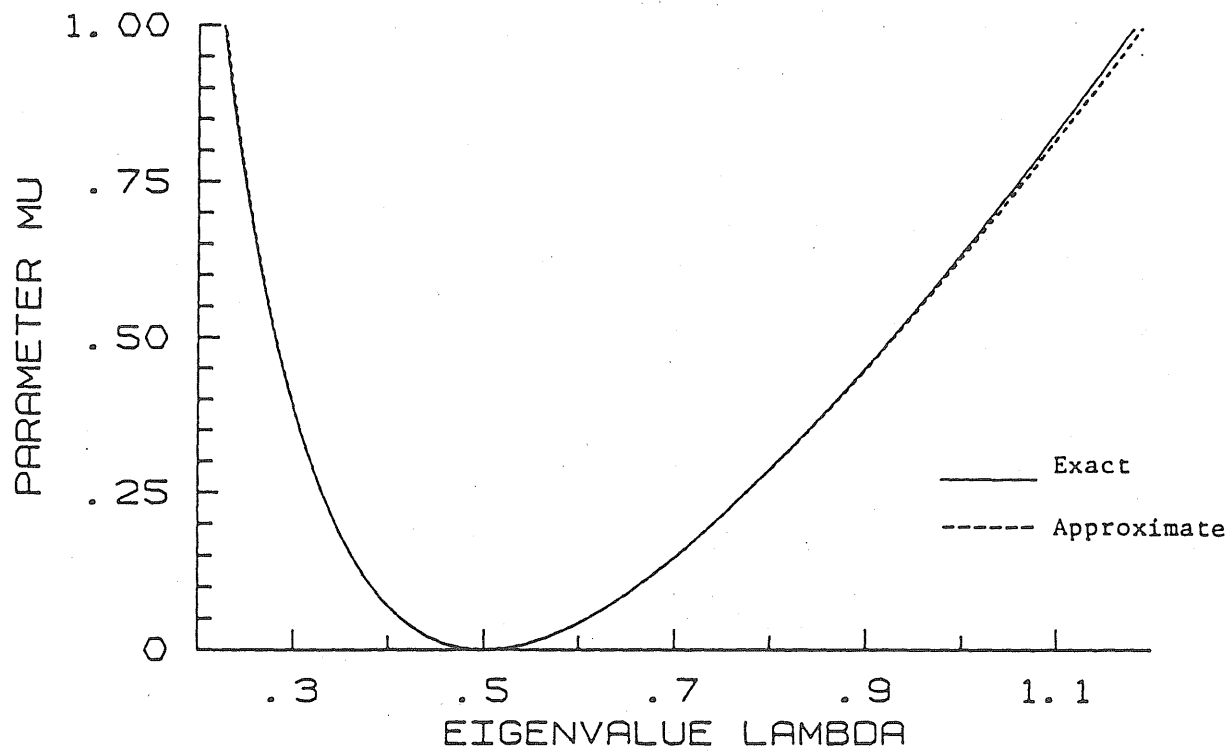


Fig. 4.22 Variation of λ_1 and λ_2 for Third Order Perturbation

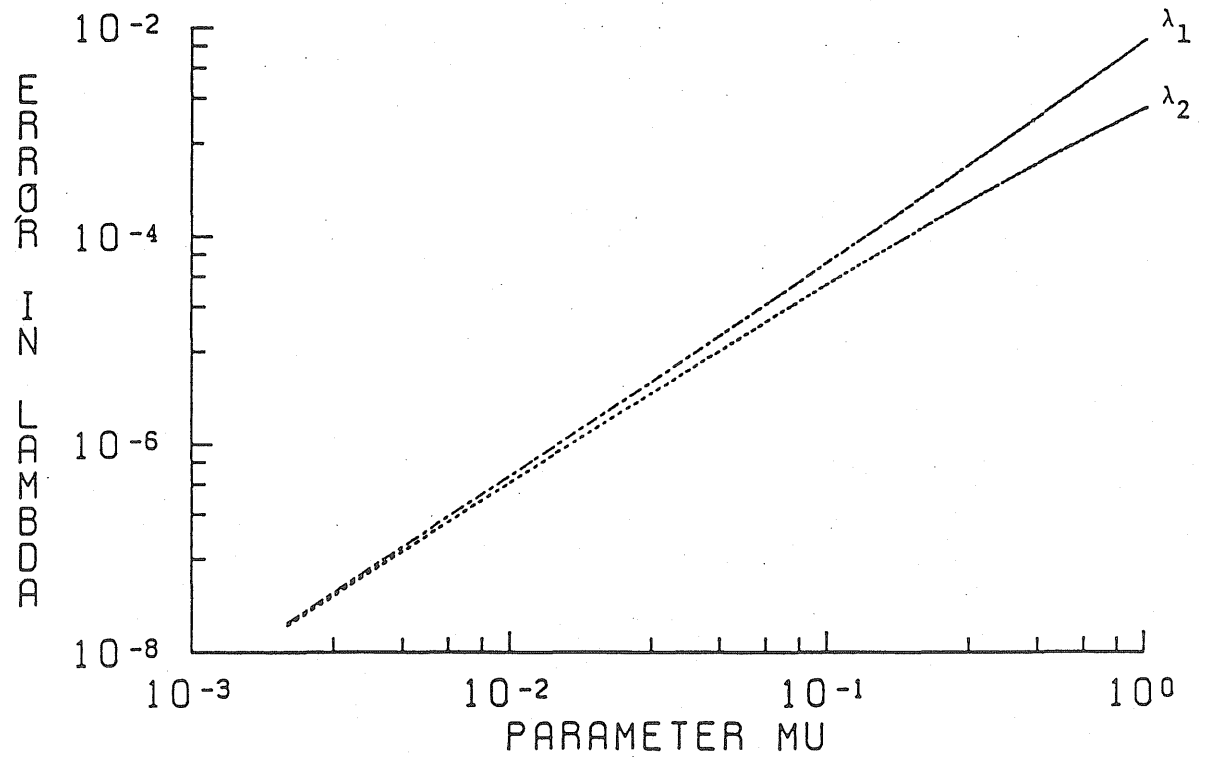
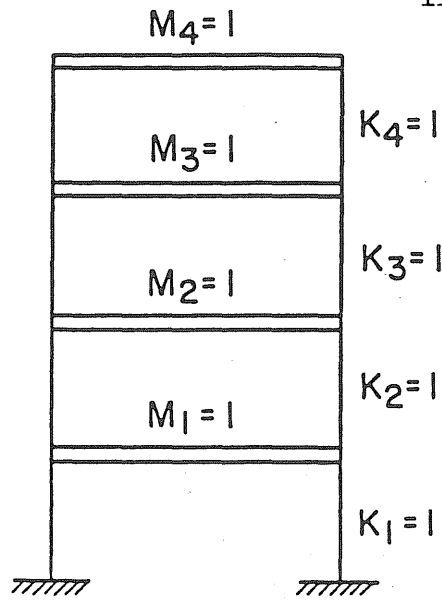
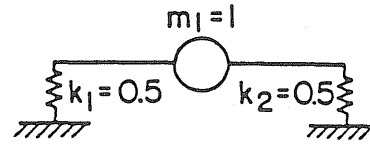
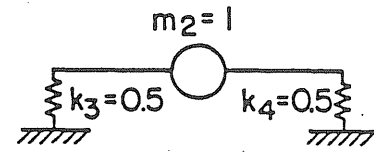


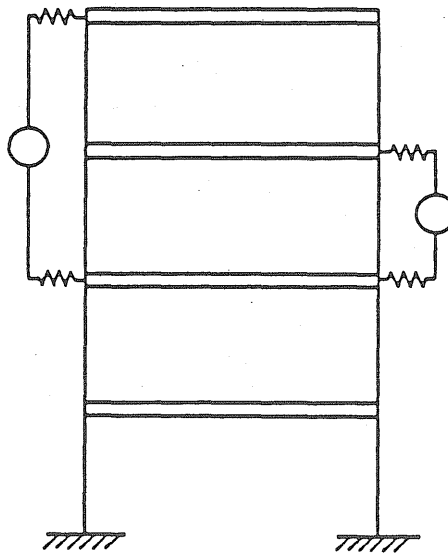
Fig. 4.23 Comparison of the Error in Approximation Between the Exact and the Third Order Perturbation for λ_1 and λ_2



Primary Subsystem



Secondary Subsystems



Combined Structural System

Fig. 4.24 Combined Structural System for Example 4

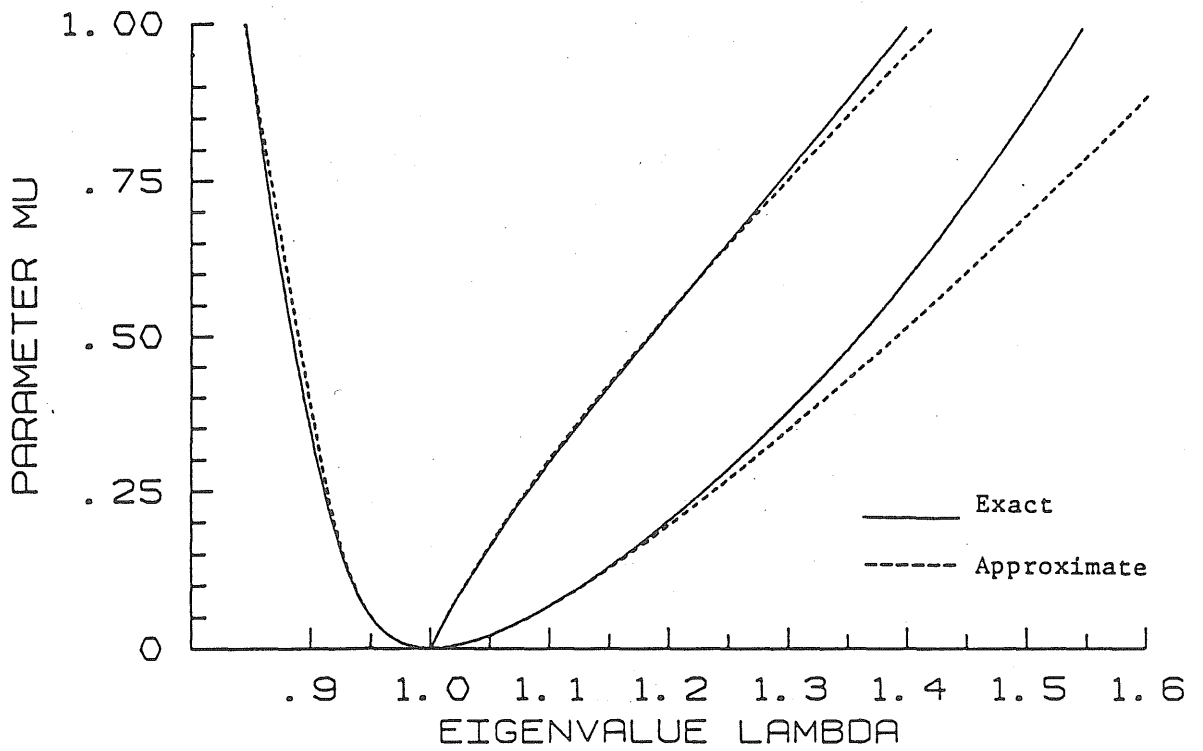


Fig. 4.25 Variation of λ_1 , λ_2 and λ_3 for Third Order Perturbation

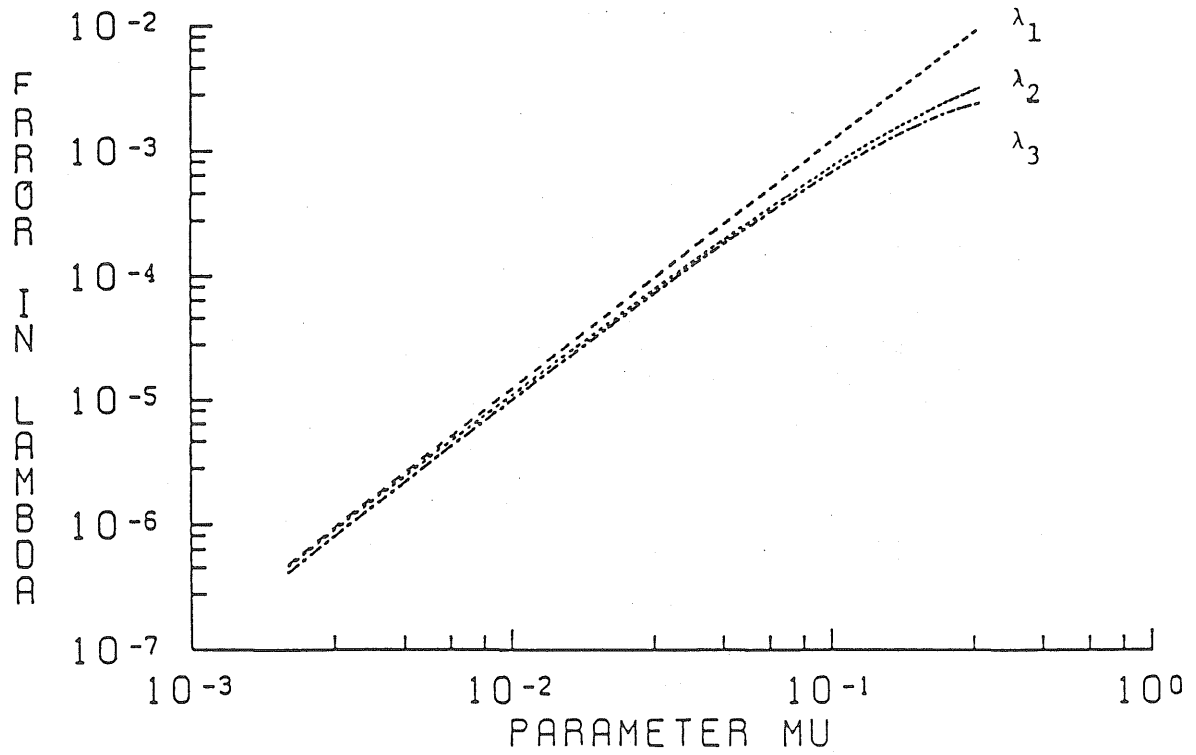


Fig. 4.26 Comparison of the Error in Approximation Between the Exact and the Third Order Perturbation for λ_1 , λ_2 and λ_3

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