Tampere University

HELEN ALFARO VÍQUEZ

Are they ready?

A study about preservice mathematics teachers' education in Costa Rica

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ACADEMIC DISSERTATION
To be presented, with the permission of
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for public discussion
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The real voyage of discovery consists not
in seeking new landscapes,
but in having new eyes.

El verdadero viaje de descubrimiento no consiste
en buscar nuevas tierras,
sino en tener nuevos ojos.
Marcel Proust

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When I started this journey six years ago, I had high expectations of what the country, the people, the university, Tampere would be like and what I would learn. However, over time my expectations were transformed and, in some cases, exceeded. In Finland I learned to be resilient, to walk despite the freezing cold and to be grateful with every sunshine. I learned to find peace and joy in its pine forests and in the calm waters of its beautiful lakes. I met people of many nationalities who showed me the beauty of differences and the importance of kindness. I tried dishes that I would never have dared to eat and enjoyed landscapes that leave me speechless. But most importantly, I learned a lot about myself, about my weaknesses and strengths, about how much I love my family, my country and my people. So thank God and life for giving me the courage to start and finish this learning journey.

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On a very hot day in Milano, June 13, 2022

Helen Alfaro Víquez

ABSTRACT

The knowledge, professional skills and beliefs of mathematics teachers significantly influence their quality of teaching. Teacher education programs (TEPs) offer pre-service teachers (PSTs) opportunities to acquire the knowledge and competencies they need to teach effectively. In Costa Rica, however, little is known about mathematics TEP content, quality, and outcomes, and there are no selection processes that assess the knowledge and aptitudes of teachers before they are hired. Recent reports have urged universities to update their TEPs to address the deficiencies observed in in-service teachers. This study reports on the characteristics of the mathematics TEPs in Costa Rica by investigating the TEP contents and teaching methods, the beliefs on mathematics education by the PSTs and teacher educators, and the relevant knowledge and competencies of the pre-service mathematics teachers at the end of their studies.

The knowledge necessary for teaching mathematics has been studied by different theoretical frameworks (e.g., Ball et al., 2008; Carrillo et al, 2018) which consider the knowledge categories defined by Shulman (1986) about content knowledge and pedagogical content knowledge. However, professional competence in mathematics is integrated by the cognitive abilities and the affective-motivational characteristics. In this study the cognitive abilities component is approached with the Knowledge for Teaching Mathematics framework (Tatto et al, 2008) informed by Shulman's (1986) categories of CK, PCK and general pedagogical knowledge. In addition, the affective component is studied considering the beliefs about the nature of mathematics and mathematics teaching and learning.

The results of this dissertation are informed by qualitative and quantitative data, collected using the instruments of Teacher Education and Development Study in Mathematics (TEDS-M) international study. The study was conducted in Costa Rica during autumn 2019 with participants from three public universities. In total, 80 future mathematics teachers in their last year of preparation and 19 teacher trainers collaborated as participants. Data from preservice teachers was collected using a paper-and-pencil questionnaire, while teacher educators answered an online questionnaire. The statistical analysis of the learning opportunities, the beliefs, and the performance of the participants in the items, was complemented with a content analysis of the solutions to the items to have a more holistic understanding of the question under study.

The results showed that the TEPs taught more tertiary-level mathematics subject matter topics than mathematics education and general pedagogy topics using various methods such as lectures, pre-service teacher presentations, reading of related research, and solving math problems. They also taught instructional planning and assessment, but little critical and reflective skills to serve students from different backgrounds or to offer meaningful feedback. The TEPs trained PSTs well in applying skills but poorly in reasoning. In addition, significant weaknesses were observed in participants' monitoring of their own work and in modeling solution strategies and connecting results for solving problems. Moreover, the PSTs and teacher educators had dynamic constructivist beliefs but neglected teacher-centered practices and mathematics as a set of rules and procedures. Besides, they believe that mathematics can be learned by everyone despite of their culture, gender, or background.

This study revealed differences in the way TEPs distribute their topics and the teaching methods experiences they offer. Differences were also found in the performance of the preservice teachers at the different universities, especially in the items of mathematical content knowledge, although the number of the topics studied was not correlated with the participants' performance.

This research has several contributions. First, it contributes to the knowledge gap about preservice mathematics teachers in Costa Rica, providing insights about where they stand at the end of their preparation programs, regarding knowledge and competencies for teaching mathematics, and what needs to be improved. It also reaffirms previous results about differences in TEPs but goes further pointing out how those differences are evident in the opportunities to learn and the preservice teachers' knowledge. The study also makes visible the preservice teachers and teacher educators' beliefs about mathematics nature, mathematics teaching, and achievement, which have been understudied in Costa Rica and Latin America.

Keywords: knowledge for teaching mathematics, PSTs, mathematics teacher education, TEDS-M, opportunities to learn, mathematics beliefs

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ABBREVIATIONS

CK	Content Knowledge
COACTIV	Cognitive Activation in the Classroom
IEA	Association for the Evaluation of Educational Achievement
LMT	Learning Mathematics for Teaching
MCK	Mathematical Content Knowledge
MEP	Ministry of Public Education
MKT	Mathematical Knowledge for Teaching
MPCK	Mathematical Pedagogical Content Knowledge
MTSK	Mathematics Teacher’s Specialized Knowledge
MUST	Mathematical Understanding for Secondary Teaching
OTL	Opportunities to Learn
PCK	Pedagogical Content Knowledge
PEN	State of the Nation Program
PST	Pre-service teachers
TEDS-M	Teacher Education and Development Study in Mathematics
TENK	Finnish National Board on Research Integrity
TEP	Teacher Education Program
U1	University 1
U2	University 2
U3	University 3
U4	University 4

ORIGINAL PUBLICATIONS

Publication I	Alfaro, H., & Joutsenlahti, J. (2020). What Skills and Knowledge Do University Mathematics Teacher Education Programs Give Future Teachers in Costa Rica? *European Journal of Science and Mathematics Education*, 8(3), 145-162. DOI: https://doi.org/10.30935/scimath/9553
Publication II	Alfaro, H., & Joutsenlahti, J. (2021). Mathematical Beliefs Held by Costa Rican Pre-Service Teachers and Teacher Educators. *Education Sciences*, 11(2), 70. DOI: https://doi.org/10.3390/educsci11020070
Publication III	Alfaro, H. (2022) Costa Rican Preservice Mathematics Teachers' Readiness to Teach. *International Electronic Journal of Mathematics Education.* 17(2), em0676. DOI: https://doi.org/10.29333/iejme/11712

The above publications are referred to in the text by the Roman numerals I-III. The full version of the articles has been included at the end of this thesis.

AUTHOR'S CONTRIBUTION

As a researcher, I am the first and corresponding author of all publications. I was in charge of the studies' conception and design, data collection and processing, data analysis and interpretation, and the preparation, writing and editing of the manuscripts. Docent Jorma Joutsenlahti supervised the doctoral research and provided feedback and comments for all the manuscripts.

1 INTRODUCTION

1.1 Rationale

Many nations share an interest in improving students' learning in mathematics. Several studies have found that the "teaching quality" is the school-related factor that has the greatest influence on students' achievement (e.g., Hsieh et al., 2011; Organisation for Economic Co-operation and Development or OECD, 2005). More specifically, about mathematics teachers, it has been found that elements such as the contents studied in teacher education programs (TEPs) influence teachers' knowledge (Schmidt, Houang, et al., 2011), the beliefs of teachers about mathematics and its teaching influence their practice (Nespor, 1987; Speer, 2005; Voss et al., 2013), and together, they inform what and how teachers teach, which affect students' learning (Hill et al., 2005). Therefore, to achieve high math teaching quality, it is important to know and regulate mathematics teachers' knowledge and beliefs.

In the OECD report "Teaching Matters" (2005), it was revealed that many countries are concerned about "whether enough teachers have the knowledge and skills to meet school needs" (OECD, 2005, p.10). Adler et al. (2005) stated that for students to achieve mathematics proficiency, their teachers must be prepared to foster such proficiency in them. However, TEPs have been seen as ineffective in honing teachers' professionalism (Kaiser et al., 2017). Similarly, math TEPs have been criticized for not fulfilling the knowledge needs of pre-service teachers (PSTs) for effective math teaching in school (Alfaro et al., 2013; Koponen et al., 2016). Thus, TEPs should be designed or updated with the aim of providing quality training for math PSTs.

Such a task is a big challenge, considering the lack of "a shared knowledge base for building more effective teacher preparation programs" (Hiebert et al., 2003, p. 202) and the differences in mathematics teacher education traditions from country to country (Blömeke, 2012). Many efforts have been made to design frameworks that identify the knowledge that is considered nessesary and sufficient for teaching mathematics, based on observations of teaching practice and conversations with in-service teachers and experts (e.g., Ball et al., 2008; Carrillo et al., 2018). Nevertheless, there is no shared framework nor well-defined theoretical grounds yet (Hoover et al., 2016). However, many frameworks consider, to some extent, the categories defined by Shulman (1986) of content knowledge, pedagogical content knowledge, and pedagogical knowledge.

Considering the Costa Rican context, the problem regarding the deficit in mathematics education is worrying. In the Program for International Student Assessment (PISA) tests from 2009 to 2018, the performance of Costa Rican students has been below the OECD average by approximately 90 points, and it has not changed significantly (OECD, 2019). A 90-point gap is interpreted as a three-year gap between Costa Rican students and the students of other OECD member countries (Programa Estado de la Nación, PEN, 2017). In 2012, the Costa Rican Ministry of Education (MEP) introduced a new mathematics curriculum that had constructivist foundations and focused on working with students on the mathematical processes of solving problems, reasoning and argumentation, as well as representing and connecting concepts or mathematical objects, and communicate mathematical ideas (MEP, 2012). However, the change did not improve the students' PISA results (OECD, 2019) or school achievement (PEN, 2019).

To investigate what was happening in math lessons, PEN (2019) conducted classroom observations and interviews with teachers. The results showed that the teachers were unable to implement what was established in the curriculum, and classes continued to be teacher-centered. In addition, it was found that the teachers had serious weaknesses in their initial preparation, which made it difficult for them to implement the new methodologies, and that the training offered by the MEP could not correct these deficiencies (PEN, 2019).

The poor preparation of mathematics teachers for implementing the new curricula is not the only problem regarding teaching quality. In 2010, the MEP diagnosed in-service mathematics teachers' knowledge of mathematics topics taught in secondary school. The results showed that 43.4% (N = 1,733) of the teachers performed below the average, which suggested differences in the teachers' mathematical knowledge. There were also differences between the mathematics teachers who graduated from public universities and those from private institutions, which indicated differences in their TEPs (MEP, 2011).

Recent reports on Costa Rican teacher policies and issues (PEN, 2019; Roman & Lentini, 2018) highlighted key recommendations for improving teaching quality. Among them are the establishment of a national framework for teacher qualification, the introduction of in-service teacher assessments to identify professional development needs, the revision of the poor and obsolete teacher recruitment policies that do not require assessment of teachers' knowledge and aptitudes before they are hired, and control over the variation and quality of TEPs.

Studies related to the knowledge and beliefs for teaching mathematics have been focused on some populations while leaving others aside. In Costa Rica, for instance, most of the studies conducted with mathematics teachers focused on in-service teachers (e.g., Chávez, 2013; MEP, 2011). In Latin America very few articles cover the topic (Hoover et al., 2016), and only one country has participated in international studies regarding mathematics teachers' knowledge (Tatto et al., 2008). Moreover, most of the studies were conducted with primary school teachers, and there is a need for more large-scale studies (Hoover et al., 2016).

The mentioned arguments point out the relevance of studying the knowledge and beliefs of Costa Rican pre-service mathematics teachers as they will provide insights regarding the content, quality of training, and effectiveness of different mathematics TEPs in Costa Rica and because such teachers' knowledge and beliefs will define their future teaching approaches.

1.2 Mathematics teacher education in Costa Rica

In Costa Rica, both public and private universities offer education programs for mathematics teachers. Due to the government's lack of control over teaching policies, TEPs vary depending on the type of institution. Currently, there are eight public and private universities that offer major programs for becoming a mathematics teacher, that is, a bachelors' degree program in the teaching of mathematics, with 120 to 144 credits (Alfaro et al., 2013). These major programs could take four years in public universities and two years and a half in private institutions, with the option of a licentiate degree that requires more courses and writing a thesis. Although TEPs differ widely in duration, focus, and content, all of them include courses in mathematics, education, and mathematics education, with different numbers of courses in each area and different levels of integration.

Universities have not established specific requirements for people who wish to be admitted to a bachelor's program in mathematics education. The general requirement is to have a high school certificate and to perform the corresponding administrative procedures to enroll, with the exception of three public universities that also require passing an admission exam (Alfaro et al., 2013). Thus, the entry profile of future mathematics teachers is not filtered at all.

Training is delivered differently depending on the institution. In some universities, there is a *school of mathematics* in charge of math courses and a *school or department of education* in charge of pedagogical courses. Mathematics education courses may be taught by one school or the other depending on the availability of faculty. However, there are universities that have only one school attending all the TEP courses, the *school of mathematics* or the *school of education*. The last case is observed in private universities.

As for teacher educators in charge of preparing mathematics teachers, there are no general standards for their academic preparation. The selection of teacher educators depends on the university policies and the approach of its TEPs. Therefore, there are teacher educators with doctoral degrees in mathematics, mathematics education, or education and teacher educators with only master's or licentiate's degrees in the same areas. Mathematics courses are typically taught by teacher educators who are mathematicians, while teacher educators with degrees in mathematics education may be assigned to teach any course, including school experience or the practicum. Teacher educators specializing in education teach courses related to general pedagogy. However, depending on the university, it is possible to find differences in the responsibilities of teacher educators.

The main job opportunity for professionals in the teaching of mathematics is teaching in secondary school, that is, in grades 7 to 11, in public or private high schools. In this case, the main hiring entity is the MEP. The MEP teacher hiring system dates from 1970. It requires a bachelor's or licentiate's degree in teaching, in this case, in teaching mathematics, and affiliation with the respective professional association (Roman & Lentini, 2018). According to the degree earned (a licentiate's degree is higher than a bachelor's degree), the number of years of teaching experience, and the professional development of the teacher, a score is assigned to the teacher. The teaching positions are filled first with the teachers with the best score and according to their geographical preferences. Differences in teacher training or in the person's vocation or ability to teach are not considered (Roman & Lentini, 2018). Teachers who work for the MEP may have an interim or proprietary position, and a full-time job means teaching 48 lessons of 40 minutes each per week. The time for planning the lessons, designing assessments, or grading them are not included in the paid time. Although teachers working for the MEP can have job stability, a good salary, and incentives, teachers who work in public institutions constantly complain of non-pedagogical workloads that shorten their teaching time (Actualidad Educativa, 2018), a situation that has worsened with the pandemic.

Currently, the excess supply of teachers and thus, the reduced demand by the MEP (Roman & Lentini, 2018) have caused mathematics teachers to seek other job opportunities, for example, as teachers in private schools, private high schools, universities, or as tutors.

1.3 Aims and research questions

The aim of this study is to describe the knowledge that Costa Rican pre-service mathematics teachers acquire in their TEPs. Such knowledge involves the content that they had the opportunity to study, the beliefs they hold about mathematics and its teaching, and how they use such knowledge in solving tasks about mathematical content knowledge (MCK) and mathematical pedagogical content knowledge (MPCK). Thus, the findings from this study are aimed at filling the knowledge gap in the country regarding the knowledge gains of pre-service mathematics teachers from their TEPs. Gain insights from these findings on the weaknesses of such mathematics TEPs that could be improved and their strengths that could be reinforced, is also expected. To achieve the aim, I pose four main research questions and nine sub-questions, which had been answered in three studies.

1) What are the opportunities to learn (OTLs) offered in the Costa Rican TEPs? (**Article I**)
    a) How are the OTLs distributed in the knowledge areas? (**Article I**)
2) How did the pre-service mathematics teachers perform in the assessment of their knowledge for mathematics teaching? (**Articles I & III**)
    a) How did they perform in the geometry, algebra, numbers, and data subdomains? (**Article I**)
    b) What is their performance in the knowing, applying, and reasoning subdomains? (**Articles I & III**)
    c) How did they perform in the enactment and curriculum and planning skills domains? (**Articles I & III**)
    d) How was their understanding for teaching mathematics demonstrated in the test items? (**Article III**)
3) What are the beliefs of Costa Rican PSTs and teacher educators about the nature of mathematics, mathematics teaching and learning, and mathematics abilities? (**Article II**)
    a) What factors influence the beliefs of PSTs? (**Article II**)
    b) What factors influence the beliefs of teacher educators? (**Article II**)

4) How are the OTLs, beliefs, and performance in the knowledge for teaching mathematics assessment related? (**Articles I and II**)

    a) How are the OTLs and the results of the knowledge for teaching mathematics assessment related? (**Article I**)

    b) How are the beliefs and the results of the knowledge for teaching mathematics assessment related? (**Article II**)

The abovementioned research questions were derived from the integration of the questions that guided each of the cited articles. The integration and the need for two articles to answer a research question are due to the complexity of the topic under study and of how the OTLs, beliefs, and knowledge for teaching mathematics are intertwined in the body of professional competencies needed for teaching mathematics. In this way, the first results about OTLs and the PSTs' performance in the knowledge for teaching mathematics tasks were used, for instance, to understand and extend the studies about beliefs and allow to approach PSTs' performance in MKT and MPCK tasks from different perspectives.

1.4 Research process

According to Tripodi and Bender (2010), the process of building social knowledge can be seen as a continuum, where exploratory research first identifies the phenomenon and variables that need to be investigated, then descriptive research understands the characteristics of that phenomenon and examines possible relationships between its variables, and finally, explanatory research draws causal inferences. In this study, I considered the State of Education (PEN, 2019) and the *Costa Rican Teaching Policy* (Roman & Lentini, 2018) reports as exploratory sources that indicate the needs that must be addressed with respect to TEPs, specifically the differences in the quality, quantity, and duration of the TEPs, as well as in their suitability to meet the needs of the Costa Rican education system. Therefore, following that continuum, I performed a descriptive study of such TEPs from the experiences and performance of pre-service mathematics teachers.

A descriptive study answers who, what, when, where, and how questions to describe a social phenomenon, that needs to be described (Tripodi & Bender, 2010) to have a better understanding of it. In this study, I aim to describe what knowledge the pre-service mathematics teachers acquire in their TEPs, how this knowledge is acquired (methodological mediation), and how much they know about mathematics for teaching.

A mixed, qualitative and quantitative, methods approach was used in this study. After a literature review regarding mathematics teachers' knowledge frameworks and measurement instruments, in this study the Teacher Education and Development Study in Mathematics (TEDS-M) questionnaire is used as the instrument for collecting data. TEDS-M is the first large-scale international study with future mathematics teachers and was developed under the auspice of the International Association for the Evaluation of Educational Achievement (IEA). TEDS-M was chosen because it was designed to meet international standards, has a section specifically for secondary mathematics PSTs, which allows to describe TEPs in terms of the OTLs that they offer, the beliefs of the PSTs and their educators, and the assessment of the PSTs' knowledge for teaching mathematics.

The first step in this study was getting the permission of the IEA (see Appendix 1) to use and translate the questionnaire. When its permission was granted and the questionnaire, provided, it was translated to Spanish, which is the researcher's first language, and its translation and contextualization was validated by three Costa Rican mathematics educators who were not part of the project. For the application, the researcher went to Costa Rica in the fall of 2019 to ensure good communication with the participants and uniform application conditions. It is important to mention that that was the only window of time to collect the data from the PSTs and the teacher educators from the four universities. After the data were collected, coded, and cleaned, different analyses were performed according to the information in the different sections of the questionnaire. Figure 1 shows the research process, including the topic, the participants, and the treatment of the data in each of the articles. A summary of the articles is also provided in the next section.

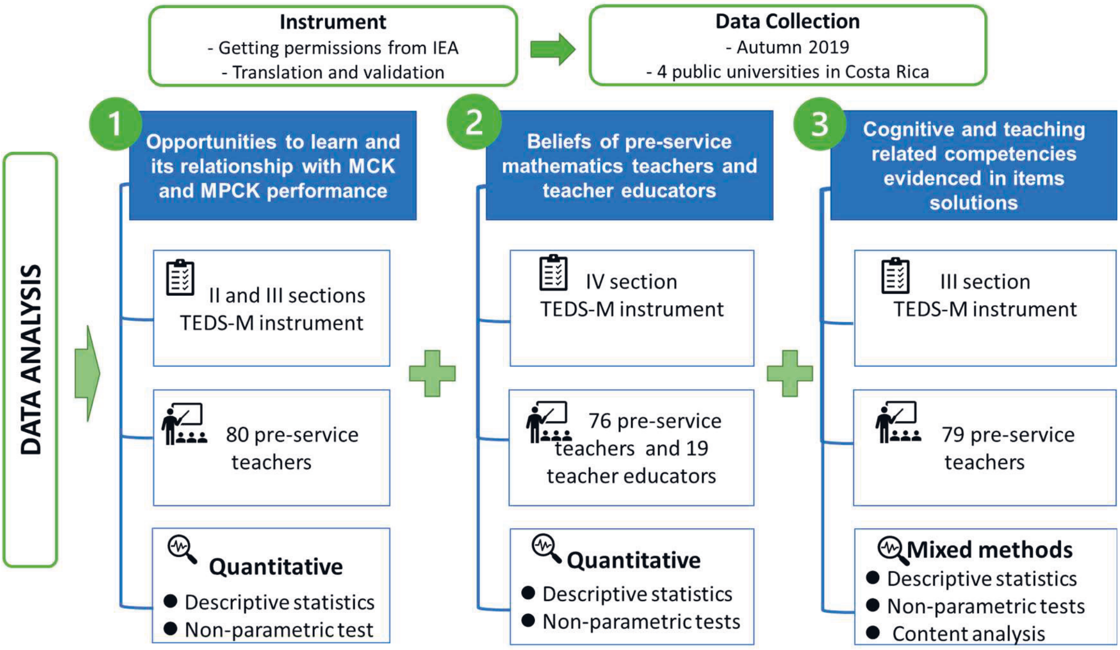


Figure 1. Research Process

Article I

Article I studies the learning opportunities offered in the TEPs of four universities in Costa Rica according to the OTLs experienced by the PSTs. It also includes a quantitative analysis of the results of the MCK and MPCK assessment. Both inputs made it possible for us to study, for example, the relationships between the MCK results obtained by the PSTs and the mathematical contents studied, or between the PSTs' MPCK performance and teaching methods experienced. Besides, it was possible to identify which topics receive more attention in Costa Rican TEPs and if there were differences in the way the studied topics were distributed in each TEP and if the PSTs' performance differed among the universities and how.

The results obtained from Article I served to begin the description of the TEPs and the possible contents that the PSTs could have studied in them. This is an important input for analyzing the knowledge acquired by the PSTs, which is expected to be observed in the knowledge assessment section. However, considering that beliefs influence the way in which teachers approach knowledge and their practices, I decided to first analyze the beliefs, to evaluate the participants' responses to the knowledge items in a more informed way.

Article II

Article II explores the beliefs of PSTs and their teacher educators regarding the nature of mathematics, its teaching and learning, and mathematical abilities. The results revealed the belief patterns of both groups of participants. In addition, the relationships of the PSTs' beliefs with the TEP to which they belong, with the references about their school performance in mathematics and with the results of the TEDS-M knowledge assessment section were analyzed. Regarding the teacher educators, the relationships between their beliefs and their years of experience in training mathematics teachers, their academic background, and their special preparation to train teachers were studied.

Having collected the information on the participants' OTLs and beliefs as well as some facts about their performance in the MCK and MPCK knowledge assessment, qualitative content analysis of their solutions of the tasks was performed, to gain more information about their knowledge for teaching mathematics.

Article III

Article III takes a more detailed look at the PSTs' answers to the TEDS-M items. Since there were three types of items in the test—multiple-choice, complex multiple-choice, and complex-response—I considered that a deep analysis of the solutions could expand the description of the knowledge for mathematics teaching of future Costa Rican teachers. A theory-driven content analysis was performed using the TEDS-M (Tatto et al., 2008) and Mathematical Understanding for Secondary Teaching (MUST) frameworks (Kilpatrick et al., 2015). More specific results on errors and weaknesses in the content domain or in the teaching-related skills that cannot be determined merely by evaluating an answer as correct or incorrect, were obtained.

Adding the results of the three studies will allow a better understanding of the knowledge for teaching mathematics of Costa Rican PSTs.

2 THEORETICAL BACKGROUND

Teaching is a very complex profession due to the multiple tasks that teachers must perform on a typical school day. For example, in addition to their tasks that are strictly related to teaching, they must also participate in school activities and perform administrative tasks. However, the core activity inherent to the profession is teaching, for which teachers need to develop professional competencies in both the cognitive and affective aspects (Döhrmann et al., 2012). According to Potari and Ponte (2017), "teachers need to know about the subject that they teach, they need to know how to teach it, and they need to know how to act and behave as teachers" (p. 3). Nevertheless, the conception of teacher competencies has evolved from that which considered only the cognitive abilities, known as professional knowledge, and the affective-motivational characteristics such as beliefs, motivation, and self-regulation (Döhrmann et al., 2012), to a more complex model that considers competencies a "continuum with dispositions closely related to observable performance […] fully or partially mediated by situation-specific cognitive skills" (Blömeke & Kaiser, 2017, p. 786). In this new model, the professional knowledge and the beliefs-motivational facets are considered the dispositions that inform the performance.

The cognitive abilities or professional knowledge specific for teaching mathematics has been studied for several years now and from different perspectives. Hoover et al. (2016), after reviewing literature on mathematics knowledge for teaching, identified three main lines of the research: (1) the nature and composition of mathematical knowledge for teaching, (2) the development of teachers' mathematical knowledge for teaching, and (3) the impact of mathematical knowledge for teaching. The elaboration of frameworks about mathematics teachers' knowledge was included in the main line of the nature and composition of the specialized knowledge for teaching mathematics. In this regard, the ideas of Shulman (1986) about the importance of the specific knowledge related to the teaching profession and the subject to teach are crucial. Shulman proposed that knowledge for teaching can be divided into different types, one of them being content knowledge for teaching. Then, he proposed to divide the content knowledge into three categories:

content knowledge (CK), pedagogical content knowledge (PCK), and curricular knowledge.

Since this seminal work of Shulman, many frameworks have been developed based on the aforementioned three categories of content knowledge. Some frameworks define subcategories or subdomains of knowledge to provide a more detailed description of knowledge categories, such as the Mathematical Knowledge for Teaching (MKT) framework of Ball et al. (2008), which divides content knowledge into common content knowledge and specialized content knowledge. Others were built from modifications in the definition of the categories of other theoretical frameworks or by adding new subdomains of knowledge. For example, the framework Mathematics Teacher's Specialized Knowledge (MTSK) by Carrillo et al. (2018) considers the conceptions of the MKT framework, identifies deficiencies in terms of the delimitation of the categories and the knowledge that corresponds only to the math teacher, and establishes new categories. It is possible to identify many other frameworks that are not necessarily based on each other but are built from different contexts and generate categories that appear similar but approach the category topic differently (e.g., Knowledge Quartet by Rowland et al., 2005 and Professional Knowledge of Secondary School Mathematics Teachers by Baumert et al., 2010). The existence of new and different theoretical frameworks of the mathematical knowledge needed to teach reveals the lack of agreement among scholars on definitions and basic concepts (Hoover et al., 2016) as well as problems with the blurred boundaries between the established categories.

In this regard, Hoover et al. (2016) pointed out that instead of developing new frameworks, "the one avenue of work that represents progress on the field is the development of instruments […] as they serve to operationalize ideas about mathematical knowledge for teaching and test assumed models of the role it plays" (p. 9). Here, it is possible to mention three important standardized instruments developed to study the knowledge for teaching mathematics (see Figure 2). First, the Learning Mathematics for Teaching (LMT) instrument was designed in the US to understand practicing elementary and middle school teachers' knowledge for teaching and to improve it by means of professional development programs (Ball et al., 2008). Second, the Cognitive Activation in the Classroom (COACTIV) project was created in Germany to examine "the implications of CK and PCK for processes of learning and instruction in secondary level mathematics" (Baumert et al., 2010, p. 135) with in-service secondary teachers. Third, TEDS-M is an international study that investigates the preparation of mathematics teachers, both primary and secondary (Tatto et al., 2008), in which the participants are PSTs.

Each of these projects uses a different framework for designing the items and knowledge categories to measure. In a study of the comparison of the three frameworks, Kaarstein (2014) noticed that all of them “build on or use Shulman’s categories as part of their theoretical background” (p. 38), specifically CK and PCK, and that although the frameworks cover Shulman’s description of the categories, they do it using different subdomains. Moreover, Kaarstein (2014) found that an item designed to measure PCK in the LMT project could be classified as measuring CK in the TEDS-M CK description. This issue highlights the existence of blurred boundaries between CK and PCK and reinforces the idea that these categories are not mutually exclusive, as, for instance, PCK usually depends on CK (Döhrmann et al., 2012; Potari & Ponte, 2017).

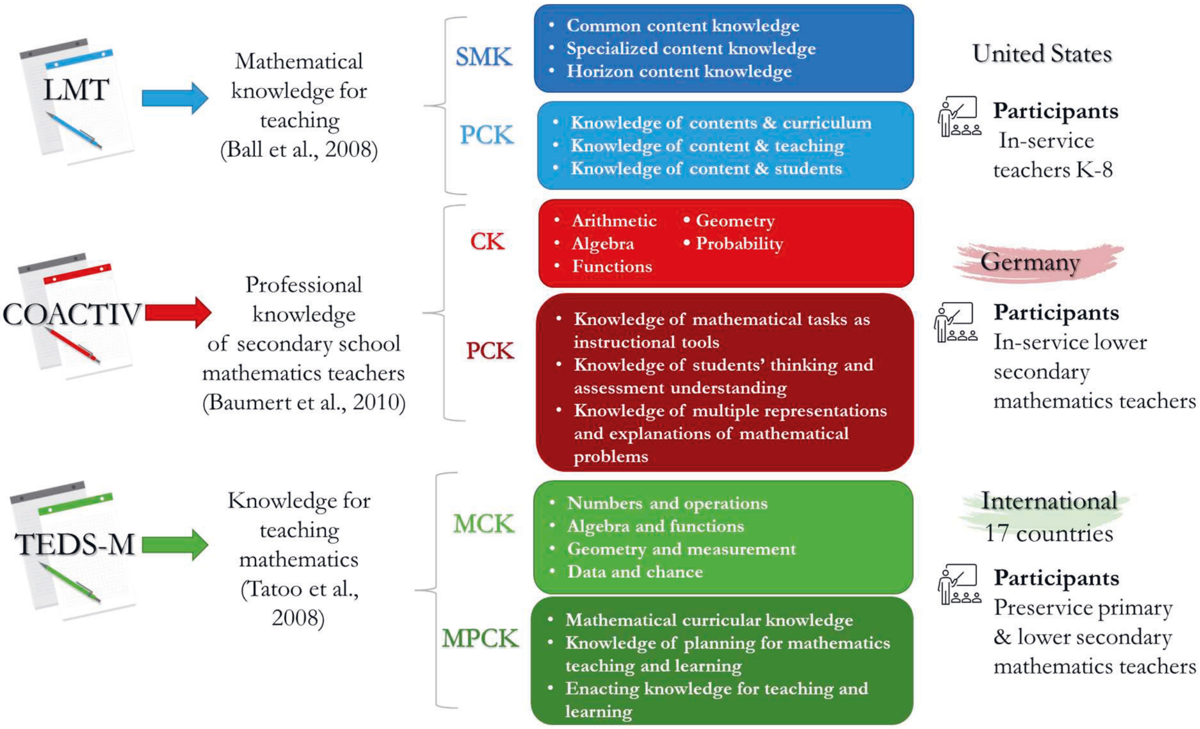


Figure 2. Projects Measuring Knowledge for Teaching Mathematics

Considering the complexity of delimitating which knowledge to categorize as CK and which as PCK in teaching practice, Kilpatrick et al. (2015) presented a different approach. They developed the MUST framework, which adopts a more dynamic position. First, the authors differentiate understanding from knowledge and choose the former because they consider it a growing construct, that is, as evolving and becoming deeper during a teacher's career. In addition, they stated that understanding can be "viewed as the use of [the] knowledge [that] one has" (p. 10), and thus, is observable, instead of knowledge that is difficult to see. Another important characteristic of the MUST framework is that the kind of understanding that it defines "is not a [simple] matter of 'knowing the mathematics' adjoined to 'knowing how to teach'" (Kilpatrick et al., 2015, p. 10).

From the different theoretical frameworks and instruments presented in this section, this study will elaborate on the theoretical framework of the TEDS-M with regard to the professional competencies for teaching mathematics, including the learning opportunities, beliefs, and knowledge for teaching mathematics. Furthermore, this study will consider the MUST framework to complement the theoretical background.

2.1 Teachers' professional competencies for teaching mathematics

The TEDS-M study aims to investigate and compare teacher preparation across countries by considering teachers' professional competencies as outcomes of TEPs. According to Blömeke and Delaney (2012), the way in which teacher competencies are understood in the TEDS-M framework is associated with the notions of Niss (2003) regarding teacher competence in mathematics and with the conception of Schoenfeld and Kilpatrick (2008) of proficiency in teaching mathematics. Professional competencies can be understood as "having the cognitive ability to develop effective solutions for job-related problems and, in addition, having the motivational, volitional and social willingness to successfully and responsibly apply these solutions in various situations" (Blömeke & Delaney, 2012, p. 227). Two important dispositions integrate the competencies: the cognitive abilities, which, in the case of the TEDS-M study, are informed by Shulman's (1986) categories of CK, PCK, and general pedagogical knowledge; and the affective-motivational characteristics, which, in the TEDS-M context, include beliefs about mathematics and its teaching and learning. This theoretical orientation takes a multidimensional approach to "come as closely as possible to real behavior in the classroom that is supposed to be guided by both types of dispositions" (Blömeke & Delaney, 2012, p. 227), but it does not consider the situational or practical aspects that inform and form the process of becoming proficient in specific teaching domains (Kaiser et al., 2017).

Overall, TEDS-M investigates "the opportunities provided and taken by preservice teachers while engaged in teacher preparation toward developing the competencies deemed by the literature to be relevant to quality classroom instruction" (Schmidt, Cogan, et al., 2011, p. 139). In doing so, the study collects information from three sources: teachers' professional knowledge, teachers' beliefs, and OTLs in TEPs.

2.1.1 Opportunities to learn

The OTLs were covered in Article I. In TEDS-M, they correspond to the contents studied in the TEPs, and the teaching methods by which the contents were taught, and teaching skills were trained. The contents and structure of TEPs respond to the context needs, that is, to what a mathematics teacher is expected to know to teach effectively in a specific country and context and are conditioned by cultural and political norms (Blömeke & Kaiser, 2014). In the context of the TEDS-M study, the OTLs are considered central to explain how teacher preparation impacts teacher learning (Tatto et al., 2008). Following Blömeke (2012), three types of OTLs that have impacts on teachers' outcomes can be mentioned. One type is the OTL in terms of mathematical knowledge, which according to the author, are the basis for the teaching of mathematics and presenting mathematical content in a meaningful way to students. The professional preparation on how learners acquire mathematical knowledge and how to design classes and instruments to teach diverse students is another type. Finally, the quality of teaching methods experienced during the TEPs, meaning class participation and practice-teaching opportunities, are also associated with the outcomes of teacher education.

These three types of OTLs were investigated in TEDS-M. TEDS-M also surveys the topics related to general pedagogy and mathematics education pedagogy. Figure 3 summarizes the OTLs in the TEDS-S study that are important for this research.

Opportunities to Learn

Mathematics	**a)Tertiary-level mathematics** (linear algebra, number theory, advanced calculus) **b) School mathematics** (Numbers, measurement, functions and equations)
Mathematics Education Pedagogy	**a) Foundations** (Context of mathematics education, development of mathematics activity and thinking) **b) Instruction** (Mathematics instruction, developing teaching plans)
General Pedagogy	**a) Social Science** (History of education and educational systems, philosophy of education) **b) Application** (Theories of schooling, methods of educational research)
Teaching methods experienced	**a) Class participation** (Class discussions, teach a class) **b) Class reading** (research on mathematics, mathematics education, teaching and learning) **c) Solving problems** (Write mathematical proofs) **d) Instructional practice** (learn how to explore multiple solution strategies with pupils) **e) Instructional planning** (Create learning experiences) **f) Assessment uses** (use assessment to provide feedback to parents and to guide teaching decisions) **g) Assessment practice** (Assess learning goals, analyze pupils' assessment to improve assessment)
Professional Preparation	**a) Teaching for diversity** (Develop strategies for teaching students with learning disabilities, diverse cultural backgrounds) **b) Teaching for reflection on practice** (Develop strategies to reflect about the own professional knowledge, teaching effectiveness…) **c) Teaching for improving practice** (Develop and test new teaching practice, identify appropriate resources for teaching)

Figure 3. Opportunities to Learn Categories
Note. Tatto et al. (2008).

Thus, for the TEDS-M survey, the OTLs in mathematics, mathematics pedagogy and general pedagogy, are studied according to the topics studied in each area, while the OTLs related to the experienced teaching methods and professional preparation are examined through the frequency with which each action was practiced.

2.1.2 Teachers' beliefs

Defining the belief construct has been considered difficult, as discussed in Article II. It is possible to find different interpretations and meanings (Speer, 2005) as well as associations with other constructs such as conceptions, opinions, attitudes, or knowledge, which make the task of giving a specific definition of beliefs difficult. However, there are characteristics such as the existential feature of beliefs that define them as personal truths (subjective), that differentiate them from other constructs such as knowledge, which is considered an objective social construct shared by the general public (Boz, 2008; Furinguetti & Pehkonen, 2002). Various definitions of beliefs can be found. In the field of mathematics education, one widely accepted definition of belief is that offered by Schoenfeld (1992): "an individual's understandings and feelings that shape the ways that the individual conceptualizes and engages in mathematical behavior" (p. 358).

Considering this definition of beliefs and considering mathematics teachers, beliefs can be seen as the bridge that connects knowledge to action (Blömeke & Delaney, 2012). In teaching, the beliefs or the ways teachers conceive the world inform their practices (Boz, 2008; Speer, 2005) in different aspects. For example, in teacher-student relationships, beliefs could influence the way teachers interact with students, as well as the perception and development of student skills (Barkatsas & Malone, 2005; Pajares, 1992; Voss, 2013). On the other hand, regarding the teaching of mathematics, beliefs can influence the way teachers approach the contents, their methodological choices, and their assessment practices (Tang & Hsieh, 2014; Tatto et al., 2012). Therefore, in the field of mathematics education, teachers' beliefs are studied based on the idea that beliefs can explain how mathematics is taught and learned (Skott et al., 2018) and because it can provide "insight into the way teachers understand and carry out their job" (Ponte, 1999, p. 43).

According to Voss et al. (2013), teachers' beliefs can be grouped into three levels of belief systems. However, in mathematics education, the focus has been on studying beliefs about the immediate context of teaching and learning, specifically beliefs about the nature of mathematics and its teaching and learning (Speer, 2005). The TEDS-M study investigates the beliefs of PSTs and teacher educators with regard to the nature of mathematics and its teaching and learning, together with the teachers' perceptions of their students' mathematical abilities (see Figure 4; Tatto et al., 2008).

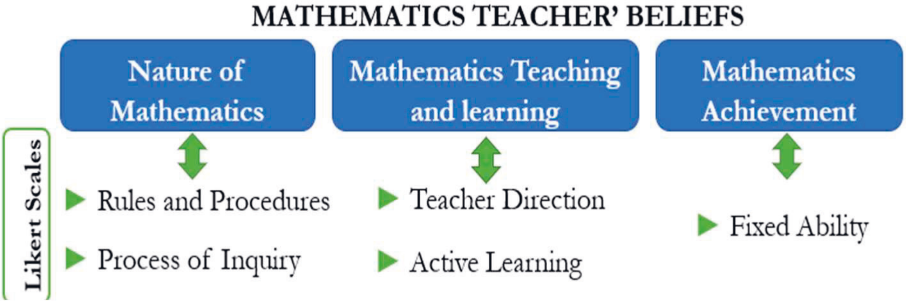


Figure 4. Teacher Education and Development Study in Mathematics (TEDS -M) Study Structure of Beliefs
Note. Tatto et al. (2008). Figure adapted from Alfaro and Joutsenlahti (2021).

The nature of mathematics explores the way teachers perceive mathematics as a subject (Tatto et al., 2008). These beliefs have been classified in different ways—for instance, as instrumentalist, Platonist, and problem-solving, or as traditionalist, formalist, and constructivist—views that can coincide respectively (Blömeke & Kaiser, 2014). However, the TEDS-M study is informed by the approach developed by Grigutsch et al. (1998), which has two fundamental beliefs regarding the nature of mathematics: the static view and the dynamic view. For the static view, in which mathematics is considered an unalterable unified entity (Tang & Hsieh, 2014), TEDS-M includes the scale of mathematics as a set of rules and procedures. On the other hand, the dynamic view is when mathematics is seen as something that is in a constant process of change and revision, which also requires the activation of creativity to generate new knowledge or solution paths (Tang & Hsieh, 2014). Thus, the correspondent scale in TEDS-M is named *process of inquiry*.

As for beliefs regarding teaching and learning, the TEDS-M framework is informed by the work of Peterson et al. (1989), from which two major categories are obtained: transmission and constructivist. Teachers with a transmissive view consider themselves the possessor and transmitter of information and knowledge, and students as the passive receivers who must obey the teacher's instructions (Blömeke & Kaiser, 2014). In TEDS-M, the scale of teacher direction is associated with this category. In contrast, the constructivist view gives the student greater responsibility in the process of building knowledge and meaning, so the teacher must promote the active participation and commitment of students in learning (Voss et al., 2013). In TEDS-M, the scale used for this is the scale of active learning.

Finally, the third area of beliefs is about teachers' conception of students' abilities to learn mathematics. For instance, whether gender and culture influence the learning of mathematics is considered. For this area, TEDS-M considers only the scale called fixed ability, which is anchored on the belief that the ability to learn mathematics is stable and cannot be changed despite efforts to improve it. On the contrary, it is the belief that learning mathematics requires a body of skills that can be built through the learning process (Wang & Hsieh, 2014).

At this point, it is important to note that belief systems do not necessarily have a logical order due to their nature as psychological constructs. Thus, contradictions or inconsistencies are possible (Boz, 2008). In fact, according to Voss et al. (2013), categories such as constructivist and transmissive beliefs are not mutually exclusive. Moreover, it is possible to find inconsistencies between teachers' stated beliefs and those that they practice (Speer, 2005), which highlights the complexity of studying teachers' belief systems.

2.1.3 Knowledge for teaching mathematics

Knowledge for teaching mathematics is regarded under the TEDS-M framework as encompassing the cognitive abilities that teachers need to be considered competent teachers. Such abilities were studied in Articles I and III. Since there is no single definition of knowledge for teaching mathematics, the TEDS-M team developed a framework for the mathematical knowledge needed for teaching that was informed by other domain-specific conceptions of teaching frameworks (Ball et al., 2008; Schmidt et al., 2007) and the teacher training standards of the participating countries (Tatto et al., 2008). This theoretical framework has two main categories: MCK and MPCK (see Figure 5), which have an approximate correspondence with Shulman's (1986) categories of content knowledge for teaching (Tatto et al., 2008).

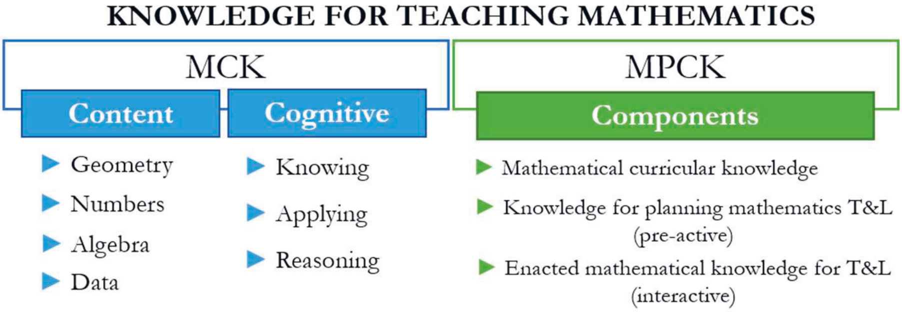


Figure 5. Subdomains of Mathematical Content Knowledge and Mathematical Pedagogical Content Knowledge Categories in the TEDS-M Framework
Note. Tatto et al. (2008). Figure from Alfaro (2022).

Following Shulman (1986), subject matter content knowledge, in this case, MCK, is defined as the amount and organization of the knowledge—including the fundamental assumptions, definitions, concepts, and procedures—that constitutes the ideas to be learned. Shulman highlights, however, that MCK involves more than concepts and facts, but it also requires an understanding of the structure, rules, and operation of the subject.

In the TEDS-M framework, MCK has content and cognitive subdomains, and both are informed by the Trends in International Mathematics and Science Study (TIMSS) data for lower secondary teaching (Tatto et al., 2008). The content subdomain includes the topics that are taught in lower and upper secondary schools as well as in tertiary schools (see Figure 5). The items included in this study assess irrational numbers; number theories; linear algebra; algebraic expressions; equations; formulas and functions; geometric shapes and measurements; data organization, representation, and interpretation; and chance.

The cognitive domain investigates three skills levels: knowing, applying, and reasoning (see Figure 5). The knowing subdomain considers actions such as recalling definitions and properties, performing algorithmic procedures, recognizing mathematical objects, and classifying them according to their properties (Tatto et al., 2008). The applying subdomain examines the skills of selecting appropriate solution strategies or methods to solve routine problems and using different representations of mathematical objects depending on the context. Finally, the reasoning subdomain assesses the most demanding tasks that require analysis of situations, provision of justifications, and solution of non-routine problems (Tatto et al., 2008).

The other mathematical knowledge category corresponds to MPCK. In the TEDS-M framework, this construct focuses on "the temporal dimension of teaching, moving from what mathematics to teach, to planning to teach it, to carrying out instruction" (Senk et al., 2008, p. 5). It was designed considering the correlation between teaching competencies and classroom situations (Blömeke & Delaney, 2012). In addition, as TEDS-M is an international study, for the conception of MPCK, the national teaching standards of the participating countries had to be considered and the contextual differences (Kaiser et al., 2017) between such countries had to be met, which represented a big challenge.

Table 1. Mathematical Pedagogical Content Knowledge Components

Mathematics curricular knowledge	**Knowledge of planning for mathematics teaching and learning**	**Enacting mathematics for mathematics teaching and learning**
-Establishing appropriate learning goals -Knowing different assessment formats -Selecting possible pathways and seeing connections within the curriculum -Identifying the key ideas in learning programs -Knowledge of mathematics curriculum	-Planning or selecting appropriate activities -Choosing assessment formats -Predicting typical students' responses, including misconceptions -Planning appropriate methods for representing mathematical ideas -Linking didactic methods and instructional designs -Identifying different approaches for solving mathematical problems -Planning mathematics lessons	-Analyzing or evaluating students' mathematical solutions or arguments -Analyzing the content of students' questions -Diagnosing typical students' responses, including misconceptions -Explaining or representing mathematical concepts or procedures -Generating fruitful questions -Responding to unexpected mathematical issues -Providing appropriate feedback

Note. Tatto et al. (2008).

As shown in Figure 5, MPCK has three components: mathematical curricular knowledge, knowledge for planning mathematics teaching and learning that is considered pre-active or before the teaching moment, and the interactive moment of enacting mathematical knowledge for teaching and learning (Tatto et al., 2008). The skills included in each component are described in Table 1.

2.2 The Mathematical Understanding for Secondary Teaching (MUST) framework

This framework proposed by Kilpatrick et al. (2015) was studied in Article III. The authors perceived the teaching of mathematics as a complex process that goes beyond mathematical knowledge and knowledge about how to teach. For them, knowing how to teach mathematics in secondary school requires specialized knowledge that is different from the mathematical knowledge of the engineer or the primary school teacher. Furthermore, Kilpatrick et al. did not refer to the mathematical knowledge of the teacher as such but preferred to call it the mathematical understanding of the teacher for the following reason:

> Knowledge may be seen as static and something that cannot be directly observed, whereas understanding can be viewed as the use of the knowledge one has (….) Also, because of its nature, a teacher's understanding grows and deepens on the course of his or her career (Kilpatrick et al., 2015, p. 10).

With this idea, Kilpatrick et al. stated that MUST can be characterized as understanding the general mathematical abilities relevant to teaching, having the competencies to execute the typical actions of the teaching work, and understanding the environments in which the mathematics skills will be used and the actions will be practiced. Therefore, the MUST framework is organized from three perspectives: mathematical proficiency, mathematical activity, and mathematical context of teaching (Figure 6), which are interwoven rather than isolated from the other perspectives, since they make sense as a whole (Kilpatrick et al., 2015).

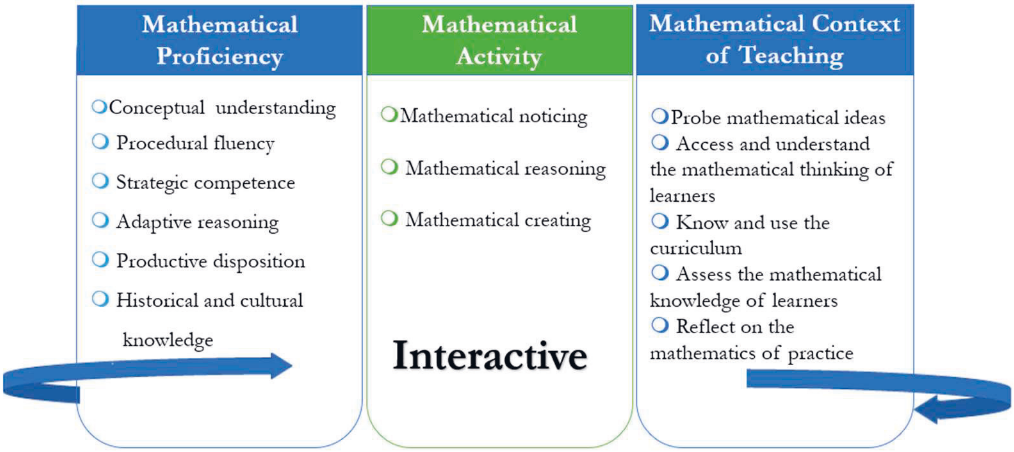


Figure 6. Framework of Mathematical Understanding for Secondary Teaching
Note. Kilpatrick et al. (2015). Figure adapted from Alfaro (2022).

Mathematical proficiency. The mathematical proficiency perspective, as shown in Figure 6, refers to six mathematical competencies that schools aim to develop in their students but using a deeper and more detailed approach that the teacher must know to be able to "guide students toward greater proficiency in mathematics" (Kilpatrick et al., 2015). These mathematical competencies are as follows. *Conceptual understanding* means understanding and using mathematical concepts in various contexts; monitoring one's own work and that of one's students' work; understanding, identifying, and using connections in math; formulating proofs when working by themselves; and remembering and reconstructing methods, it is described as "knowing why". *Procedural fluency* is the ability to quickly recall and accurately execute procedures and algorithms. *Strategic competence* is being able to select strategies for solving problems; solve problems using flexible approaches; generate, evaluate, and implement problem-solving strategies; and know various solution strategies. *Adaptive reasoning* calls for recognizing assumptions and adjusting to them and requires being able to provide valid explanations and justifications. *Productive disposition* is the belief that one will benefit from performing mathematics activities and the confidence that one will succeed in mathematical tasks. Finally, historical and cultural knowledge refers to understanding the origin and significance of mathematical ideas, recognizing the current developments in the field of math, and being aware of the ways in which different cultures conceptualize and express mathematical ideas.

Mathematical activity. Kilpatrick et al. (2015) conceived this perspective as the actions (notice, reason, and create) that are performed with the mathematical objects— and that a teacher should keep in mind when planning and implementing the lessons, to enhance the students' understanding of mathematical ideas (see Figure 6). The authors defined *mathematical noticing* as recognizing the structures of mathematical systems, symbols, and arguments and noticing their connections within and outside math. On the other hand, *mathematical reasoning* includes the skills of justifying, demonstrating, conjecturing, generalizing, restricting, and expanding. Finally, *mathematical creation* pertains to the skills to represent, define, and manipulate mathematical objects in the most appropriate way according to the learning situation (Kilpatrick et al., 2015).

Mathematical context of teaching. This perspective is about applying the knowledge from the mathematical proficiency perspective and the skills in the mathematical activity perspective into the classroom (see Figure 6) to help students develop their mathematical understanding (Kilpatrick et al., 2015). Therefore, this perspective is strictly related to the teacher-student interaction typical of mathematics learning, where teachers pose appropriate questions to access and understand the students' mathematical thinking. Also, they know the curriculum and teaching materials and use that knowledge to plan the classes, assess the mathematical knowledge of the students, determine their level of understanding, and reflect on the mathematics in one's practice (Kilpatrick et al., 2015).

3 METHODOLOGY

3.1 Research approach

Research has been described as a systematic inquiry performed to gain information or answer research questions by means of data collection, analysis, and interpretation (Assalahi, 2015). As a systematic inquiry, research has been associated with paradigms. A research paradigm can be understood as a set of beliefs shared by scientists in a certain discipline, which influence, for instance, how the subject matter is interpreted and tackled (Weaver, 2018). In this research, a pragmatic approach was taken, which is a "worldview that focuses on 'what works' rather than what might be considered absolutely and objectively 'true' or 'real'" (Weaver, 2018, p. 1287).

According to Biesta (2010), the pragmatic approach should not be considered a philosophical position but a "philosophical tool" that can be used to solve problems and answer questions. There have been many dilemmas regarding the pragmatic approach, from whether it should be called a paradigm to the definition of its ontological and axiological elements (Biesta, 2010; Maarouf, 2019). Maarouf (2019) attempted to provide an integrated vision of the pragmatic paradigm with its ontological, epistemological, and axiological elements. She defined the ontological position of pragmatism as a reality cycle that considers "one reality and multiple perceptions of this reality in the social actors' minds" (Maarouf, 2019, p. 7). Such reality exists in a certain environment and in a certain point of time; therefore, it can be different if it is considered in a different context or later in time. That position about reality is coherent with Dewey's position, presented by Morgan (2014), that a reality that exists apart from human experience can only be encountered through human experience, and that such reality is actively constructed (Weaver, 2018).

Epistemologically speaking, pragmatism has been criticized for its "what works" position, in which researchers can choose a subjectivistic or objectivistic view depending on which one solves the research problem, although in other philosophical positions they have considered incompatible (Biesta, 2010). In this regard, Morgan (2014) pointed out that in pragmatism, all knowledge is based on experience and thus, "each individual's knowledge is unique because it is based on individual experience, … [but] much of this knowledge is socially shared because it comes from socially shared experiences" (Morgan, 2014, p. 39). Morgan thus suggested that knowledge is intersubjective. Maarouf (2019) proposed the epistemological stance of pragmatism as double-faced knowledge, which means that "any type of knowledge can be seen as observable or unobservable based on the instantaneous ontological position of the pragmatic researcher" (p. 10). Thus, knowledge is conceived and approached according to the stance of the reality that is being considered, the one reality that exists conditioned by context and time, or the multiple realities in the social actors' minds.

Regarding the axiological stance, Maarouf (2019) stated that the necessary bias principle is adequate for the pragmatic paradigm. The principle "permits the researcher to be biased only by the degree necessary to enhance his research and helps to reach his research objectives" (Maarouf, 2019, p. 10).

Considering the pragmatic position about "what works," many options are opened for methodological choices, allowing the researcher to choose the best methodological approach to answering the research questions. Hence, pragmatism is considered an appropriate and most common philosophical support for mixed-methods research, the approach used in this study.

The mixed-methods approach to research integrates the advantages of quantitative and qualitative methods to gain a better understanding of the research problem and to achieve the research goals. For instance, quantitative methods provide precise and quick results, and qualitative research is best for analyzing complex phenomena because such phenomena can be examined in-depth and in detail (Maarouf, 2019). Creswell (2009) proposed six strategies for research design, among which the concurrent embedded design was used in this study. In such design, quantitative and qualitative data are collected at the same time in one data collection phase. In the concurrent embedded design, there is a primary method and a secondary method, and they answer different questions or analyze the phenomenon at different levels, thus offering complementary information (Creswell, 2009). In this study, the primary method was quantitative, and it was used to study the OTLs, the participants' beliefs, and their knowledge for teaching mathematics. The qualitative method was used to analyze the solutions of the PSTs to the assessment of their knowledge for teaching mathematics so that their skills could be better understood.

In conclusion, pragmatism is the research paradigm compatible with the choices and worldviews of this study and which, at the same time, is compatible with the mixed-methods research approach that guided this work; and the concurrent embedded design was chosen as the best method for answering the posed research questions.

3.2 Research design

As mentioned above, this research followed a mixed-methods approach with a concurrent embedded design where the quantitative method was the principal source of information. To collect both quantitative and qualitative data, a survey research design by means of a pencil-and-paper questionnaire was used. The questionnaire had closed questions in the form of Likert scales to investigate the OTLs and the beliefs. In addition, it had multiple-choice items to collect background information. It had a section with multiple-choice, complex multiple-choice, and complex-response items that showed tasks associated with knowledge for teaching mathematics. Those items were coded and analyzed quantitatively. The complex responses, which required the participants to develop a written solution, represented the qualitative data.

Survey research is extensively used in the social sciences (Given, 2008; Lavrakas, 2008) to collect standardized information because it gathers the same information from all the participants. Among the different instruments used to collect data in survey research, questionnaires are the most common and consist of "a set of standardized questions, often called items, which follow a fixed scheme in order to collect individual data about one or more specific topics" (Lavrakas, 2008, p. 652). Questionnaires are also implemented in the same fashion for all participants—in this case, for the PSTs, it was a paper-and-pencil implementation, and all the participants had three hours to answer it. In the case of the teacher educators, it was an online questionnaire. The use of a questionnaire also implies that the presence of the researcher during the data collection has minimal effect on the participants. Finally, the use of a questionnaire is convenient for studying the knowledge needed for teaching math, as it

> provide a crucial tool for investigating the nature and composition of mathematical knowledge needed for teaching. They serve to operationalize ideas about mathematical knowledge for teaching and test assumed models of the role it plays. They are used to investigate the teaching and learning of such knowledge, relationships with other variables, and other questions important for practice and policy (Hoover et al., 2016, p. 9).

The TEDS-M test was the instrument utilized for the data collection; therefore, I did not have to construct the items. As it is already used in international studies, issues about the reliability of the scales and the testing of the items were performed by the TEDS-M expert team. The reliability of the questionnaire scales ranges from

0.78 to 0.97, and the items have been internationally tested and examined by expert panels (Tatto et al., 2008), considering language and context differences.

3.3 Research Context

To become a mathematics teacher in Costa Rica, it is necessary to have a bachelor's degree in the teaching of mathematics from a private or public university. This study considers the cases of three public universities that offer the degree. The duration of the TEP for obtaining a bachelor's degree is four years, and one more year if continuing with the licentiate's degree. All three institutions administer an entrance examination that, combined with the student's average grades in high school, represent the admission grade. The places for each major are filled according to the demand and the admission grade; and in the case of mathematics education, there are no special entry requirements.

The three institutions have different administrative structures, which influence their TEP approach. One university offers the major in two campuses; and while the two campuses have the same TEP, the conditions of the campuses' contexts vary with regard to the specialty of the teacher educators who teach the courses or the number of students admitted. In another university, the only education major offered is the degree in the teaching of mathematics, and thus, all the courses are specific for mathematics education. In the other institutions, however, there are other education majors, and some pedagogy courses are shared with them. However, in all the institutions involved, the TEPs are designed in such a way that courses are not taken in separate blocks depending on the area but are so distributed that students attend courses in mathematics and mathematics education or pedagogy each semester.

Mathematicians and education professionals mainly undertake the preparation of mathematics PSTs. More teacher educators specializing in mathematics education recently joined them.

3.4 Data collection and sample

The data were collected in Costa Rica in the fall semester of 2019, of four provinces according to the location of the universities. All eight public and private universities that currently have a TEP for training mathematics teachers were invited to participate in this study. Only the five public institutions accepted. However, one of them was excluded because it had a distance teaching modality that did not offer PSTs an option for face-to-face application of the instrument. In addition, one university was considered two universities in this study because each of its two campuses has a different TEP and the conditions for implementing the two TEPs are different. Therefore, four universities took part in this study, which are referred to in this paper as U1, U2, U3, and U4. (In Articles I, II, and III, the universities are referred to as Univ. A, B, C, and D, respectively).

This study targeted Costa Rican mathematics PSTs who were taking the courses in the last year of their TEPs, and thus, have already experienced almost the entire program. Teacher educators of the same institutions were also included for the analysis of beliefs, to contrast the results across the teacher educator of four universities, with those of the PSTs in the same universities.

The data were collected using the TEDS-M questionnaire, which investigates PSTs' OTLs, beliefs, and knowledge for teaching mathematics. The paper-and-pencil questionnaire was administered during class time in a TEP course given by the responsible teacher and took three hours. The PSTs were informed that participation in this study was voluntary and that answering the questionnaire would indicate their agreement to participate. They were also informed that all the data would be treated confidentially and that the results of the questionnaire survey would not affect their grade in the course. The numbers of participants per university are shown in Table 2.

Table 2. Distribution of Participants by University

University	Number of groups	PSTs	Teacher educators
U1	2	24	7
U2	1	8	3
U3	2	19	5
U4	2	29	4
Total	7	80	19

A total of 80 PSTs participated in this study, but some of them did not have time to answer all parts of the questionnaire. Thus, for the study of knowledge for teaching mathematics, 79 answered questionnaires were considered, and for the study of beliefs, 76. Of the 80 participants, 44 were male and 36, female, and their average age was 23.8 years (SD = 2.89 years).

Regarding the teacher educators, they were asked to answer the survey using an online form. Nine of them were female and 10, male, and all of them had between 2 and 20 years of experience in preparing mathematics teachers (M = 9.3 years, SD = 5.2 years). Six of the teacher educators have a PhD degree, three in mathematics, two in education, and one in mathematics education; 13 have a licentiate's degree, one in education and 12 in mathematics education; and 10 have a master's degree, one in mathematics, four in education, and five in mathematics education. Their participation was also voluntary.

3.5 Instruments

The data were collected using the survey of the international TEDS-M study (Brese & Tatto, 2012, 2012b). Two different instruments were used—one for PSTs and another for teacher educators. The PSTs' questionnaire had four sections, and the teacher educators', three (see Table 3).

Table 3. Sections of the Surveys of the Pre-service Teachers and the Teacher Educators

Survey section	Pre-service teachers' survey	Teacher educators' survey
I	Background information	Background information
II	Opportunities to learn experienced	Opportunities to learn offered (not included in this study)
III	Knowledge for teaching mathematics items	Beliefs about mathematics and its learning
IV	Beliefs about mathematics and its learning	----

The first section in both instruments included questions on the participant's background information. The second section covered the OTLs experienced by the PSTs and the OTLs offered or taught for teacher educators, but the teacher educators' OTLs section was not considered in this study. The beliefs section included the same statements for both groups. Finally, the fourth section of the survey for the PSTs used items and tasks to assess the participants' knowledge for teaching mathematics. At the end of the survey, a new question was added: whether or not the PSTs believed that their TEPs had missing elements such as more courses in mathematics, mathematics education, or education, and more practice opportunities, or if the TEP was good as it was. In the next subsections, each survey section will be explained.

3.5.1 Background information

In the PSTs' survey, the questions on the background information were adapted from the TEDS-M international version of the questionnaire (Brese & Tatto, 2012). They included the following: the institution enrolled in; the age and gender; the level of education of the parents; the experience in mathematics (for example, the most advanced mathematics course taken in high school); the general performance in high school; the reasons for becoming a mathematics teacher; expectations of his or her future as a mathematics teacher; if there was a previous profession; and difficulties (financial or work-related) during the PST's studies, if any. This information is useful for studying relationships between variables. For example, it was studied if the results of the knowledge for teaching mathematics items were associated with the TEPs, meaning the institutions.

The online form that the teacher educators were asked to fill out also included a background information section adapted from the TEDS-M international version of the questionnaire (Brese & Tatto, 2012). The teacher educator were asked the following: the institution for which they worked; the number of years they have been working for it; the number of years they have been teaching mathematics in high school and university; the number of years they have been training mathematics teachers; their academic degrees and areas of specialization (i.e., mathematics, education, mathematics education, or another); their experience as a researcher and as a practical experience supervisor; and if they had special preparations for training mathematics teachers.

3.5.2 Opportunities to learn

Section II of the survey investigated the OTLs offered in the TEPs in eight categories, which are shown in Table 4. For some categories, the PSTs were asked if they have studied a topic or not, while for other categories, they were asked how often they had the opportunity to do or learn to do a certain action. Therefore, in this section, there were four types of questions (see Table 5).

Table 4. Items and Question-type Distribution by Opportunities to Learn Category

Category	Number of items	Question type (see Table 5)
1) Tertiary-level mathematics	19	1
2) School-level mathematics	7	1
3) Mathematics pedagogy	48	1- 2- 3
4) General pedagogy	8	1
5) Teaching for diversity	6	3
6) Reflecting and improving practice	12	3
7) School experience and the practicum	17	2-4
8) Coherence of the TEPs	6	4

As shown in statement D of the question type 1 example in Table 5, when the participants were asked about mathematics, mathematics education, or general pedagogy topics, examples were given that further detailed such topics so that it would be clear to the respondents what the item was referring to. This is because a topic could have different titles in the TEPs. The respondents could mark "studied" if they had studied at least one of the contents detailed in the item. The topics did not each represent a university course, but a group of topics could be studied in one course.

Table 5. Types of Questions Used to Investigate the Opportunities to Learn

Type 1	Consider the following list of mathematics topics that are often taught at the primary or secondary school level. Please indicate whether you have studied each topic as part of your current teacher preparation program.	
	D. Functions, Relations, and Equations (e.g., algebra, trigonometry, analytic geometry)	(Studied, not studied)
Type 2	In the mathematics education courses that you have taken or are currently taking in your teacher preparation program, how frequently did you do any of the following?	
	C. Participate in a whole class discussion	(Never, rarely, occasionally, often)
Type 3	In your current teacher preparation program, how frequently did you engage in activities that gave you the opportunity to learn how to do the following?	
	G. Create learning experiences that make the central concepts of subject matter meaningful to pupils	(Never, rarely, occasionally, often)
Type 4	To what extent do you agree or disagree with the following statements about the school experience or practicum you had in your teacher preparation program?	
	D. I learned the same criteria or standards for good teaching in my courses and in my school experiences /practicum.	(Disagree, slightly disagree, slightly agree, agree)

Note. Examples taken from TEDS-M questionnaire (Brese & Tatto, 2012).

Tertiary-level mathematics (Table 4). This category included 19 items all under question type 1. Each item referred to a different mathematics topic studied at the university level. In the TEDS-M study (Brese & Tatto, 2012a), the topics were grouped into four scales: geometry (4), discrete structures and logic (6), continuity and functions (5), and probability and statistics (2) according to the contents. For instance, in the geometry scale, the following items were included (Brese & Tatto, 2012):

> A. Foundations of Geometry or Axiomatic Geometry (e.g., Euclidean axioms);
>
> B. Analytic/Coordinate Geometry (e.g., equations of lines, curves, conic sections, rigid transformations or isometrics);
>
> C. Non-Euclidean Geometry (e.g., geometry on a sphere); and
>
> D. Differential Geometry (e.g., sets that are manifolds, curvature of curves, and surfaces).

The topics topology and the real and/or complex functions were not categorized, but they were considered in the analysis.

School-level mathematics (Table 4). This category included topics that are usually studied in primary or secondary school, and thus, that the participants may have to teach in the future. The seven items under question type 1 fell under two scales (Brese & Tatto, 2012a). One scale included the items related to numbers (e.g., whole numbers, fractions, decimals, real numbers, …), measurement (e.g., units, perimeter, area, volume, …), and geometry (e.g., one- and two-dimensional coordinate geometry, Euclidean geometry, three-dimensional geometry, congruence, and similarity). These topics are usually studied in lower school levels. The other scale consisted of the following items: (a) functions, relations, and equations; (b) data representation, probability, and statistics; (c) calculus; and (d) validation, structuring, and abstracting (Boolean algebra, induction, logical connectives, …). All these topics were meant to be studied at higher school levels.

Mathematics pedagogy (Table 4). The items in this category covered different aspects. As shown in Table 5, in this study, four types of questions were used to collect information. The question type 1 items were about topics studied regarding mathematics pedagogy. These items were nine in all, of which three fell under the *foundations* scale, which covered topics on foundations of mathematics, the context of mathematics education, and development of mathematics ability and thinking. The other six items were clustered under the *instruction* scale, which considered topics about mathematics instruction, developing teaching plans, mathematics teaching, mathematics standards and curriculum, specific didactics, and affective issues in mathematics (Brese & Tatto, 2012a).

The question type 2 items were about the teaching methods that the PSTs experienced in the TEP courses. The 13 items with this characteristic were grouped into three scales, following the TEDS-M study (Brese & Tatto, 2012a). One of the scales was about *class participation* (e.g., asking questions during the class, making presentations for the rest of the class, and teaching a class session using methods of my own choice). The second scale grouped the items according to the *readings done in the class* (e.g., research on mathematics, mathematics education, and teaching and learning). Finally, the third scale, on *solving problems*, covered the skills of writing mathematical proofs or solving mathematics problems using multiple strategies.

For question type 3, there were 26 items classified into four scales (Brese & Tatto, 2012a). These items were associated with the teaching methods that the PSTs should learn for teaching mathematics. The *instructional practice* scale had seven items with topics such as learning how to explore multiple solution strategies with students and integrating mathematical ideas from different areas of mathematics. The *instructional planning* scale included 10 items related to the organization of several abilities in each lesson, dealing with learning difficulties, and using students' misconceptions to plan lessons. The *assessment uses* scale had five aspects, as among them, the use of evaluations to give feedback to students and parents and to inform decisions on what and how to teach. The last scale, on *assessment practices*, consisted of four items on the assessment of lower- and higher-level objectives and the analysis of the students' assessment data to improve the PST's own assessment practices.

General pedagogy (Table 4). This category considered general education topics that mathematics teachers must learn. All its eight items fall under question type 1 and are classified into two scales (Brese & Tatto, 2012a). The *social science* scale covered three items on the history, philosophy, and sociology of education. The *application* scale had five items on educational psychology, theories of schooling, methods of educational research, knowledge of teaching, and the theory and practice of assessment.

Teaching for diversity (Table 4). This category investigates the preparation that the PSTs have had for training students from different backgrounds. This category included six items under question type 3 and explored how often the participants had the opportunity to learn to develop specific strategies for teaching students with behavioral and emotional problems and learning disabilities, gifted students, or students from diverse cultural backgrounds.

Reflecting and improving practice (Table 4). This category was explored using 12 items under question type 3. The scale *teaching for reflection on practice* had four items (Brese & Tatto, 2012a) about developing strategies to reflect on the PST's own teaching effectiveness, professional knowledge, and learning needs. Correspondingly, the scale *teaching for improving practice* had eight items (Brese & Tatto, 2012a) about topics such as developing and testing new teaching practices, using research findings to improve teaching and learning, and identifying appropriate teaching resources.

School experience and practicum (Table 4). This category studied the experiences of the PSTs teaching in secondary school and the role of their supervisor in providing feedback and reinforcing the objectives of the university. There were three scales in this category (Brese & Tatto, 2012a). The first scale, *connecting classroom learning to practice*, included eight items under question type 2 that investigated how much of the teaching strategies, theories, and ideas learned in the courses could be applied during the experience of teaching in high school. The second scale, *supervising teacher reinforcement of university goals for practicum,* had five items under question type 4 that investigated the supervisor acceptance and reinforcement of what the PSTs learned in their university courses. Finally, the third scale, s*upervising teacher feedback quality*, included four items under question type 4 regarding the quality of the feedback provided by the supervisor in terms of helping the PSTs to improve their teaching methods, their understanding of the students and the curriculum, and the knowledge of mathematics content.

Coherence of the TEPs (Table 4). This category was studied with six items under question type 4. The items analyzed, for instance, if there were clear links between the courses in the TEPs and if the TEPs trained the participants in all that they needed to learn to become effective teachers. This category offered the participants an opportunity to evaluate their TEPs.

All the information collected under the OTL categories and how the PSTs experienced them in their TEPs provide insights on the topics being studied herein concerning Costa Rican mathematics PSTs and the teaching practices they have experienced. This will help us to understand the training of PSTs and their performance with regard to the knowledge to teach mathematics items.

3.5.3 Knowledge for teaching mathematics items

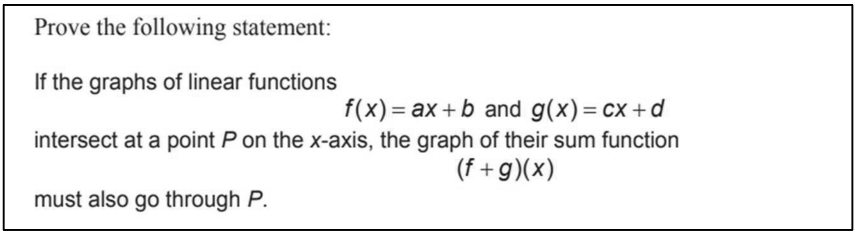
This section consisted of a set of items, tasks, or exercises for assessing the MCK and the MPCK of the PSTs. The items were distributed into four content domains and three cognitive and two teaching-related skills subdomains, as shown in Table 6, and corresponded only to the TEDS-M released items (see Brese & Tatto, 2012b). The items used in the TEDS-M study were subjected to international trials and considered "clarity, correctness, cultural relevance, classification within the framework of domains and subdomains, relevance to teacher preparation, and curricular level" (Tatto et al., 2008, p. 35).

Table 6. Distribution of the Released TEDS-M Items Used in the Questionnaire

Content subdomains	MCK Cognitive subdomains			MPCK Teaching-related skills	
	Knowing	Applying	Reasoning	Implementing teaching & learning	Curriculum & planning
Algebra	-	5	2	1	4
Numbers	4	-	4	3	-
Geometry	2	4	-		-
Data	-	1	-	1	-

Note. MCK: mathematical content knowledge; and MPCK: mathematical pedagogical content knowledge (Alfaro, 2022).

There were items with three different formats: constructed-response (seven items), multiple-choice (two items), and complex multiple-choice (22 items). The constructed-response items could involve writing proofs for mathematical statements, writing the solution to geometry or algebra problems, or providing explanations, such as for why one algebra word problem is more difficult than another (see Figure 7). All these items required the PSTs to present their own written solutions.



Prove the following statement:

If the graphs of linear functions

$$f(x) = ax + b \text{ and } g(x) = cx + d$$

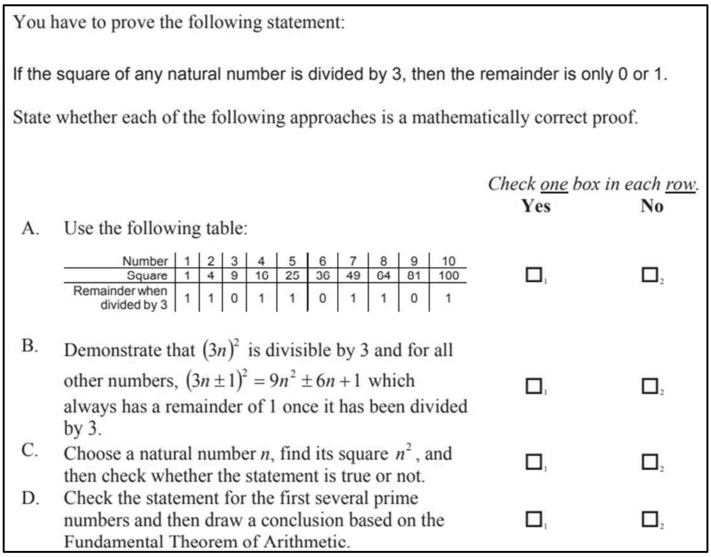
intersect at a point P on the x-axis, the graph of their sum function

$$(f + g)(x)$$

must also go through P.

Figure 7. Exercise 7, Example of a Constructed Response Item, TEDS-M Released Items
Note. Brese & Tatto (2012b).

In multiple-choice items, for a given statement, the respondents had to choose only one answer from the given options. On the other hand, the complex multiple-choice items had a question statement and several response options, but for each response option, the respondent could choose a different answer (see Figure 8). For example, the respondent could indicate if each option is valid or invalid, or if the situation described in the option is true always, sometimes, or never.



You have to prove the following statement:

If the square of any natural number is divided by 3, then the remainder is only 0 or 1.

State whether each of the following approaches is a mathematically correct proof.

Check one box in each row.

		Yes	No
A.	Use the following table:		
		☐1	☐2

Number	1	2	3	4	5	6	7	8	9	10
Square	1	4	9	16	25	36	49	64	81	100
Remainder when divided by 3	1	1	0	1	1	0	1	1	0	1

		Yes	No
B.	Demonstrate that $(3n)^2$ is divisible by 3 and for all other numbers, $(3n \pm 1)^2 = 9n^2 \pm 6n + 1$ which always has a remainder of 1 once it has been divided by 3.	☐1	☐2
C.	Choose a natural number n, find its square n^2, and then check whether the statement is true or not.	☐1	☐2
D.	Check the statement for the first several prime numbers and then draw a conclusion based on the Fundamental Theorem of Arithmetic.	☐1	☐2

Figure 8. Example of a Complex Multiple-Choice Item, TEDS-M Released Items
Note. Brese & Tatto (2012b).

In the complex multiple-choice items, the respondents had to analyze each option carefully instead of choosing only one option and ignoring the rest. As most of the items in the questionnaire were complex multiple-choice items and constructed response items, the PSTs were required to take a careful and analytical stance to solving the items correctly.

3.5.4 Beliefs about mathematics and its learning

The section of the survey dedicated to studying the beliefs was the same for both groups of participants: the PSTs and the teacher educators. The section consisted of three categories: the nature of mathematics, learning mathematics, and students' achievement in mathematics. The type of question was the same for all the categories; only the topic was changed correspondingly. For instance, a question could be: "*To what extent would you agree or disagree with each of the following statements about learning mathematics*?" (Brese & Tatto, 2012) and six response options were given that indicated the level of agreement: *strongly disagree, disagree, slightly disagree, slightly agree, agree, strongly agree.*

1. Beliefs about the nature of mathematics. This category studied the respondents' perception of mathematics. It had 12 items divided into two scales: *rules and procedures* (6 items) and *process and inquiry* (6 items; Brese & Tattoo, 2012a). Agreeing to the items of the first scale would indicate a static view of mathematics, and on the contrary, agreement to the items in the second scale would represent a dynamic view of mathematics that allows creativity and new ideas.

2. Beliefs about the learning of mathematics. For this category, there were 14 items and two scales (Brese & Tatto, 2012a). The first scale, *teacher direction*, had eight items and studied the respondents' agreement that when learning mathematics, the teacher always has to have the last word and the students' role is passive. The second scale, *active learning*, had six items and explored the PSTs' and teacher educators' agreement that when learning mathematics, the students' active participation is required and therefore, promoted.

3. Beliefs about students' achievement in mathematics. This category had eight items that all corresponded to a single scale, *fixed ability* (Brese & Tatto, 2012a). The agreement to all the items of this scale denotes the belief that only people born with mathematical abilities or only specific groups of people (e.g., men) can be successful in mathematics and that mathematical abilities cannot be acquired through effort.

The level of agreement to the statements in each category provide information on the belief systems of mathematics PSTs and teacher educators. As mentioned in subsection 2.1.2, the level of agreement to the *nature of the mathematics* statements could reveal static or dynamic views. Transmissive or constructivist views could be observed from the level of agreement to the *learning of mathematics* scale. Finally, the *students' achievement in mathematics* scale revealed if the respondents considered mathematics skills fixed abilities or not.

3.6 Analysis

The analysis of the data had two stages. The first stage was the quantitative analysis of the results of the Likert scales used in the OTLs and beliefs sections of the survey, and of the numerical results of the knowledge for teaching mathematics items. The second stage was the theory-driven or direct content analysis of the respondents' written solutions to the knowledge for teaching mathematics items. After the data were collected and coded, they were screened. As the sample was small, data was treated separately for each section with regard to the missing data. Each stage is described as follows.

3.6.1 Quantitative analysis of data

Starting with the data screening, decisions were made regarding the missing data. The method was decided upon depending on the question type and the section of the survey. Thus, in section II, with the question type 1 scales that consisted of factual data, the missing values (maximum of 2 per variable) were not imputed. In this case, the count of positive answers was added to know how many topics in each scale the participants had studied, then percentage of the studied topics in each OTL category were computed because the categories and TEPs were easier to compare by percentage.

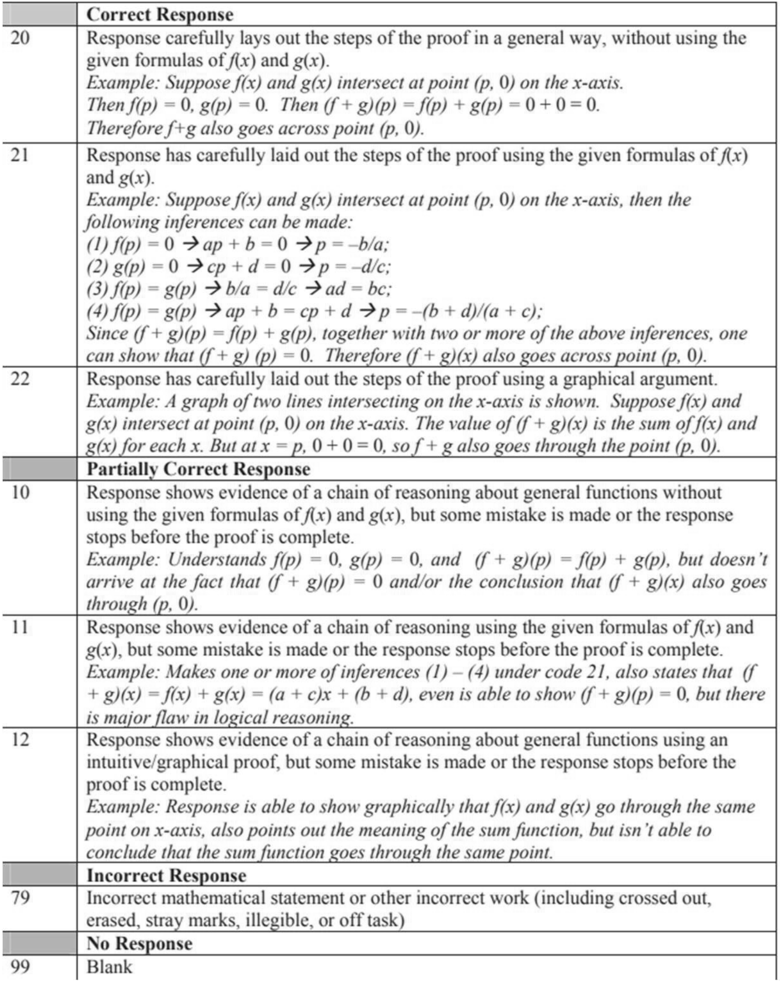
In the same section, for question types 2 and 3 that collected information based on the respondents' perception of the OTLs, the missing data were handled using the median imputation method, where the missing values were replaced by the median in the items with the Likert scales, and the scales were computed using the mean value.

The same imputation method was used for the Likert scales of the beliefs section, both for the data of the PSTs and of the teacher educators. The belief scales were computed using the mean and then grouped following the TEDS-M method, where answers 5 and 6 (*agree* and *strongly agree*) were regarded as endorsing the statement and answers 1 to 4 (*strongly disagree* to *slightly agree*) were seen as failing to endorse the statement (Tatto, 2013).

Due to the time limit, some PSTs were unable to complete sections III (one participant) and IV (four participants). However, considering the small size of the sample, I decided to delete these cases only for the analysis of the respective sections.

Therefore, out of a total of 80 PSTs, 79 surveys were considered for the analysis of section III, and 76 for section IV.

The TEDS-M user guide (Brese & Tatto, 2012b) provided instructions for coding the solutions to the section III items. The correct answers to the multiple-choice and complex multiple-choice items were indicated, and each correct answer was given 1 point and each incorrect answer, 0 point. On the other hand, a scoring guide was designed for grading the constructed response items (see an example in Figure 9). The scoring guide was created to ensure greater objectivity of the different international entities participating in the study in assessing the answers to those tasks.



	Correct Response
20	Response carefully lays out the steps of the proof in a general way, without using the given formulas of $f(x)$ and $g(x)$. *Example: Suppose $f(x)$ and $g(x)$ intersect at point $(p, 0)$ on the x-axis.* *Then $f(p) = 0$, $g(p) = 0$. Then $(f + g)(p) = f(p) + g(p) = 0 + 0 = 0$.* *Therefore $f+g$ also goes across point $(p, 0)$.*
21	Response has carefully laid out the steps of the proof using the given formulas of $f(x)$ and $g(x)$. *Example: Suppose $f(x)$ and $g(x)$ intersect at point $(p, 0)$ on the x-axis, then the following inferences can be made:* *(1) $f(p) = 0 \rightarrow ap + b = 0 \rightarrow p = -b/a$;* *(2) $g(p) = 0 \rightarrow cp + d = 0 \rightarrow p = -d/c$;* *(3) $f(p) = g(p) \rightarrow b/a = d/c \rightarrow ad = bc$;* *(4) $f(p) = g(p) \rightarrow ap + b = cp + d \rightarrow p = -(b + d)/(a + c)$;* *Since $(f + g)(p) = f(p) + g(p)$, together with two or more of the above inferences, one can show that $(f + g)(p) = 0$. Therefore $(f + g)(x)$ also goes across point $(p, 0)$.*
22	Response has carefully laid out the steps of the proof using a graphical argument. *Example: A graph of two lines intersecting on the x-axis is shown. Suppose $f(x)$ and $g(x)$ intersect at point $(p, 0)$ on the x-axis. The value of $(f + g)(x)$ is the sum of $f(x)$ and $g(x)$ for each x. But at $x = p$, $0 + 0 = 0$, so $f + g$ also goes through the point $(p, 0)$.*
	Partially Correct Response
10	Response shows evidence of a chain of reasoning about general functions without using the given formulas of $f(x)$ and $g(x)$, but some mistake is made or the response stops before the proof is complete. *Example: Understands $f(p) = 0$, $g(p) = 0$, and $(f + g)(p) = f(p) + g(p)$, but doesn't arrive at the fact that $(f + g)(p) = 0$ and/or the conclusion that $(f + g)(x)$ also goes through $(p, 0)$.*
11	Response shows evidence of a chain of reasoning using the given formulas of $f(x)$ and $g(x)$, but some mistake is made or the response stops before the proof is complete. *Example: Makes one or more of inferences (1) – (4) under code 21, also states that $(f + g)(x) = f(x) + g(x) = (a + c)x + (b + d)$, even is able to show $(f + g)(p) = 0$, but there is major flaw in logical reasoning.*
12	Response shows evidence of a chain of reasoning about general functions using an intuitive/graphical proof, but some mistake is made or the response stops before the proof is complete. *Example: Response is able to show graphically that $f(x)$ and $g(x)$ go through the same point on x-axis, also points out the meaning of the sum function, but isn't able to conclude that the sum function goes through the same point.*
	Incorrect Response
79	Incorrect mathematical statement or other incorrect work (including crossed out, erased, stray marks, illegible, or off task)
	No Response
99	Blank

Figure 9. Example of the Scoring guide for Exercise 7.
Note. Brese & Tatto (2012b).

The points were allocated using the following method.

> • One-point constructed response (CR) items were scored as correct (1 score point) or incorrect (0 score point).
>
> • Two-point CR items were scored as fully correct (2 score points), partially correct (1 score point), or incorrect (0 score point). For example, a response to an MCK item that contained an incorrect solution but a mathematically appropriate reasoning and procedure received partial credit. A response to an MPCK item that was incomplete or lacking some clarity was awarded partial credit (Tatto, 2013, p. 41).

The points assigned to each item were added, and the score was calculated using different filters, for example, the grade in the items of the applying subdomain, in the MCK items, or in all the items. In addition, the differences and relationships of those scores were analyzed with the TEP and the quartiles of performance.

After the data coding and screening, descriptive statistics were used to show the distribution of the OTLs or beliefs by TEP, or the behavior towards the belief endorsement by PSTs and teacher educators. Because the sample was small and did not meet the recommended size for testing normality (Siebert & Siebert, 2017), this study used non-parametric analysis methods such as the Kruskal-Wallis H test (Huizingh, 2007), the Wilcoxon signed-rank test (Huizingh, 2007), and Spearman's analysis of correlation (Scott & Mazhindu, 2005). Chi-square test of independence (Salkind, 2007) was also used.

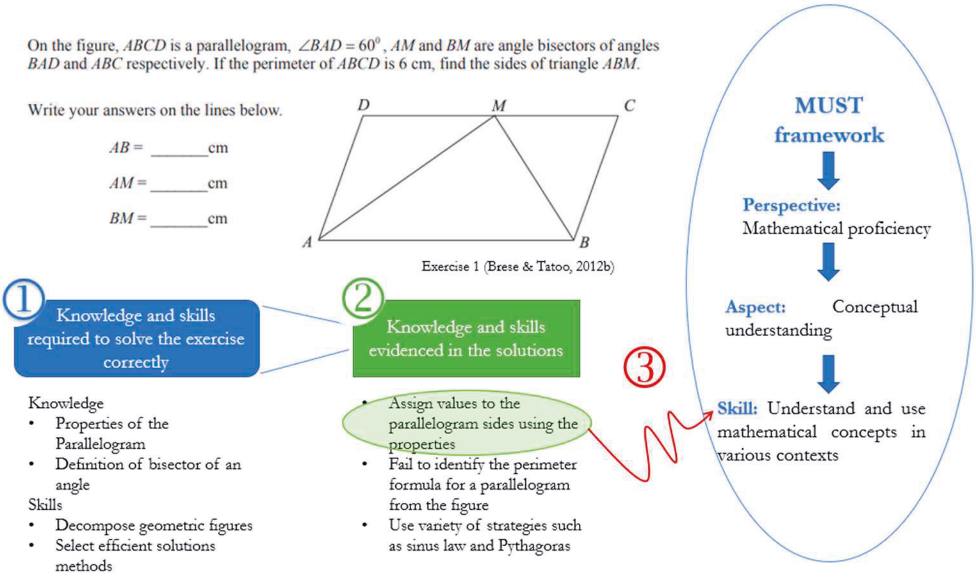
The Kruskal-Wallis H test was used to determine if there were significant differences between dependent continuous variables such as the PSTs' performance in the applying subdomain items and the independent variable, the TEP. This non-parametric test is suitable for use in comparing two or more independent groups (Huizingh, 2007), such as the groups of participants in each TEP. In contrast, the Wilcoxon signed-rank test is used to compare two related groups (Huizingh, 2007) and to "compare two sets of scores that come from the same participants" (Lund Research, 2008, para. 1). Thus, it was a suitable option for testing differences in the performance of items of the applying and reasoning subdomains.

Spearman's analysis of correlation assesses the correlation between two variables of the same research unit (Scott & Mazhindu, 2005). In this study, it was used to study the correlation between the belief scales and the performance in the knowledge for teaching mathematics items, and between the variables related to the topics studied and the performance in the knowledge for teaching mathematics items. For example, the existence or non-existence of a correlation between the number of topics studied in geometry and the performance in the MCK items was studied.

Finally, the Chi-square test of independence was used to test the significance of the relationship between two categorical variables (Salkind, 2007), such as the TEP and the endorsement or non-endorsement of the belief scales.

3.6.2 Direct content analysis

Qualitative analysis was performed using a direct (Hsieh & Shannon, 2005) or theory-based approach, which means that categories already defined in literature were used as lenses to analyze the data. This study used the MUST framework (Kilpatrick et al., 2015) and the description of its perspectives to analyze the solutions to the 13 exercises in the 79 questionnaires of the participants. Figure 10 give an example of the process followed.



On the figure, *ABCD* is a parallelogram, $\angle BAD = 60^0$, *AM* and *BM* are angle bisectors of angles *BAD* and *ABC* respectively. If the perimeter of *ABCD* is 6 cm, find the sides of triangle *ABM*.

Write your answers on the lines below.

$AB =$ _______cm

$AM =$ _______cm

$BM =$ _______cm

Exercise 1 (Brese & Tatoo, 2012b)

Knowledge
- Properties of the Parallelogram
- Definition of bisector of an angle

Skills
- Decompose geometric figures
- Select efficient solutions methods

- Assign values to the parallelogram sides using the properties
- Fail to identify the perimeter formula for a parallelogram from the figure
- Use variety of strategies such as sinus law and Pythagoras

Perspective: Mathematical proficiency

Aspect: Conceptual understanding

Skill: Understand and use mathematical concepts in various contexts

Figure 10. An Example of Content Analysis Process

First, for each exercise, considering the categories of content knowledge and the cognitive domain, or of content knowledge and the teaching-related skills included in the TEDS-M study, the knowledge and skills that the PST should mobilize to solve the exercise was identified. Second, the 553 solutions to the constructed response items were read, as were all the drawings and annotations made by the participants in items of other kind. As all these constituted a huge amount of information, the previous step was used to focus on what to read from those solutions, and I observed in each exercise the knowledge and skills that the PSTs used to arrive at their solution strategies, concepts, and procedural difficulties. After such pre-screening, the third step was to link the information obtained from the solutions to the aspects of each MUST perspective.

3.7 Ethical considerations

When conducting research, there are important ethical principles that must be followed. Sometimes, those principles are specific to the research field and the kind of sample involved in the study. In education research, according to Hammersley and Traianou (2012), there are five ethical principles that must be met. The first is minimizing the harm, in this study, to the participants or institutions involved. Respecting their autonomy is also an important principle, which means allowing them to decide for themselves, for example, if they should participate in this study or not. Third, the privacy of the participants must be protected by concealing their identity, such as by referring to them with numbers or pseudonyms. Hammersley and Traianou (2012) also mentioned as ethics principles offering the participants reciprocity as a way of thanking them for their participation and treating them equitably, that is, fairly and without discrimination.

Considering the mentioned ethical principles and the recommendations stated of the Finnish Advisory Board for Research Integrity (TENK, 2009, 2012), several steps were followed to ensure good ethical research practice.

1. The permission of the IEA for the access, use, and translation of the TEDS-M materials was secured. The permits were granted through the form IEA18-093 (Appendix 1).

2. All the institutions in Costa Rica with a TEP for mathematics teachers were invited to participate in this study. The directors of the TEPs were contacted using the contact details—that is, the email addresses and phone numbers—on the websites of their respective institutions. Each director was sent a letter asking for permission for data collection in the director's institution. The letter explained the aims of this research, the specific request for data collection, and a brief description of the instrument.

3. As a way of offering reciprocity, an agreement was reached with the directors of the participating institutions, that they would receive a report on the performance of their students and the main results of the research.

4. The time of the data collection was adjusted to the time most convenient for each institution.

5. Each participant was given a survey booklet with a letter that explained the objectives of the research, the role of the participants, the confidential treatment of their data, and that by filling out the questionnaire, they indicate their consent to participate in the study. The participants were also orally informed of the content of the letter at the start of the study implementation in case they did not read it.

6.The preservice teachers were asked not to write their name or email on the questionnaire sheet but only the institution name and its student ID number, to which the researcher did not have access. When the data were coded, the student ID number were deleted, and an ID number was given to each questionnaire sheet instead to guarantee the anonymity of the data. Similarly, the online questionnaire for teacher educators did not request for any identification, the email that was not mandatory, was deleted when the data was coded, each teacher educator's questionnaire was given an identification number

7. The data were used only for the research purposes indicated, and the results were interpreted thoroughly and published accurately.

8. The use of other authors' ideas was recognized and referenced correctly to avoid plagiarism.

By following such steps, I ensured my adherence to ethical principles in research and upheld the integrity of my research.

4 MAIN RESULTS

4.1 Costa Rican mathematics teacher education programs (TEPs): Studied topics and teaching methods

The results for the OTLs offered by the TEPs in Costa Rica, are presented in Article I. To organize this section, I describe the statistical results regarding the topics studied and the teaching methods used.

Studied topics

Three areas of topics studied at the university level were analyzed: mathematics topics, general pedagogy, and mathematics education pedagogy. Regarding the mathematics topics, the data showed that participants (N = 80) studied an average of 14.9 (78.6%) mathematics topics at the university level out of the 19 options. They studied the most topics in the discrete structures & logic and continuity & functions categories—5.2 out of 6 and 3.9 out of 5 on average, respectively. The analysis showed that on average, the TEPs dedicated 45% of their topics to tertiary-level mathematics (see Table 7). Thus, it can be observed that the TEPs have strong mathematics foundations.

Table 7. Mean Number of Topics Studied in Knowledge Areas by University

University	Mathematics		Mathematics education pedagogy		General education pedagogy	
	Mean	%	Mean	%	Mean	%
U1	15.1	47	11	34	6.2	19
U2	15.1	48	10.3	32	6.3	20
U3	16.4	47	12.7	36	6.1	17
U4	13.7	43	13.1	41	5.3	16
General results	**14.9**	**45**	**12.1**	**37**	**5.8**	**18**

Turning now to the mathematics pedagogy topics, the PSTs reported having studied 5.6 of the 8 topics listed. The topics that fewer participants studied, 60% and 53% (N = 80), respectively, were development of mathematics ability and thinking and affective issues in mathematics. The lack of knowledge of PSTs on those topics could present future challenges for PSTs when they start teaching. On the other hand, there were positive results for the school-level mathematics topics. The PSTs indicated that they have studied 6 or more of the 7 topics presented (93%). The school-level mathematics topics were included in the mathematics education pedagogy area because they are directly related to PSTs' future work in secondary schools. Taken together, these categories represented, on average, 37% of the topics in the TEPs (see Table 7). However, the PSTs continue clamoring for more courses in mathematics education, as shown in Figure 11.

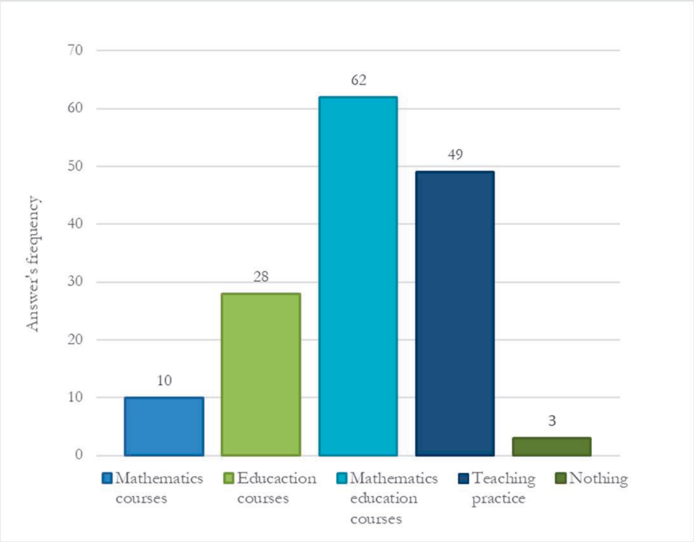


Figure 11. Courses that Pre-service Mathematics Teachers Felt Should Be Added to their TEP
Note. Alfaro & Joutsenlahti (2020).

For the last area, the general pedagogy topics, the outcomes showed that of the 8 topics listed in the survey, the TEPs included 5.8 topics. A closer observation of the data revealed that the topic philosophy of education was studied only by 56% (N = 80) of the PSTs, and the topic history of education and educational systems was studied only by 55% of them (N = 80). The average percentage dedicated in the TEPs to general pedagogy topics was 18% (Table 7), as this was the area with fewer topics.

After knowing the general way in which the topics were distributed according to the area of knowledge, the existence of differences in the way the TEPs distributed the topics were studied. Using the Kruskal-Wallis H test, no significant difference in the distribution of the tertiary-level and general pedagogy topics across the institutions were found. However, as shown in Table 7, U3 topped the list of institutions with the highest number of tertiary-level mathematics topics in their TED, with a difference of 2.7 topics compared to U4, the lowest in the table. The results of the same Kruskal-Wallis H test for the mathematics education pedagogy topics showed significant differences in the distribution of those topics across the universities [$\chi 2(3, N = 80) = 17.82, p \leq 0.00$]. U4 had the most topics, 2.8 more than U2, which had the least number. Altogether, the results showed variations between the TEPs, as reported in literature (PEN, 2019; Roman & Lentini, 2018).

Teaching methods

Two kinds of questions were asked for the study of the teaching methods used in the TEPs. One question was how often the participants experienced a teaching method, for example, ask questions in the class, and the other question was how often they had opportunities to learn how to do a teaching method, for instance, assessment practice. Considering this difference, the results are presented separately.

For the first question, the actions listed were related to the activities that the PSTs were asked to do during their TEP classes in their role as students. Thus, in the list, there are items about class participation, class reading, and solving problems. The average frequencies of the performance of the activities are shown in Figure 12. The figure reveals that the classroom activities were varied in a balanced way.

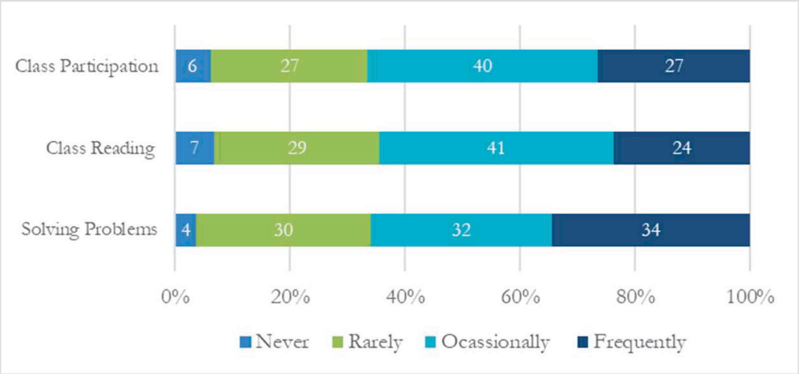


Figure 12. Frequency of Activities Done During Class
Note. N = 80.

The question on the frequency of the occurrence of opportunities for the participants to learn how to do something was related to the participants' role as PSTs. Instructional planning and practice, assessment uses and practice, teaching diverse students, and reflecting and improving practice were addressed. As Figure 13 shows, most of the PSTs reported having occasionally or frequently had the opportunity to learn instructional planning (74%) and practice (63%), and assessment practice (63%). On the contrary, most of the participants had rare or no opportunities to learn about assessment uses, teaching for diversity, reflection on teaching practice, and improvement of teaching practice. These are disturbing results, because such practices are crucial for teaching, and without knowledge of them, PSTs will have to improvise, ignore, or underperform in aspects like providing students and parents assessment feedback or developing strategies for teaching students from different backgrounds.

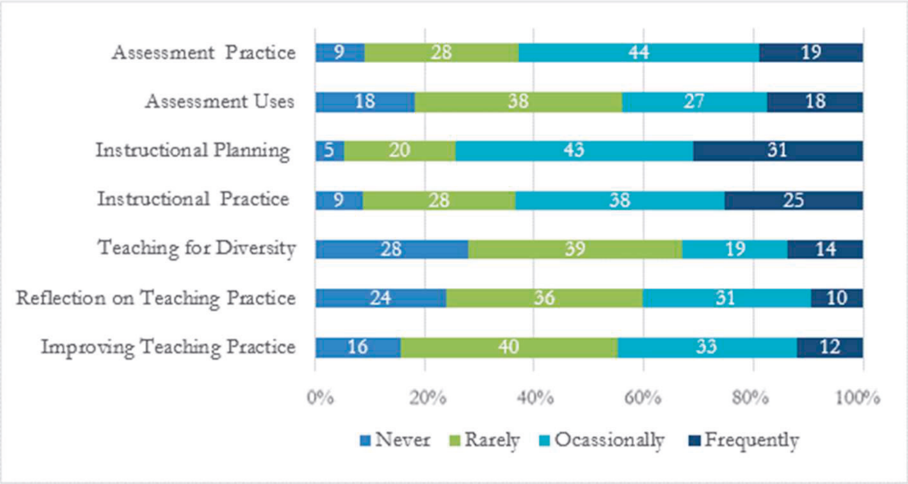


Figure 13. Frequency of Activities to Learn about Teaching Practices
Note. N = 80.

In addition, the Kruskal-Wallis H test revealed that the perspectives of the PSTs on the frequencies of their experience of the teaching methods were not the same in all the institutions. The methods of class reading [$\chi2(3, N = 80) = 9.886, p \leq 0.05$], solving problems [$\chi2(3, N = 80) = 8.249, p \leq 0.05$], assessment practice [$\chi2(3, N = 80) = 9.463, p \leq 0.05$], assessment use [$\chi2(3, N = 80) = 8.67, p \leq 0.05$], instructional planning [$\chi2(3, N = 80) = 19.870, p \leq 0.00$], instructional practice [$\chi2(3, N = 80) = 11.693, p \leq 0.05$], and teaching for diversity [$\chi2(3, N = 80) = 18.185, p \leq 0.00$] showed significant differences (see Appendix 2). These results indicate that the TEPs not only differ in terms of the topics taught but also in terms of the teaching methods offered.

4.2 Performance of the pre-service teachers in the knowledge for teaching mathematics items

The knowledge for teaching mathematics items provided insights about the performance of the participants in the different subdomains and were discussed in Articles I and III. In this section, the results are presented considering the PSTs' scores in the MCK and MPCK items, as well as in the respective cognitive, content, and teaching-related skills subdomains, first in a general way considering all the participants, and then, by TEPs.

From a general point of view, the results showed that the participants (N = 80) correctly answered 66.9% of the 22 MCK items and 79.2% of the nine MPCK items. Figure 14 reports the percentage of correct answers of each university out of the total test score as well as the MCK and MPCK scores.

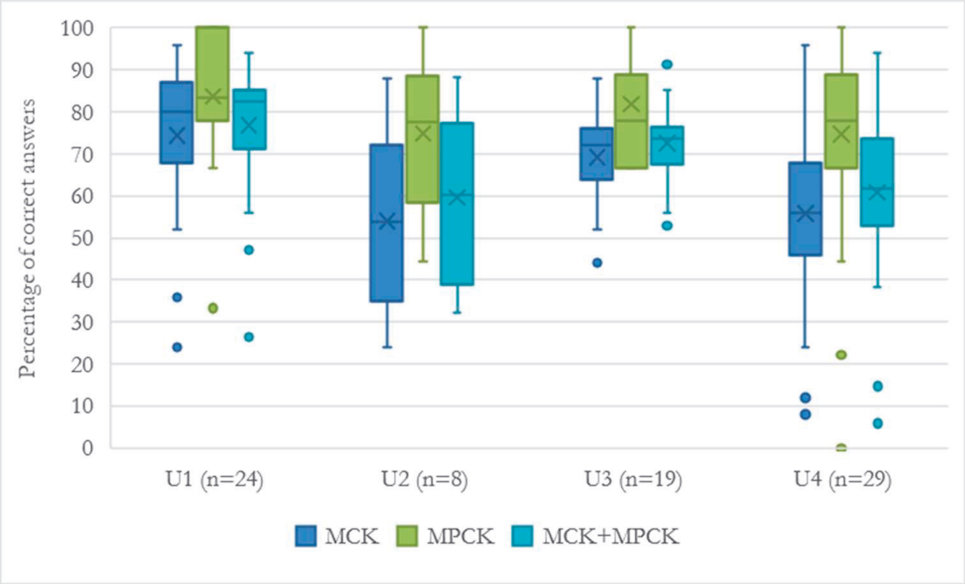


Figure 14. Pre-service Teaches' Performance on the MCK and MPCK Items by University
Note. N=80. Adapted from Alfaro and Joutsenlahti (2020).

Figure 14 shows that U2 had better scores in both constructs than the other institutions, while the participants from U2 and U4 had the lowest performance, with only around 60% correct answers. In fact, the Kruskal-Wallis H test showed that there was a statistically significant difference in the distribution of the total correct answers between the universities [$\chi2(3) = 17.079$, $p \leq 0.001$]. This result reflects the differences between the TEPs in terms of the training of PSTs and is consistent with the differences in the OTLs highlighted in the previous section. To gain a better understanding of those differences and of the knowledge strengths and weaknesses of the PSTs, their performance by subdomain was analyzed finding some patterns.

In the content subdomain, the participants (N = 80) performed better in algebra, followed by the numbers and geometry items. This leaving aside the two data items, which does not offer representative results, but were successfully solved by 82% (N=79) of the participants. Figure 15 shows the outcomes of each TEP in the content subdomains. They reveal varying patterns across universities. Indeed, the Kruskal-Wallis H test resulted in significant differences in the distribution of scores for algebra [$\chi2(3) = 15.102$, $p \leq 0.005$] and numbers [$\chi2(3) = 8.2$, $p \leq 0.005$] items among the universities. Additionally, the Wilcoxon signed-rank test bared a significant difference ($Z = -2.7$, $p < 0.05$) between the numbers and geometry scores, with the numbers results significantly better. Thus, it can be inferred that in general, PSTs are better prepared in the numbers area than in geometry.

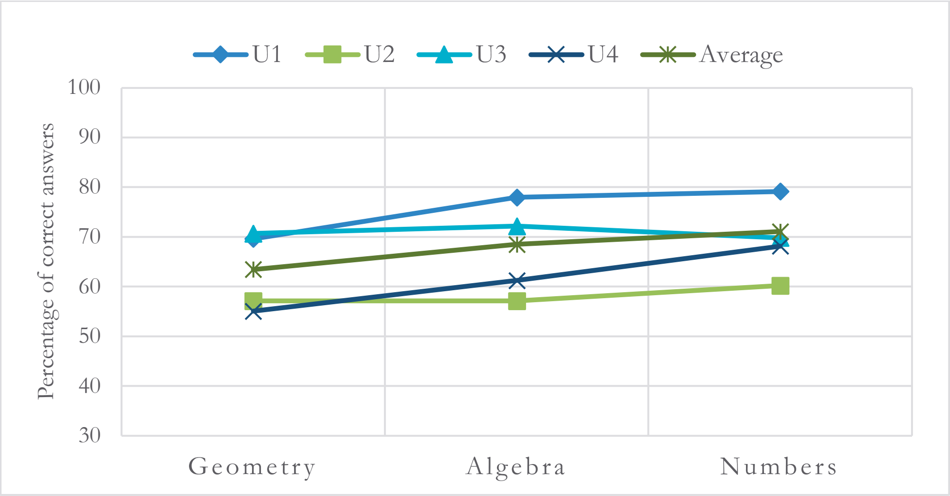


Figure 15. Pre-service Teachers' Average Performance Patterns in the Content Subdomain by Universities.
Note. N=79.

On the other hand, the pattern for the cognitive domain (Figure 16) was the same for all the TEPs except for that of U1. The general pattern was applying-knowing-reasoning, in a descending way, but the pattern of U1 was knowing-reasoning-applying, performing best in the reasoning and knowing subdomains. Further analysis using a Kruskal-Wallis H test showed that the distribution of the results for the knowing subdomain [$\chi 2(3, N = 79) = 9.093, p \leq 0.05$] and the reasoning subdomain [$\chi 2(3, N = 79) = 17.242, p \leq 0.001$] significantly differed across the universities. In addition, the Wilcoxon signed-rank test showed that the general results for the applying items were significantly higher than those for the reasoning items ($Z = -3.45, p < 0.05$). Likewise, the general outcomes for the knowing items were significantly higher than those for the reasoning items ($Z = -2.4, p < 0.05$). Thus, it is possible to conclude that the performance in the reasoning subdomain was significantly lower than that in the applying and knowing subdomains. These results suggest that the preparation of mathematics teachers in the TEPs in Costa Rica focus on the lower-order competencies of applying and knowing and must be redirected towards higher cognitive levels such as reasoning.

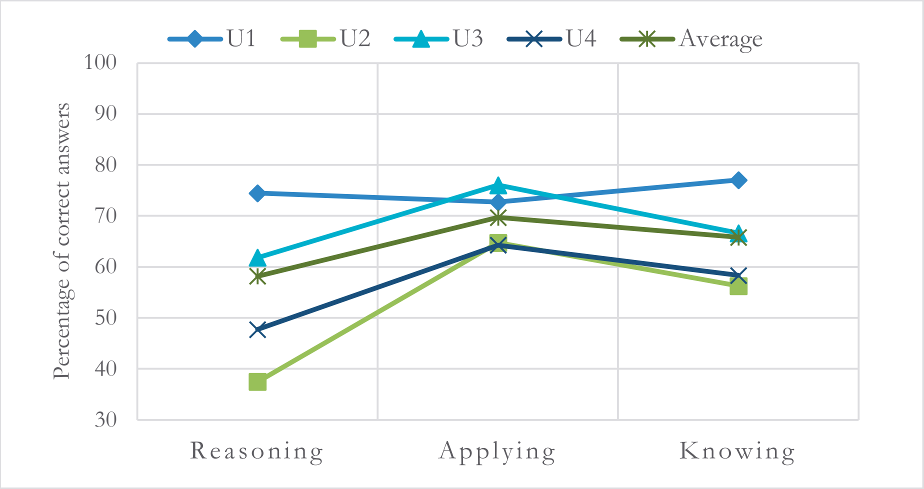


Figure 16. Pre-service Teachers’ Average Performance Patterns in the Cognitive Subdomain by Universities.
Note. N=79

Finally, the PSTs’ performance in the teaching-related skills items had a descending pattern from the curriculum and planning subdomain to the enacting subdomain (see Figure 17) for all the TEPs except that of U3, which presented the opposite trend. The Kruskal-Wallis H test proved that there were significant differences in the enacting subdomain [$\chi 2(3, N = 79) = 9.821, p \leq 0.05$] among the universities. The general results did not bare a significant difference between the two areas of teaching-related skills.

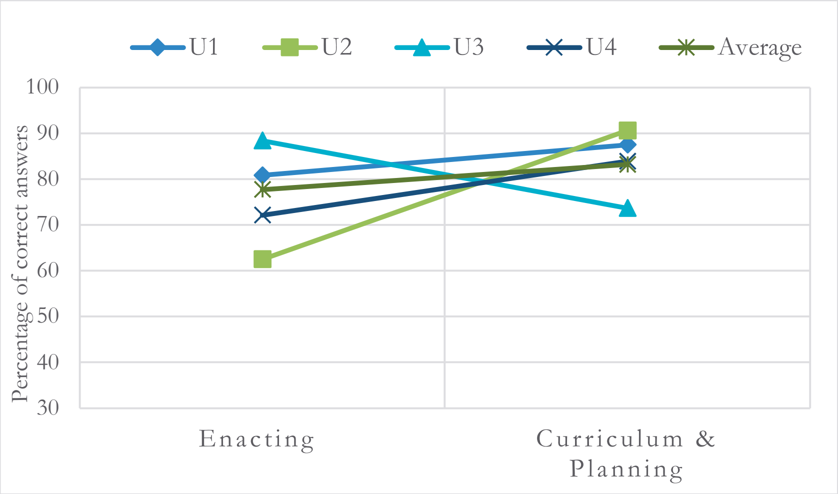


Figure 17. Pre-service Teachers' Average Performance Patterns in the Teaching-related Skills Subdomain by Universities.
Note. N=79

In conclusion, the results in this section showed that the Costa Rican PSTs have some weaknesses in the content subdomain of geometry and are better at solving items in the applying cognitive subdomain than in the other two. According to the framework used in the TEDS-M study, this means that PSTs are better at selecting, representing, modeling, implementing, and solving routine problems than in the knowledge skills of recalling, recognizing, computing, measuring, and ordering. Moreover, they performed poorly in the reasoning skills of analyzing, generalizing, integrating, justifying, and solving non-routine problems. The weakness of the participants in the reasoning items stands out. Additionally, in skills related to teaching, they turned out better at the curriculum and planning tasks after they were asked to identify the background knowledge needed to teach a given topic, than at the enacting tasks, for which they were asked to think about the possible difficulties of secondary school students in order to evaluate solutions or understand the reasons for students' errors.

At this point, it is appropriate to study whether the OTLs were related to the results obtained by the PSTs in the knowledge for teaching mathematics items, following the results of previous studies that linked PSTs' performance in the knowledge for teaching mathematics items with the number of topics that they studied (e.g., Qian & Youngs, 2016; Schmidt, Houang et al., 2011). In Table 7, the mean number of topics studied in each area (mathematics, mathematics education pedagogy, and general education pedagogy) can be observed. If those variables are related in the case of the PSTs of Costa Rica, it could be hypothesized that the participants from U3, who had the most mathematics topics in their TEP (see Table 7), would have a better result for the MCK items, or that the U4 PSTs would excel in the MPCK items because they had more mathematical education pedagogy topics (see Table 7) in their TEP. However, considering the results presented in Figure 14, this is not the case. The Spearman's analysis of correlation showed no significant relationship between the numbers of correct MCK answers and the number of topics studied. The same analysis was conducted on the number of correct MPCK answers and the number of MPCK topics studied, and again, no significant correlation was found.

Overall, the outcomes demonstrated significant differences in the performance of the PSTs from different universities, specifically in the numbers, algebra, reasoning, knowing, and enacting subdomains. These can be interpreted as differences in the quality of the programs for mathematics teachers that are offered in Costa Rica. However, it was revealed that the results for the MCK and MPCK items were not associated with the differences in the number of topics studied.

4.3 In-depth analysis of the participants' solutions to the knowledge for teaching mathematics items

Using the MUST framework, an in-depth analysis of the PSTs' responses to the knowledge for teaching mathematics items was conducted and presented in Article III. This analysis allowed us to identify the strengths and weaknesses of the participants in a more detailed way, expanding the results of the previous section. The analysis specifically yielded evidence of such strengths and weaknesses of the participants from the perspectives of mathematical understanding (in the aspects of conceptual understanding, procedural fluency, strategic competence, and adaptive reasoning) and mathematical context for teaching (from the aspect of assessing the mathematical knowledge of learners).

Conceptual understanding

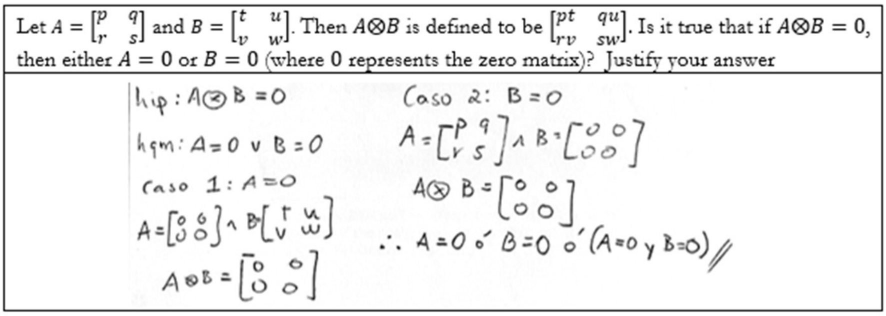
The evidence of conceptual understanding was classified into four skills: understanding and using mathematical concepts in various contexts, monitoring one's own and students' work, formulating proofs, and understanding, identifying, and using connections in mathematics. However, there was evidence that bared more than one skill.

For the skill of understanding and using mathematical concepts in various contexts, it was possible to identify the strengths of the participants and their basic errors. For example, among the strengths are explaining mathematical situations using the participant's own words. This was observed when participant P78 explained his answer to how many possible ways there are of choosing 2 and 8 students out of 10 as follows: "it is the same to choose the 2 that stay or the 2 that leave" in Exercise 804. This shows a deep understanding of the mathematical situation involved. The participants also demonstrated that they knew mathematical properties and how to use them, that they could understand a given mathematical statement in natural language and write its meaning in symbolic form to facilitate computation, and that they could identify errors in students' attempted solutions when the conditions of the statement were not met. However, they also showed errors in recalling definitions to decide on the truth of a statement or its correct use, and poor abstraction skills by needing the explicit form of a function to prove a general statement.

With regard to the PSTs' abilities to monitor their work and that of the students, the results are disturbing. The PSTs were strict in reviewing student work and giving evidence of mistakes, but they were not careful with their own procedures. They made mistakes similar to those made by the students, such as not verifying in their procedures that the sum of the angles they found for the rhomboid was greater than 360°. They also failed to read the conditions of an exercise to propose valid counterexamples, and they used incorrect words to refer to mathematical objects. The last case was observed in Exercise 806, in which the PSTs were asked to explain why the solver made a mistake when inferring information from a histogram. For this, participant P75 wrote that the solver thought "each graphic represented a country," when the correct answer was, "each bar represented a country".

Regarding the understanding, identification, and use of connections in mathematics, two phenomena were observed. One was that the PSTs had the ability to formulate equations as a result of connections between formulas, but they could not always connect them efficiently to achieve the objective of the task. The other phenomenon occurred with connections that involved the verbal representation of a statement and its symbolic form. For example, the PSTs were able to successfully connect "the result of dividing the circumference of a circle by its diameter" with the formulas and operations involved, in symbolic language, to decide whether the result was always an irrational number or not. However, some of the PSTs (i.e., P13, P49, and P70) failed to interpret a word problem properly and to write it algebraically, since for the statement "Peter has 6 times as many marbles as David" in Exercise 604 A1, they wrote $P = 6 + D$ instead of $P = 6 \times D$.

Finally, on the ability to formulate proof, the performance of the PSTs also showed problems. One such problem was their inefficient use of a hypothesis or their poor understanding of a task, which made long proofs difficult to follow. It was also noticed that some participants tried to formulate a proof using an invalid counterexample and overgeneralized by using a case with very specific features to assume that a statement was valid for all other cases, as shown in Figure 18.



Let $A = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$ and $B = \begin{bmatrix} t & u \\ v & w \end{bmatrix}$. Then $A \otimes B$ is defined to be $\begin{bmatrix} pt & qu \\ rv & sw \end{bmatrix}$. Is it true that if $A \otimes B = 0$, then either $A = 0$ or $B = 0$ (where 0 represents the zero matrix)? Justify your answer

hip: $A \otimes B = 0$

hqm: $A = 0 \vee B = 0$

Caso 1: $A = 0$

$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \wedge B = \begin{bmatrix} t & u \\ v & w \end{bmatrix}$

$A \otimes B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Caso 2: $B = 0$

$A = \begin{bmatrix} p & q \\ r & s \end{bmatrix} \wedge B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$A \otimes B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\therefore$ $A = 0$ ó $B = 0$ ó ($A = 0$ y $B = 0$)//

Figure 18. Example of Overgeneralization Made for Participant P16
Note. Adapted from Alfaro (2022).

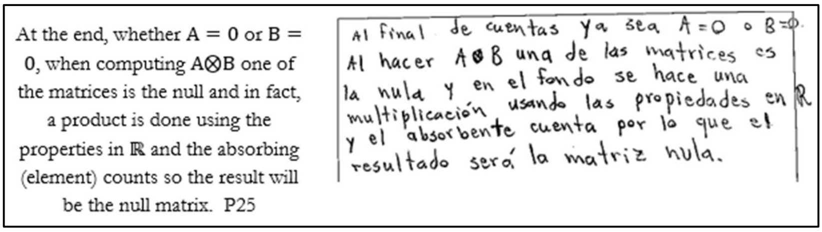
Procedural fluency

On the aspect of procedural fluency, evidence was found of the skill of quickly recalling and accurately executing procedures and algorithms. Proficiency in recalling and relating formulas was exhibited in the solutions for different exercises. For example, for an exercise in finding the measures of a rhomboid, the perimeter, the Pythagoras formula, the special triangle 30°-60°-90° relation, and the law of sines were used. However, proficiency requires not only recalling the formulas but also using them in the correct context and without calculation mistakes, but basic computation errors in the solutions that were similar to those committed by the students were observed. For example, P22 failed to solve the algebraic expression

"$x + x + 2x + 2x = 6 \leftrightarrow 8x = 6$." As mentioned, these errors may be related to their poor monitoring of their work.

Adaptive reasoning

The adaptive reasoning aspect was observed in two moments or directions: when the PSTs performed the role of teachers who had to explain the reasoning of students and when they were the ones who had to offer justifications for their procedures. From the perspective of their role as teachers, the participants offered valid reasons for explaining students' difficulties and indicated what was the error in each mistake. Nevertheless, their explanations lacked clarity and depth. When they performed the role of solvers, there were cases in which they were not successful, and their proofs turned into a maze of unconnected equations. There were also cases when they succeeded, however, and the constant recurrence of their justifications for their conclusions was observed, as seen in Figure 19.



At the end, whether A = 0 or B = 0, when computing A⊗B one of the matrices is the null and in fact, a product is done using the properties in ℝ and the absorbing (element) counts so the result will be the null matrix. P25

Al final de cuentas ya sea A = 0 o B = 0. Al hacer A⊗B una de las matrices es la nula y en el fondo se hace una multiplicación usando las propiedades en ℝ y el absorbente cuenta por lo que el resultado será la matriz nula.

Figure 19. Example of an Explanation Given by Participant P25
Note. Adapted from Alfaro (2022).

In the explanation of participant P25, it can be seen that she referred to the operation properties in real numbers and the absorbent element 0, which she used as justifications for her reasoning.

Strategic competence

For this aspect, the solutions showed evidence of the skills of selecting strategies for solving problems, having a flexible approach, knowing various solution approaches, and generating, evaluating, and implementing problem-solving strategies. Many of these skills were observed in the rhomboid exercise. For instance, the ability to select strategies was shown when the participants used their knowledge that a right triangle formed one side of the rhomboid and that there were two bisector lines to choose strategies such as the Pythagoras formula or the trigonometric ratios. In the same case, some participants demonstrated the skill of having a flexible approach for solving the problem when using the strategy of dividing the rhomboid figure or extracting the triangles from it which allow them to observe properties and relationships more clearly. This way, they could solve the problem in different steps, noticing that different strategies could solve different values.

Assessment of the mathematical knowledge of learners

This aspect can be seen in the items that required the PSTs to evaluate students' work or to analyze the reasons for students' mistakes. Some important inferences can be made from the participants' performance. For example, in their explanations of why students wrongly interpreted a histogram about the frequency of the adult female literacy rate in Central and South American countries, they referred only to obvious reasons such as that the students only counted the bars, or typical errors such as not reading the axes or the title of the graph. This phenomenon of providing superficial explanations for students' difficulties was also noticed in item 604B, where the PSTs had to explain why a particular word problem was more difficult for secondary students to solve than other problems. In this case, the participants did not refer to the difficulties associated with the structure of the equation systems involved in the solution of the problem, which would have been more specific. Instead, they explained that "students generally have trouble interpreting the relations of proportionality, regarding the translation from natural language to mathematical" (P80), which explained more closely typical errors faced by students when solving word problems.

4.4 Beliefs of Pre-service teachers and teacher educators' beliefs about mathematics and learning mathematics

Three categories of beliefs of PSTs (N = 76) and teacher educators (N = 19) were analyzed, and the analysis results were presented in Article II. The results showed some patterns that matched those in previous studies (Wang & Hsieh, 2014) as well as relevant information on the relationship between the beliefs of the participants and variables such as the TEPs or the academic background of the participants.

Belief patterns

In the category of the beliefs about the nature of mathematics, there was clear endorsement of the process of inquiry view (see Appendix 4) by 92.2% of the PSTs (N = 76) and 89.5% of the teacher educators (N = 19) which considers that mathematics is dynamic and can be discovered using creativity and different strategies. Conversely, the participants rejected the view of mathematics as a set of rules and procedures (see Appendix 3), as only 35.6% of the PSTs and 26.4% of the teacher educators supported it. In this sense, both groups belonged to the inquiry-preferring profile. According to Wang and Hsieh (2014), teachers and teacher educators with that profile view mathematics not as a set of rules and procedures but as a process of inquiry and creativity. Nevertheless, the data by university showed that the PSTs from U2 had a comprehensive profile, as the majority of them endorsed both the view of mathematics as rules and procedures and the view of mathematics as a process of inquiry.

Regarding beliefs about learning mathematics, there was full support (96.1% of PSTs and 94.7% of teacher educators) for the scale that considers mathematics an active learning process (Appendix 6). In contrast, the teacher direction scale (Appendix 5) did not receive support at all. These results reveal that both groups of participants are convinced that the learning process of mathematics requires the active participation of students. The corresponding profile for this category is active-learning-preferred (Wang & Hsieh, 2014).

With respect to the category on students' achievement in mathematics (see Appendix 7), the positioning was also radical. None of the participants supported the belief that mathematical abilities are fixed or subject to gender or cultural or birth characteristics, which positions the participants in the incremental-view endorsement profile (Wang & Hsieh, 2014). Thus, both the PSTs and teacher educators believed that mathematics skills can improve with practice and time and are not strictly associated with the students' characteristics.

For all the categories, the patterns identified in the responses of the PSTs were similar to those of their teacher educators. Moreover, the Kruskal-Wallis H test found a homogeneous distribution of the five belief scales among the four universities. This suggests that the PSTs' belief patterns were the same even though they participated in different TEPs and had different teacher educators in each university.

In conclusion, the PSTs and university teachers who were considered in this analysis believed that mathematics is a process of inquiry where creativity and new ideas are allowed and where multiple solutions can be used to solve everyday problems, contrary to the static view of mathematics as a set of established rules and procedures that cannot be changed. Furthermore, the teacher educators and the future teachers believed that learning processes should encourage the active and engaged participation of students instead of promoting lessons in which students are only recipients of the teacher's instructions. Finally, the participants showed that they were totally against the idea that mathematical skills are linked to cultural or gender aspects, or even that they stemmed solely from natural talent.

Association of variables with participant beliefs

Several statistical tests were performed to study the association of the belief scales with other variables. The Chi-square test revealed that none of the results on endorsement of the belief scales—be they results of the PSTs or of the teacher educators—was associated with the TEPs. For example, the result of the test of the PSTs' endorsement of the scale of rules and procedures was not associated with the TEPs of the universities [$X2(3) > = 6.380$, $p = 0.095$]. This finding suggests that neither the TEPs nor the universities, as study and work places, respectively, had a great influence on the beliefs of the participants. Spearman's analysis of correlation was performed to study the relationship between the PSTs' high school grades and their beliefs, and only a small negative correlation ($rs = 0.27$, $n = 76$, $p \leq 0.05$) with the belief scale of mathematics as a fixed ability was found. This suggests that respondents who had higher grades in high school supported less the idea that mathematical abilities do not change and that only some people have them. The same test was used to study the correlations among the belief scales and the participants' mathematical knowledge, and only a small negative correlation was found ($rs = -0.24$, $n = 76$, $p \leq 0.05$) between the test scores and the belief in mathematics as rules and procedures. This means that the participants who performed better in the test agreed less with that view of mathematics. The exploration of these relationships was informed by literature (Tatto et al., 2012). However, it is important to emphasize that the discovered relationships should not be taken as definitive conclusions.

Correlations between the teacher educators' beliefs and their academic background, number of years of teaching, and special preparation for teaching were also analyzed. The outcomes indicated a correlation only between the academic background and the learning about mathematics scale, where the teacher educator who reported a higher academic degree in mathematics education endorsed more strongly than the rest the belief that active learning is required to learn mathematics ($r_s = 0.48$, $n = 19$, $p \leq 0.05$). This result suggests that if teacher educators have more preparation in mathematics education, they will have a clearer conception of the importance of giving students a key role in mathematics learning.

5 RESEARCH QUALITY EVALUATION

Assessing the quality of research is a crucial exercise as it shows the decisions made by the researcher when designing and implementing the methodology in order to provide valid and reliable results. Furthermore, it "makes the knowledge claims from the study more powerful and more representative of the problem under investigation" (Plano Clark & Ivankova, 2016, p. 163). To evaluate the quality of this study as a mixed-methods research, literature states that the quality of both the quantitative and qualitative components of this study must also be assessed (Ihantola & Kihn, 2011; Plano Clark & Ivankova, 2016). Therefore, in the next section, I will delve into the quality of the quantitative, qualitative, and mixed methods separately.

5.1 Quality of quantitative methods

For the quantitative component of this study, the internal and external validity as well as the reliability were assessed. The internal validity of this study is associated with the extent to which the researcher drew valid inferences from the collected data and the research results (Plano Clark & Ivankova, 2016). The internal validity of this study was ensured by the following actions. First, to select the instrument, various frameworks related to the mathematics teachers' knowledge and competencies for teaching were studied (e.g., Ball et al., 2008; Baumert et al., 2010; Carrillo et al., 2018), as were the data collection instruments and methods used. The TEDS-M framework and instrument was chosen due to its international character, by which it considers several educational contexts, including that of Chile, which has curricular and cultural similarities with Costa Rica. Second, as TEDS-M was informed by some of the frameworks studied and as many studies have been conducted with its data (e.g., Qian & Youngs, 2016; Tang & Hsieh, 2014), current theories and study results enlightened my interpretation of this results.

In addition, data were collected from the PSTs under the same circumstances in all the institutions, using the same questionnaire, and with the same response time and the same applicator. Although in some cases the data was collected from two groups of mathematics courses at the same university, the courses were from different years in the TEP and therefore each group had different members. In that way, cross-contamination was minimized, especially regarding the knowledge for teaching mathematics items.

Although I tried to minimize all the threats to internal validity mentioned in literature (Ihantola & Kihn, 2011), there was a threat related to order bias that was not possibly to avoid. In one of the groups, the data had to be collected in two sessions, each with half of the regular response time for a session. For this reason, the participants in this group were instructed to complete first section 3 of the questionnaire on the exercises, which prevented them from having more time or resources than the other groups for solving the exercises in the two implementation sessions. However, since this condition was different from the questionnaire response order for the other groups, this may have affected the results.

External validity is considered the extent to which general conclusions can be drawn from the methods used and the data collected. In other words, it is the extent to which the results can be generalized (Plano Clark & Ivankova, 2016). In this regard, this research has several issues. First, even though I aimed to include in the sample all Costa Rican mathematics PSTs enrolled in a course on the last year of their TEP, the sample was reduced to the PSTs of only four out of the eight active TEPs in the country due to the lack of interest in participating in this study or due to the logistical aspects, such as the case of the distance learning university. Consequently, the sample size of the PSTs and the teacher educators was small, which led to statistically non-generalizable results. Furthermore, there was no private university and distance-learning university in this sample. Considering previous studies (Alfaro et al., 2013; Roman & Lentini, 2018) in which the differences between the TEPs were addressed, the results of four TEPs are not enough to expand the conclusions of this study to the entire population and in all settings. This represents an environmental validity threat. In addition, as the implementation and the sample size in this study did not follow the IEA standards, comparisons could not be made between the data of this study and the TEDS-M study data. In this study, the solution time for the questionnaire was of three hours whereas in the IEA implementation was of 90 minutes (Tatto et al., 2008). Besides, in the IEA TEDS-M study the sample was based on a multi-stage sampling design (Tatto et al., 2008), which was not use in this study. However, the results of this study are representative of the training offered by public universities in Costa Rica.

The time validity of the findings of this study is linked to the stability of the structure of the TEPs. If the institutions decide to update and improve their programs in terms of learning opportunities or teaching methods, the results of this study may lose relevance. Nevertheless, an aim of this research is to provide information that can be used to develop mathematics teacher training in Costa Rica, so I hope TEPs will be improved in light of the results of this and other studies.

Finally, the reliability of the quantitative part of this study was enhanced in different ways. The TEDS-M instrument, used for data collection was designed, implemented, and validated internationally. All the measures passed several pilot and trial processes and were reviewed by expert panels as well as the national committees of the participating countries (Tatto, 2013). The OTL and belief scales, as well as the items for the knowledge for teaching mathematics section, were evaluated in terms of "clarity, correctness, cultural relevance, classification within the framework of domains and subdomains, relevance to teacher preparation, and curricular level" (Tatto, 2013, p. 169). The psychometric quality of the items and the internal consistency of the scales were measured. The Cronbach's alpha of the Likert scales for the OTLs and beliefs ranged from 0.79 to 0.97 (Tatto, 2013), an acceptable level according to Hinton et al. (2014). As the original instrument was in English, I had to translate it to Spanish. To ensure the validity of the translation and contextualization of the items, three Costa Rican mathematics education experts were asked to assess the intelligibility and clarity of the instructions, statements, and mathematical tasks after the translation. In this way, I aimed to avoid misinterpretations that could affect the measurements.

Nevertheless, there were some threats to the reliability of the quantitative research. For instance, as this was a single-researcher study, interrater reliability measurement was not possible. This threat was minimized by following the scoring guides (see Chapter 3.6.1) provided by the TEDS-M team (Brese & Tatto, 2012b). The other threat was the failure to test the translated instruments before their use due to time limits and logistical constraints. My limited experience in conducting quantitative research can also be considered a reliability threat, which I addressed by consulting experts in quantitative research about my decisions, participating in courses about quantitative research methods, and studying relevant literature.

5.2 Quality of qualitative research

The qualitative part of this study consisted of the written solutions to the constructed response items and the annotations of the participants in the multiple-choice and complex multiple-choice items. Their analysis allowed a better understanding of the PSTs' knowledge (see Chapter 4.3). To assess the quality of this section, the contextual validity, the generalizability, and the procedural reliability were considered.

Contextual validity is defined as the credibility derived from the evidence and the conclusions drawn from it (Ihantola & Kihn, 2011). To enhance the credibility of this part of the study, excerpts of the participants' procedures and answers were provided as examples of the conclusions made. Moreover, the analysis and categorization went through the journal reviewer process. However, there are some threats to the contextual validity of this study. As this is a one-researcher study, double-blind interpretation and categorization of the data were not possible. The language was a barrier to those since the participants' solutions were in Spanish and I was the only one who could understand the data. Thus, my discussions with my supervisor about the content analysis were affected by this translation of the participants' procedures as well as the presentation of the results. A major threat is that the knowledge items for teaching mathematics among the TEDS-M items had not been qualitatively analyzed yet before. Consequently, I reflected extensively on what would be the best approach and framework for analyzing such data. Finally, I decided to use the MUST framework (Kilpatrick et al., 2015). Although I did not find studies that had used the MUST framework in the same way that it was used in this study, the MUST framework is based on the Mathematical Proficiency framework (Kilpatrick et al., 2001) for students, which has been used for this type of analysis (e.g., Alfaro, 2018; Groves, 2012; Viro & Joutsenlahti, 2018).

The qualitative part of this research was aimed at extending the understanding of the PSTs' knowledge for teaching mathematics, as shown in their solutions to the items. The results are specific to the participants and their performance in the items in the test, and thus, generalizations are not intended.

Regarding the procedural reliability of this part of the study, it was addressed by providing a thorough description of the framework used for the direct content analysis, the analysis methods, and the procedures. However, the fact that this was a single-researcher study posed a threat to the study's reliability as it reduced the objectivity of the categorization.

5.3 Quality of mixed-method research

For the evaluation of this mixed-methods research as a whole, the legitimation framework proposed by Onwuegbuzie and Johnson (2006) is used because it perceives the quality of research as a continuous and interactive process, compatible with the characteristics of mixed-methods research. Onwuegbuzie and Johnson proposed nine aspects of the evaluation of the legitimation of research, which was used in this study and will be discussed next.

Sample integration considers if the relationship between the samples used for the quantitative and qualitative methods allows meta-inferences of quality (Onwuegbuzie & Johnson, 2006). A threat to this legitimation aspect occurs when different samples are used for each part of the study (Ihantola & Kihn, 2011). In this case, this was not an issue since the data for both parts came from the same PSTs.

The inside-outside aspect refers to the different views a researcher can assume during the different research moments. For instance, the quantitative approach allows outsider perceptions, whereas the qualitative approach offers insider viewpoints. According to Ihantola and Kihn (2011), a threat is presented when those perspectives are not balanced. Onwuegbuzie and Johnson (2006) suggested that peer-reviewing, member-checking, and participant-reviewing can reduce this legitimation threat. In this study, the qualitative and quantitative data collection and analysis were conducted by one participant, who was trained as a mathematics teacher in one of the TEPs involved in this study, which allowed a better understanding of the context for accurate interpretation of the data. Thus, there was an insider perspective. Moreover, the results were presented in peer-reviewed journals, which can be consider the outsider view.

Weakness minimization refers to "the extent to which the weakness from one approach is compensated [for] by the strengths from the other approach" (Onwuegbuzie & Johnson, 2006, p. 57). In this study, this was considered in the research design. Since the statistical information provided by the quantitative part did not offer detailed information on the participants' knowledge gaps or on the weaknesses and strengths of their mathematical skills, a content analysis was performed to compensate for that. On the other hand, the qualitative analysis alone did not provide enough information on the PSTs' training, which was covered by the quantitative part.

The *sequential* aspect is related to the order in which the qualitative and quantitative data are collected and interpreted, and its possible influence on the results (Onwuegbuzie & Johnson, 2006). In this research, the qualitative and quantitative data were collected concurrently, and thus, the sequence was not an issue in this regard.

Conversion legitimation is associated with the actions of quantifying the qualitative data or qualifying the quantitative one, and how those actions can affect the quality of the meta-inferences. The content analysis in this study was not performed by counting how many times a specific topic appeared. Instead, the solution strategies, concepts, and procedural difficulties were recognized and categorized based on theory. In addition, the qualification of the quantitative data, which, as Onwuegbuzie and Johnson (2006) said, can happen when establishing narrative profiles, was managed using the mean values and the profiles that were defined in literature (e.g., Tatto et al., 2012; Wang & Hsieh, 2014).

Paradigmatic mixing is "the extent to which the researcher's epistemological, ontological, axiological, methodological, and rhetorical beliefs that underlie the quantitative and qualitative approaches are successfully (a) combined or (b) blended into a usable package" (Onwuegbuzie & Johnson, 2006, p. 57). In this sense, for this research, the qualitative and qualitative parts were treated as separate, and the results of each part complemented those of the other, which is an approach to legitimation (Onwuegbuzie & Johnson, 2006).

Commensurability legitimation refers to the researcher's ability to switch between the qualitative and quantitative lenses and to provide meta-inferences that reflect a mixed worldview (Onwuegbuzie & Johnson, 2006). In the case of this research, the mixed worldview was considered by integrating the qualitative and qualitative findings to construct an informed profile of the Costa Rican mathematics PSTs. However, my inexperience in conducting mixed-methods research could represent a threat to this legitimation type.

Multiple validities legitimation refers to "the extent to which all relevant research strategies are utilized" (Onwuegbuzie & Johnson, 2006, p. 59). In this sense, the study will have multiple relevant 'validities' (Onwuegbuzie & Johnson, 2006). It is also related to the contribution made by each research strategy, assessing the fact that the quality of the meta-inferences must be better than the quality of the sum of the inferences made from the quantitative and qualitative parts. In other words, the integration of the outcomes increases the validity of the mixed-methods research. In this research, the outcomes of the different parts were integrated to describe the profile of the PSTs who participated in this study. The quantitative outcomes for the OTLs were contrasted with the participants' knowledge for teaching mathematics shown in the items, which, at the same time, was described using the qualitative results of the content analysis. In doing this integration multiple validities were considered.

Political legitimation corresponds to "the extent to which the consumers of mixed-methods research value the meta-inferences stemming from both the quantitative and qualitative components of a study" (Onwuegbuzie & Johnson, 2006, p. 57). This type of legitimation includes the tensions that can arise when different researchers are used for each part of the study and the possible differences between the researchers regarding the procedures and interpretations. As this is a single-researcher study, the political legitimations did not represent a threat to the quality of this mixed-methods research.

6 DISCUSSION

The aim of this study is to gain information about PSTs' MCK and MPCK at the end of their TEPs in Costa Rica, as well as the curricular structure and the methodological strategies that allowed them to obtain such knowledge. In addition, the participants' beliefs that had been proven to shape teachers' practices were also considered. In this chapter, the main findings of this research are summarized. Furthermore, the research contributions are discussed and the limitations of this study as well as ideas for further studies are presented.

6.1 Research aims and findings

Considering the scant information on TEPs for mathematics teachers in Costa Rica and the many concerns about their teaching quality (Roman & Lentini, 2018), this study investigated four TEPs, including the OTLs that they offered to PSTs, their PSTs' performance in the study items on knowledge for teaching mathematics, and their beliefs. For that, this study used a sample of 80 PSTs in the last year of their TEP and 19 teacher educators, who completed a survey about the stated topics. The main results are discussed next.

The topics studied in TEPs and the teaching experiences that their participants can learn and practice will shape their knowledge (Schmidt, Houang et al., 2011) and influence their future practices, which will affect students' learning (Hill et al., 2005). The outcomes of this research revealed that the TEPs dedicate 45% of the topics in their program to teaching tertiary-level mathematics (Article I), which demonstrates a focus on subject matter knowledge, the same focus showed by top-achieving countries such as Taiwan and the Russian Federation (Schmidt, Houang et al., 2011). In addition, the TEPs allocate 18% of their topics to general pedagogy and 37% to mathematics education (Article I).

Even though the percentage dedicated to studying mathematics education topics in Costa Rican TEPs is higher than in Taiwan, the US, or Russia, which dedicate an average of 30% of their mathematics teacher education coursework to that area (Schmidt, Houang et al., 2011), the Costa Rican participants believe that more mathematics education courses should be included in their TEPs. That phenomenon may be related to the historical dissociation between courses with mathematical and pedagogical content in Costa Rican universities (Alfaro et al., 2013). It also leads us to ask: How many topics are enough in each area of knowledge for PSTs to acquire a body of knowledge and skills that allow them to face successfully the different challenges of teaching practice? Although there is no definitive answer to the question, as Koponen (2017) stated in his study on a TEP in Finland, teaching success is not about the number of topics studied but the type of topics studied and the teaching methods practiced.

Therefore, to build the professional competencies of PSTs, not only the topics they will study are important to consider but also what teaching methods will be used. In this sense, I found that the participants' class participation was varied and active—they were required to solve problems, read relevant references, and ask questions or make presentations in front of the class (Article I). However, those actions are not specific to training mathematics teachers, as they could also be required of mathematicians.

On the teaching methods strictly related to teaching, the findings showed worrying results. Most of the participants had few OTLs or no OTL about assessment uses, teaching students from different backgrounds, and reflecting on and improving their teaching practice (Article I). Those skills are important for mathematics teachers and are considered in different frameworks (e.g., Ball et al., 2008; Carrillo et al., 2018). This lack of OTLs or practice about those topics or skills can have negative effects on PSTs' future practice, and consequently, on students' learning. For instance, if teachers are not encouraged to reflect on their practice in order to evaluate their chosen teaching strategies or even their method of solving a mathematics problem, they will have few opportunities to improve and they will not become critical professionals. On the other hand, performing assessments of the participants without knowing how to use the information gained from them makes such assessments useless. Teachers need to know how to reflect on their students' performance in order to implement changes in their teaching, give their students feedback, or reinforce some topics. Finally, teachers' lack of knowledge of how to meet the specific needs of their students from different backgrounds or with different learning needs can impact those students' learning. Unfortunately, the lack of attention to students' diversity is also an international trend (Tatto et al., 2012).

Having described the OTLs offered in the Costa Rican TEPs studied and how such OTLs are distributed in the knowledge areas, I will now move on to discussing how the study participants proved what they had learned through such OTLs. The

PSTs’ performance in the items on knowledge for teaching mathematics showed acceptable results but not the best—67% of the MCK tasks and 79% of the MPCK tasks were answered correctly. These results can be broken down into scores by content and scores for the cognitive and teaching-related skills subdomains.

Regarding the participants’ performance by content area, the average results showed better performance in algebra, but the statistical analysis showed that the distribution of algebra and numbers significantly differed among the TEPs. In fact, although the general results showed the descendant pattern algebra-numbers-geometry, there were TEPs in which the PSTs’ performance scores in algebra and geometry were similar and lower than the PSTs’ performance scores in numbers, which means such results did not show shared patterns. Hence, it was not possible to identify supremacy or deficiencies in one specific area.

In terms of the cognitive domain, the general performance pattern was applying-knowing-reasoning, with significant differences between the applying and reasoning scores and the knowing and reasoning scores, and with reasoning as the cognitive domain with the lowest results (Articles I and III). Reasoning tasks also had the lowest figures in the international TEDS-M study (Hsieh et al., 2014), which means PSTs struggle with higher cognitive domain tasks.

In this sense, the results of the qualitative study allowed more specific elaboration of the deficiencies and strengths of the PSTs in the knowledge for teaching mathematics (Article III). The participants’ evidenced strengths and weaknesses were associated with the cognitive domains according to the TEDS-M framework (Tatto et al., 2008). For instance, their proficiency in recalling definitions and formulas in the relevant contexts is a sign of their knowing subdomain. However, the poor way in which the PSTs connected the definitions and formulas to obtain a model for solving a routine problem revealed issues with their applying subdomain.

In summary, regarding the knowing subdomain, the PSTs performed well in recalling definitions, geometric properties, or formulas and using them to solve a problem. However, they had difficulty in recalling a definition in order to decide if a statement was true. In addition, the PSTs committed computation mistakes by adding algebraic expressions or solving special products.

Concerning the applying subdomain, the participants showed acceptable performance in generating equivalent representations of mathematical statements in different mathematical languages, such as from a statement in a natural language to a symbolic representation. Yet, in the skills of selecting efficient strategies or generating a model for solving problems, they showed severe deficiencies.

Finally, in the reasoning subdomain, the participants showed skills in explaining mathematics situations in their own words or in providing justifications for some of their proofs. In addition, they demonstrated some analytical skills when they decomposed the rhomboid to simplify the solution process. However, their abilities to integrate procedures and results were weak, as were their monitoring practices where they failed to check the answers, consider the conditions of the statement, or understand the hypotheses.

Therefore, there is evidence that in all the cognitive subdomains, there are some skills that need to be developed, especially considering that according to the Costa Rican mathematics school curriculum, the teachers must be able to practice and develop the processes of reasoning and argumentation, problem posing and solving, communication, connection, and representation in the classroom (MEP, 2012). Going back to the TEDS-M framework, the cognitive subdomains are associated with the MCK. Thus, although the TEPs dedicated 45% of their coursework to studying tertiary mathematics and although most of the participants declared occasionally or frequently having the opportunity to solve problems in class, these efforts were seen as not enough. In fact, there was no significant correlation between the number of correct MCK answers of the participants and the number of mathematics topics studied, a relationship that was shown in previous studies (e.g., Qian & Youngs, 2016; Schmidt, Houang et al., 2011).

Here it is again evident that not only the number of topics or the amount of content covered is important but also the approach to learning mathematics. Mathematics teachers must learn during their training to be proficient in mathematics as individuals and as the mediators in their students' learning process. Hence, it is important that they not only have the knowledge but also the competencies to connect the different types of knowledge that allow them to face mathematical and didactic situations in their work. This, as Koponen (2017) stated, could be the real challenge.

As for the PSTs' performance in the teaching-related skills, although the number of items they covered was small, it was observed that the Costa Rican PSTs performed better in the curriculum and planning subdomain than in the enacting teaching and learning subdomain. Regarding their solutions, they showed deficiencies in giving feedback on their students' work. Even though they could identify the students' mistakes and point them out, when they were asked to explain why the students had difficulties with the exercises, the answers of the PSTs were superficial and not clear. This suggests that the PSTs should improve their skills in diagnosing students' misconceptions. However, as observed in the results, the efficient use of the assessment was one of the teaching practices that most of the PSTs have never or rarely had the opportunity to learn.

Another important aspect to consider about the PSTs is their beliefs about mathematics, mathematics teaching and learning, and mathematics achievement. This is because as stated, the teachers' beliefs influence their practice, the approach they take in relation to mathematics learning and learning, and their relationships with their students (Skott et al., 2018; Tang & Hsieh, 2014). The findings from this study (Article II) surfaced beliefs that have been associated with better student results and teaching strategies (Voss et al., 2013). Specifically, both the PSTs and the teacher educators believe that mathematics is a dynamic process of search and discovery in which it is important to know and study mathematical concepts and to have the teacher's guidance. However, the teacher is not the protagonist in the learning process. Instead, the participants believe that to learn mathematics effectively, the student must have an active role, that is, a constructivist conception. This dynamic constructivist orientation (Barkatsas & Malone, 2005) also matches the principles defined in the mathematics school curriculum of the Costa Rican Ministry of Education (MEP, 2012).

On the other hand, the findings about the participants' beliefs regarding mathematics achievement indicate that they are against the ideas that suggest that gender or culture affects the skills in learning mathematics. Nevertheless, they disagreed, at a lower level, with the statements about mathematical skills being fixed and results of natural talent, similar to international results (Wang & Hsieh, 2014). Overall, the beliefs of the PSTs and the teacher educators are similar and did not show significant differences among the TEPs.

The analysis of the OTLs and the participants' performance provided information that supports what Roman and Lentini (2018) and Alfaro et al. (2013) stated regarding the differences in the TEPs, but the findings from this study go further and not only show that the TEPs are different but also how they differ. For instance, the analysis of the OTLs showed that the distribution of the mathematics education topics differs across the universities, with U4 having 2.8 topics more mathematics education topics than U2. A similar difference was found between U3 and U4 as to the tertiary-level mathematics topics, even though this variable did not significantly differ between the TEPs.

There were also differences regarding the teaching methods experienced. Seven of the 10 scales showed significant differences among the universities, which means that Costa Rican mathematics PSTs have different opportunities to learn and practice solving problems, assessment and instructional practice, assessment use, teaching for diversity, instructional planning, and class reading.

In addition, significant differences were found in the performance of the participants in the knowledge items for teaching mathematics. In the MCK items, the distribution of scores differed between universities. For instance, U1 performed better than the rest, with half of its PST participants answering 80% of the items correctly, while in U2 and U4, more than half of the PSTs scored less than 60% correct items. In the content subdomain, numbers and algebra scores were distributed differently, whereas in the teaching-related skills subdomain, only the enacting items showed different distributions between universities. Major differences were found in the cognitive subdomains, with U1 exceeding U2 by more than 30 points for reasoning.

6.2 Research Contributions

Teachers' knowledge for teaching mathematics has a significant impact on the classroom dynamics and students' learning, since it affects aspects such as the choice of learning goals, curricular decisions, and even the students' beliefs about mathematics (Hill et al., 2008). Thus, to understand how to improve the quality of mathematics teaching, it is important to know the status quo, or in other words, what teachers learn, know, and believe. In this sense, the findings from this study contribute to filling the knowledge gap on the curricular and methodological characteristics of the training of mathematics teachers in Costa Rica, and on the knowledge for teaching mathematics that PSTs have at the end of their studies, which, in turn, expand the few studies with PSTs and about mathematics TEPs in the region.

This study also adds to the scarce knowledge about the beliefs of PSTs and teacher educators regarding the nature of mathematics and its teaching and learning, since previous studies on this subject in Latin American countries considered only the beliefs of in-service teachers. Making visible the beliefs of PSTs from their school experience with the teaching and learning of mathematics, positive or negative, is essential for implementing actions for their redirection or for cultivating the desired constructivist orientations (Voss et al., 2013). This research showed that the dynamic constructivist orientation of PSTs' beliefs match the declared beliefs of the in-service teachers in previous studies (Gamboa & Moreira, 2017; Mora & Campos, 2008). Nevertheless, the teaching practices identified in the PEN (2019) study contradict the dynamic constructivist orientation and reflect traditional teaching methods associated with negative student outcomes (Voss et al., 2013). Thus, it seems that there are factors that prevent teachers from practicing the beliefs they profess, which education policymakers must identify and address to ensure student-centered school environments where learning can be built through discovery and investigation.

In addition, a significant contribution of this research is the identification of the characteristics that differentiate the TEPs from each other, which point out their differences not only in the number of contents that they cover but also in the teaching methods they offer and the participants' performance, which also indicate differences in the quality of their preparation of mathematics teachers. This finding adds to the statements in literature about the diversity of the quality, contents, and duration of the TEPs in Costa Rica (Alfaro et al., 2013; Roman & Lentini, 2018) and provide clarity about the extent of those differences.

Furthermore, the finding that the number of topics studied by the participants in this study did not make a difference in their performance, as was the case in previous studies (e.g., Qian & Youngs, 2016), may suggest that the way in which these topics are addressed or the quality of the teaching methods is not the most appropriate as to make the learning of PSTs meaningful and lasting. The people in charge of designing and implementing the TEPs must consider the objectives of each course, how the topics are connected and complemented, and the overarching objectives of mathematics teachers' education (Koponen, 2017) so that future teachers could develop all the knowledge and professional skills needed to perform their work effectively. Specifically, the outcomes of this study indicate that the universities of the TEPs included in this research have to design strategies for improving PSTs' reasoning skills, for offering them OTLs to teach students from different backgrounds and learning needs, and for introducing more and better learning opportunities that equip future teachers to provide meaningful and useful feedback to students and parents. In general, such universities must put much effort into training critical and reflective professionals who have the ability to evaluate their actions and implement improvements in favor of student learning. Improving the TEPs in the way that the contents are addressed, the number of topics offered, and the methodological approaches used should result in quality mathematics teachers. Having quality mathematics teachers will improve the mathematics education that high school students receive, which could improve their performance in international tests such as PISA and their future studies.

Finally, attention must be paid to the fact that teacher educators see mathematics learning as active student learning. It is important that teacher educators reflect in their actions such belief because that facilitates the learning of mathematics. Therefore, a recommendation for policymakers and higher education institutions is that all teacher educators in charge of mathematics TEPs participate in courses or activities where they can have better knowledge of mathematics education, such as in institutional or international research groups, international congresses on mathematics education, and introductory seminars on mathematics didactics. In addition, it is important to continue supporting postgraduate studies for mathematics teachers in order to have trained personnel who can participate in the improvement and innovation of the TEPs.

Thus, to the question posed in the title, "Are they ready?" the results of this study show that they are not. Much work is needed before PSTs will be ready to teach, considering not only their mathematical knowledge but also their students' characteristics; before the TEPs will be ready to prepare teachers in all professional competencies; and before the MEP can select and hire mathematics teachers capable of teaching mathematics in high school competently.

6.3 Limitations and future research

This study has some limitations, as declared in Articles I to III, most of which are at the methodological level. Survey instruments are effective in collecting data about OTLs and some other aspects, mostly cognitive, related to the PSTs for teaching mathematics. In this study, however, there was no opportunity to observe the situated aspects of teaching, that is, teaching practices (Boston, 2012). Some professional competencies such as instructional planning, assessment practice, and the strategies for accessing students' mathematical thinking were not possible to measure. Moreover, as Kaiser et al. (2017) mentioned, due to the complex interaction between the knowledge-based facet and the situated competence facet, both of them need to be considered to capture the whole picture of teachers' knowledge. In the same way, there was no opportunity to witness if the PSTs enact the same beliefs that they declared in their answers on the questionnaire. Therefore, it is important to conduct studies where PSTs can be observed during teaching practice to gain knowledge about their skills in relation to teaching and how the OTLs and the teaching methods that they experienced during training contributed to their teaching performance.

Another limitation of this study is associated with the sample composition. The sample was small and included only participants from public universities, leaving aside private institutions or public institutions that employ the distance learning method. Therefore, the results obtained cannot be generalized even at the national level. Considering the facts indicated in the literature on the differences between the TEPs in public and private institutions (Alfaro et al., 2013) and the poor performance shown by in-service mathematics teachers from private universities in the school mathematics diagnostic test (MEP, 2011), it is crucial to learn about the OTLs, beliefs, and knowledge for teaching mathematics of PSTs from private institutions. Hence, it is important to continue the efforts to collect data from these institutions in order to have a more complete picture of the training of mathematics teachers in Costa Rica and to have evidence of the differences between the training offers.

One more limitation of this study is the fact that this research was carried out by a single researcher, which caused the lack of research triangulation that could have reduced the validity of the results, especially in the qualitative aspect. It would be relevant to analyze the data with a research group so that interpretations can be pluralized and categorizations, discussed.

More broadly, research is also needed to study the factors that cause the inconsistency between the beliefs expressed by the teachers and what they do in class. Longitudinal studies could be performed to analyze whether, how, and by what factors teachers change their beliefs. Further research should be conducted to analyze the quality of the courses in the TEPs, in terms of relevance and effectivity for developing mathematics knowledge for teaching. In addition, discussing these findings with the teacher educators from the TEPs involved could provide insights regarding the causes of the weaknesses seen in the PSTs' solutions and could generate ideas for improvement. Finally, interesting results could be derived from the implementation of the TEDS-M knowledge for teaching mathematics items with in-service teachers, to analyze if the teaching experienced has improved their performance and why.

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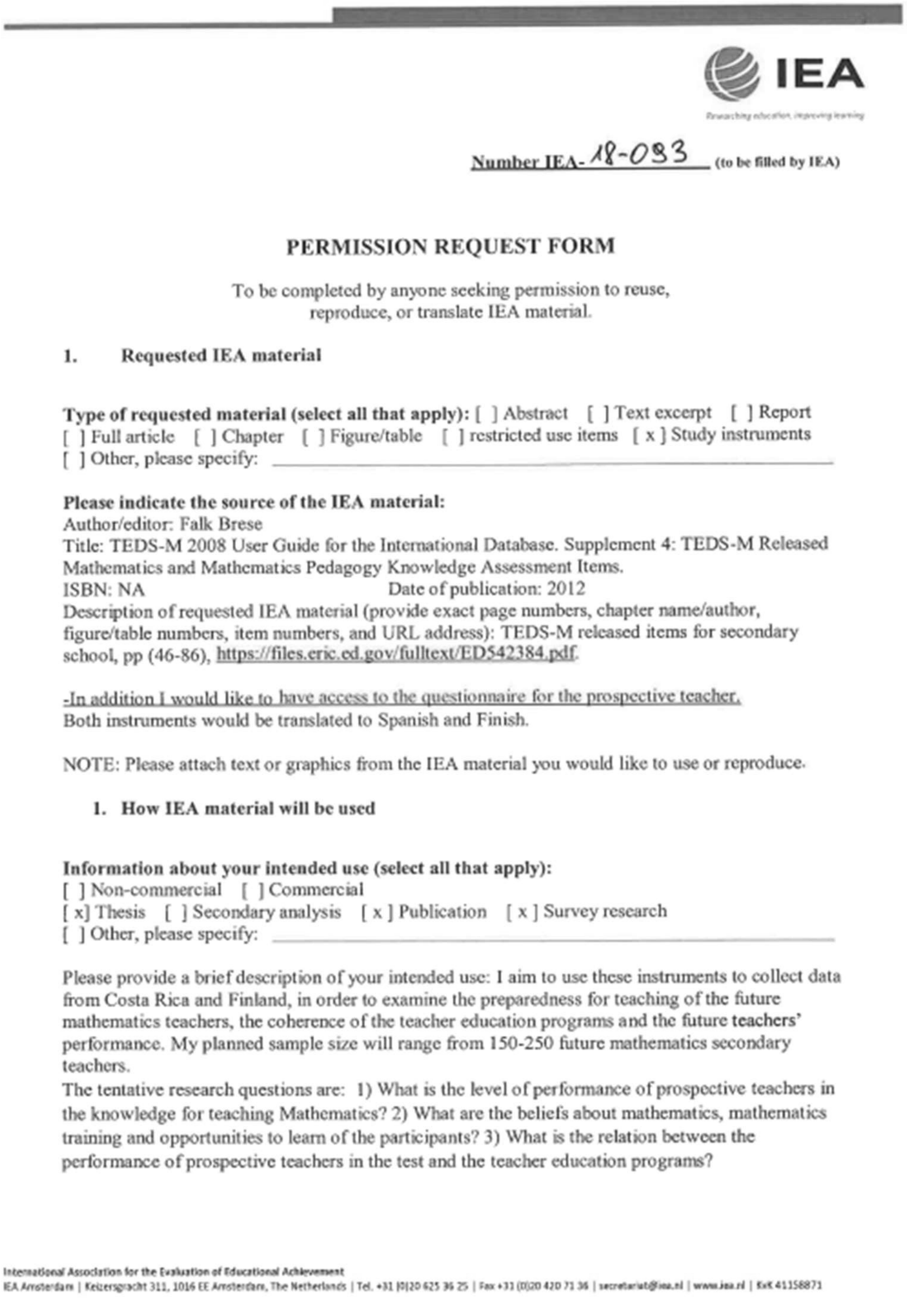
APPENDIX

Appendix 1 IEA permissions



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The tentative research questions are: 1) What is the level of performance of prospective teachers in the knowledge for teaching Mathematics? 2) What are the beliefs about mathematics, mathematics training and opportunities to learn of the participants? 3) What is the relation between the performance of prospective teachers in the test and the teacher education programs?

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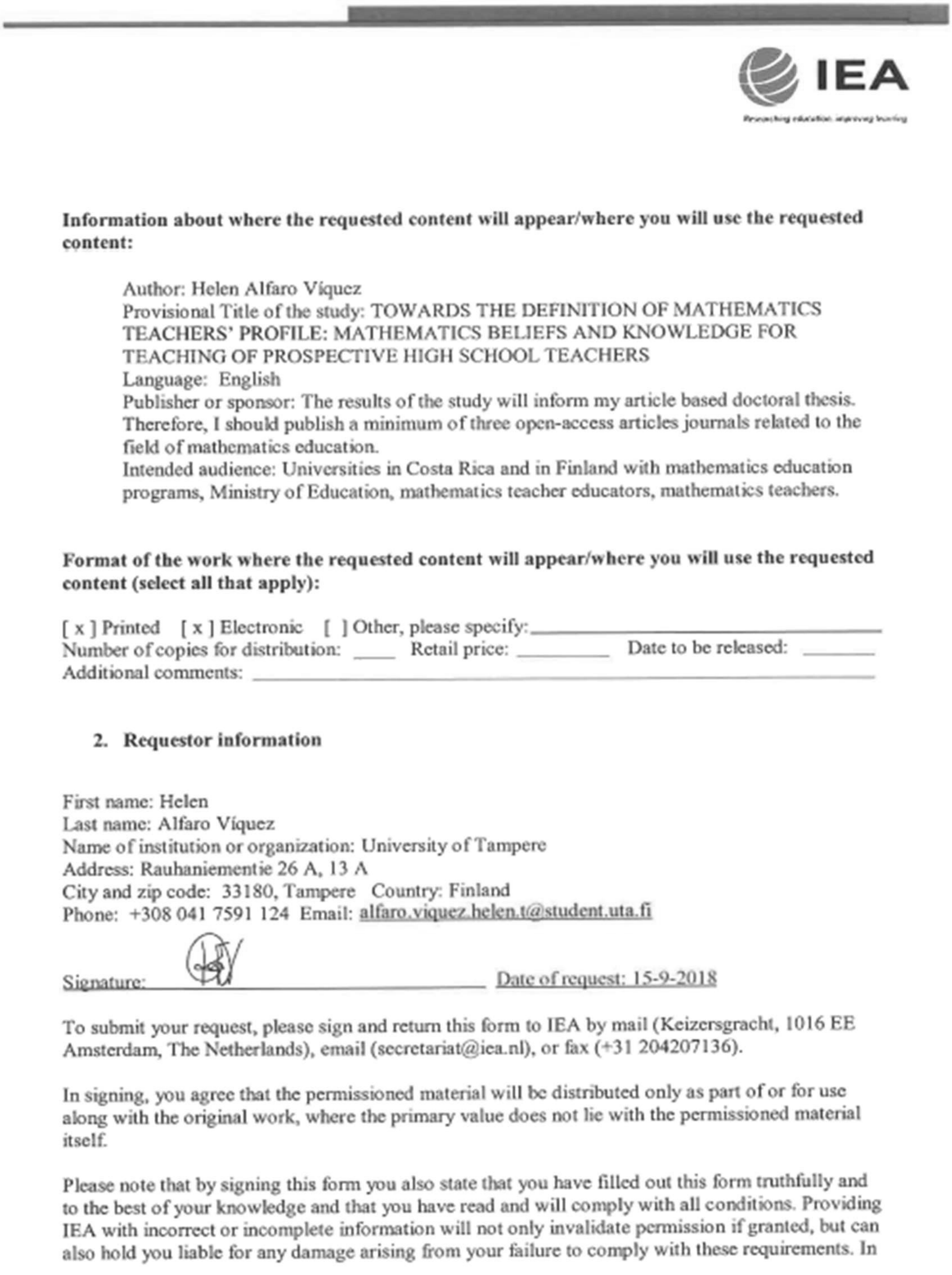

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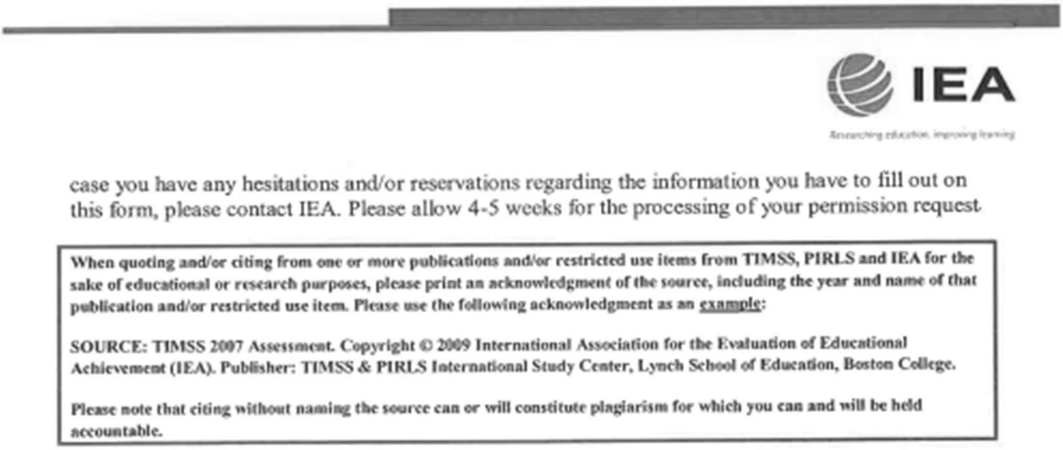

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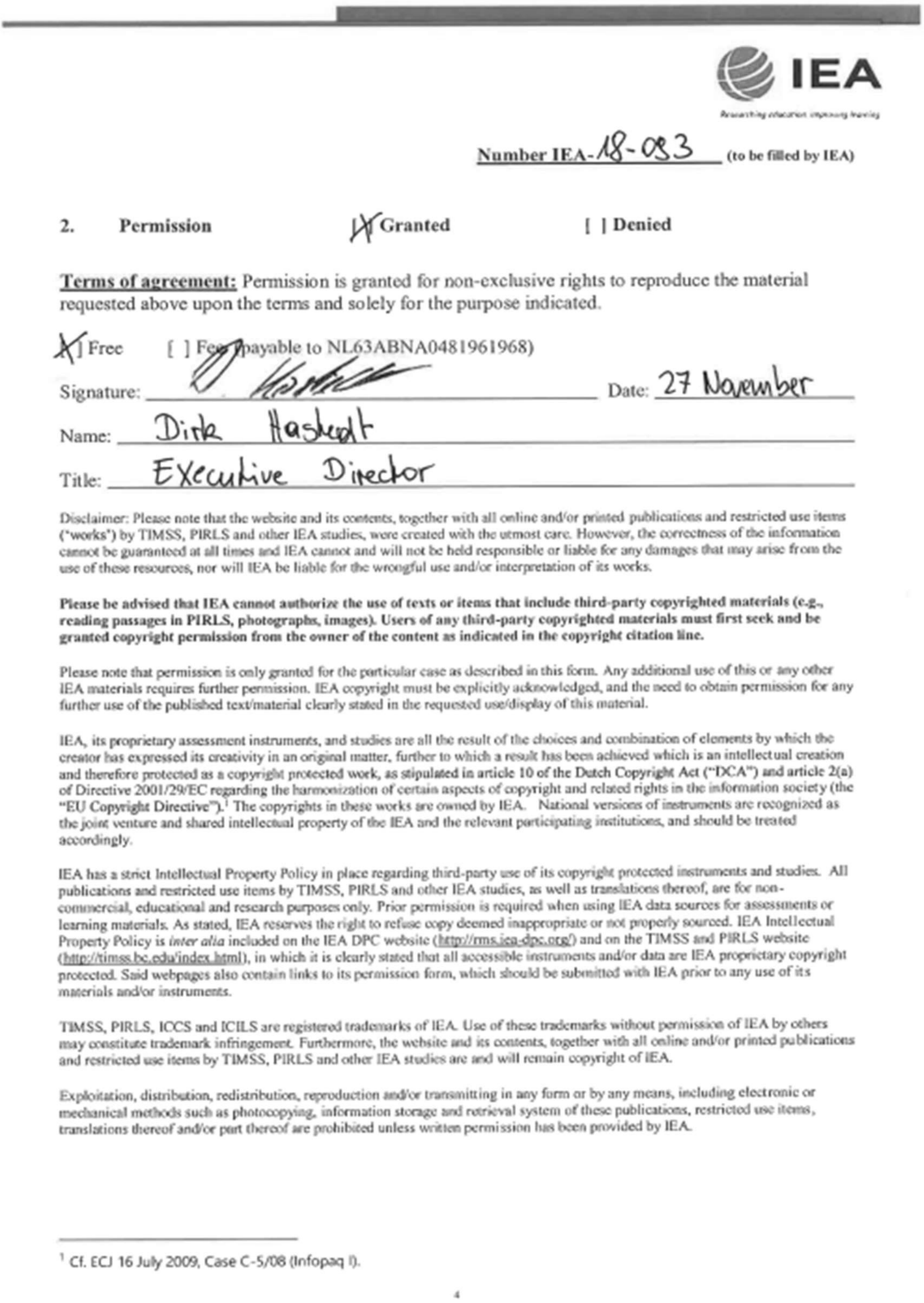

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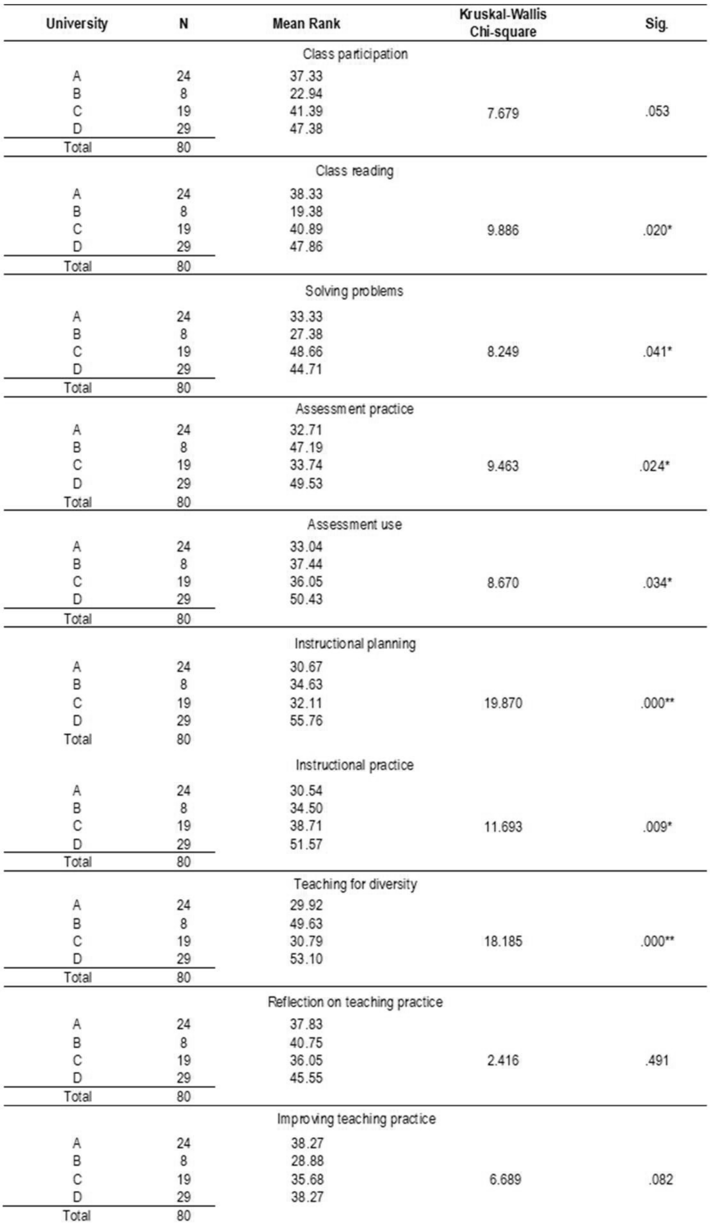
[1] Cf. ECJ 16 July 2009, Case C-5/08 (Infopaq I).

Appendix 2 Results of the Kruskal-Wallis Test of preservice teachers' perspectives on the frequency of teaching methods experienced by universities



University	N	Mean Rank	Kruskal-Wallis Chi-square	Sig.
		Class participation		
A	24	37.33		
B	8	22.94		
C	19	41.39	7.679	.053
D	29	47.38		
Total	80			
		Class reading		
A	24	38.33		
B	8	19.38		
C	19	40.89	9.886	.020*
D	29	47.86		
Total	80			
		Solving problems		
A	24	33.33		
B	8	27.38		
C	19	48.66	8.249	.041*
D	29	44.71		
Total	80			
		Assessment practice		
A	24	32.71		
B	8	47.19		
C	19	33.74	9.463	.024*
D	29	49.53		
Total	80			
		Assessment use		
A	24	33.04		
B	8	37.44		
C	19	36.05	8.670	.034*
D	29	50.43		
Total	80			
		Instructional planning		
A	24	30.67		
B	8	34.63		
C	19	32.11	19.870	.000**
D	29	55.76		
Total	80			
		Instructional practice		
A	24	30.54		
B	8	34.50		
C	19	38.71	11.693	.009*
D	29	51.57		
Total	80			
		Teaching for diversity		
A	24	29.92		
B	8	49.63		
C	19	30.79	18.185	.000**
D	29	53.10		
Total	80			
		Reflection on teaching practice		
A	24	37.83		
B	8	40.75		
C	19	36.05	2.416	.491
D	29	45.55		
Total	80			
		Improving teaching practice		
A	24	38.27		
B	8	28.88		
C	19	35.68	6.689	.082
D	29	38.27		
Total	80			

Note: * $p<0.05$, ** $p<0.001$

Appendix 3

Beliefs about the nature of mathematics: scale of rules and procedures	
MFD001A	Mathematics is a collection of rules and procedures that prescribe how to solve a problem
MFD001B	Mathematics involves the remembering and application of definitions, formulas, mathematical facts and procedures
MFD001E	When solving mathematical tasks you need to know the correct procedure else you would be lost
MFD001G	Fundamental to mathematics is its logical rigor and preciseness
MFD001K	To do mathematics requires much practice correct Application of routines, and problem solving strategies
MFD001L	Mathematics means learning, remembering and applying

Appendix 4

Beliefs about the nature of mathematics: scale of process of inquiry	
MFD001C	Mathematics involves creativity and new ideas.
MFD001D	In mathematics many things can be discovered and tried out by oneself
MFD001F	If you engage in mathematical tasks, you can discover new things (e.g., connections, rules, concepts)
MFD001H	Mathematical problems can be solved correctly in many ways
MFD001I	Many aspects of mathematics have practical relevance
MFD001J	Mathematics helps solve everyday problems and tasks

Appendix 5

Beliefs about learning mathematics: scale of teacher direction	
MFD002A	The best way to do well in mathematics is to memorize all the formulas
MFD002B	Pupils need to be taught exact procedures for solving mathematical problems
MFD002C	It doesn’t really matter if you understand a mathematical problem, if you can get the right answer
MFD002D	To be good in mathematics you must be able to solve problems quickly
MFD002E	Pupils learn mathematics best by attending to the teacher’s explanations
MFD002F	When pupils are working on mathematical problems, more emphasis should be put on getting the correct answer than on the process followed
MFD002I	Non-standard procedures should be discouraged because they can interfere with learning the correct procedure
MFD002J	Hands-on mathematics experiences aren’t worth the time and expense

Appendix 6

Beliefs about learning mathematics: scale of active learning	
MFD002G	In addition to getting a right answer in mathematics, it is important to understand why the answer is correct
MFD002H	Teachers should allow pupils to figure out their own ways to solve mathematical problems
MFD002K	Time used to investigate why a solution to a mathematical problem works is time well spent
MFD002L	Pupils can figure out a way to solve mathematical problems without a teacher's help
MFD002M	Teachers should encourage pupils to find their own solutions to mathematical problems even if they are inefficient
MFD002N	It is helpful for pupils to discuss different ways to solve particular problems

Appendix 7

Beliefs about mathematics achievement: scale of fixed ability	
MFD003A	Since older pupils can reason abstractly, the use of hands-on models and other visual aids becomes less necessary
MFD003B	To be good at mathematics you need to have a kind of "mathematical mind"
MFD003C	Mathematics is a subject in which natural ability matters a lot more than effort
MFD003D	Only the more able pupils can participate in multi-step problem solving activities
MFD003E	In general, boys tend to be naturally better at mathematics than girls
MFD003F	Mathematical ability is something that remains relatively fixed throughout a person's life
MFD003G	Some people are good at mathematics and some aren't
MFD003H	Some ethnic groups are better at mathematics than others

PUBLICATIONS

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What skills and knowledge do university mathematics teacher education programs give future teachers in Costa Rica?

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Abstract:
High-quality teaching is crucial for improving mathematics education. Teaching mathematics requires specific knowledge, including knowledge of both content and pedagogy. In this study, we analyzed the knowledge for teaching mathematics among 80 future teachers from four mathematics teacher education programs in Costa Rica. Using the Teacher Education and Development Study in Mathematics (TEDS-M) questionnaire, we studied the opportunities to learn the programs provide and the participants' performance in tests of their mathematical content and pedagogical knowledge. The results showed that all the teaching programs involved gave more emphasis to the topics covered in tertiary level mathematics than to aspects of general and mathematics pedagogy. Moreover, the results highlighted the variation among universities in the participants' performance in the tests and demonstrated that the number and content of the courses taken was not correlated with the participants' performance. These findings offer insights to the Costa Rican government and policymakers into the actual structure, variability, and characteristics of teacher education programs, which could serve as a tool for making decisions on measures to improve the quality of teaching.

Keywords: mathematics teacher education, mathematical content knowledge, mathematical pedagogical content knowledge, TEDS-M, opportunities to learn (OTL)

Introduction

In Costa Rica, the quality of mathematics teaching in primary and secondary education has been the subject of recent discussion, especially after students' poor performance in mathematics in the PISA test (see PISA 2018 results in OECD, 2019) and national tests. The report "Costa Rica: The state of public teaching policies" (Román & Lentini, 2018) and the study "The state of education" (Programa Estado de la Nación (PEN), 2019) investigated the teaching situation in Costa Rica and what was needed to improve its quality. The documents highlighted three main issues. One was the lack of control of the variation and quality of the teacher education programs. A second was poor teacher recruitment policies and ineffective measures for assessing teaching quality. Finally, the PEN (2019) study stated that more attention should be paid to teaching and education management in order to improve the education system. The study suggested, among other initiatives, the elaboration of a national framework of qualifications for education majors and the implementation of a suitability test for the recruitment of teachers. According to Schmidt (2011b), recruitment and selection are crucial for developing well-prepared and qualified teachers. Further, Tatto et al. (2012) found that strong quality assurance arrangements tend to ensure the creation and maintenance of a high-quality teaching workforce.

Hence, one of the requirements is to define the desired teaching competencies of pre-service and in-service mathematics teachers in Costa Rica. Considering a competence as a group of aptitudes and skills a person must have to master his or her job, including both cognitive abilities and beliefs, Blömeke and Kaiser (2014) affirmed that teaching competencies motivate teachers' performance in the class. Therefore, enhancing teachers' competencies is crucial to improving education. In this article, we will focus on studying future teachers' cognitive abilities.

When analyzing mathematics teacher education programs (TEPs), it is important to know what knowledge is considered necessary for the role. In 1986, Lee Shulman (1986) presented crucial ideas regarding the knowledge categories important for teaching, including subject content knowledge and subject pedagogical content knowledge. His ideas have been the basis of many studies concerning the mathematical knowledge needed for teaching (e.g., Rowland et al., 2009; Ball et al., 2008; Kilpatrick et al., 2015). However, according to Hoover et al. (2016), no "theoretically grounded, well defined, and shared conception" (p. 3) of mathematical knowledge for teaching exists. One reason is that the knowledge categories are intertwined, and it is difficult to draw a line separating one from the other. Nevertheless, the knowledge for teaching mathematics has been studied extensively, from its composition and development to the effects of teachers' knowledge on teaching and students´ learning (Hoover et al., 2016). Some studies have found that the content of the teacher education programs influences the teachers' knowledge (Schmidt et al., 2011b). Teachers' knowledge informs their ways of teaching, which in turn affects the way students learn (Hill et al., 2005). Consequently, the students' achievement is indirectly affected by the contents of the TEPs (Monk, 1994).

Despite the many studies that have been conducted on the same theme, only a few have been developed with teachers at the secondary level and only three in Latin America (Hoover et al., 2016). For instance, in the Teacher Education and Development Study in Mathematics (TEDS-M), in which 17 countries participated, only one was from Latin America, and that country ended up with the lowest performance on the test of *mathematical content knowledge* and *mathematical pedagogical content knowledge.*

Therefore, this article focuses on the problems of teaching mathematics in Costa Rica, specifically the importance of understanding the content and pedagogical knowledge of teachers and how the teaching of mathematics can be improved in order to enhance students' learning and performance. In addition, it aims to explore the lack of studies on the subject in Latin America, to describe the opportunities to learn offered by the different mathematics TEPs in Costa Rica, and to study future teachers' mathematical knowledge for teaching, using the TEDS-M questionnaire. By so doing, we hope to provide information that can be used by policymakers and university authorities to improve the educational system.

Knowledge for teaching mathematics

In his attempt to understand "the knowledge that grows in the minds of teachers," Shulman (1986, p. 9) distinguished three categories of related knowledge: *subject matter content knowledge*, *pedagogical content knowledge*, and *curricular knowledge*. According to Shulman, subject matter content knowledge concerns the amount and organization of the knowledge, involving more than just concepts and facts but requiring also a comprehension of the subject matter's structure, rules, and functioning. Pedagogical content knowledge "includes knowledge of how to represent, explain and teach the subject matter, as well as an understanding of how children learn the subject and common obstacles to this learning"(Kaarstein, 2014, p. 30). Curricular knowledge entails knowledge of the topics to teach and their organization and connections, as well as the guidelines or standards for implementing them. Moreover, it involves the books and teaching materials available for teaching the content.

Following Shulman's ideas, many researchers have set about refining the definition of the professional knowledge specific for teaching mathematics. For instance, Ball and her colleagues from the University of Michigan have developed a framework of mathematical knowledge for teaching (Ball et al., 2008), and other authors have defined knowledge for teaching using different names for the categories (e.g., Rowland et al., 2007; O'Meara, 2010). Therefore, although there is no agreement on the definitions, language, and basic concepts for the mathematical knowledge for teaching (Hoover et al., 2016), there is a consensus about the competencies for teaching mathematics. These competencies are encompassed in "(i) mathematical knowledge, (ii) pedagogical knowledge related to teaching mathematics, and (iii) general pedagogical knowledge related to instructional practices and schooling" (Schmidt et al., 2011b, p. 1266). For the TEDS-M, the authors defined a theoretical framework, which identifies three factors as quality indicators that have an impact on teacher education outcomes. They are content courses in mathematics, professional preparation for teaching mathematics, and experiences of teaching methods, and they are measured by studying opportunities to learn (OTL) (Blömeke, 2012). Moreover, they assess future teacher performance in mathematical content knowledge and mathematical pedagogical content knowledge by means of a test.

Teacher Education and Development Study in Mathematics (TEDS-M): theoretical framework

The TEDS-M is a large comparative study carried out in 2008 with the participation of 17 countries, under the auspices of the International Association for the Evaluation of Educational Achievement (IEA). It investigates "the opportunities provided and taken by future teachers while engaged in teacher preparation toward developing the competencies deemed by the literature to be relevant to quality classroom instruction" (Schmidt et al., 2011a, p. 139). For studying teacher preparation, the TEDS-M examines the participants' *opportunities to learn*, in terms of content studied and teaching methods experienced. Regarding mathematical knowledge for teaching, the TEDS-M framework entails two constructs for the test: *mathematical content knowledge* and *pedagogical content knowledge* (Tatto et al., 2008).

Opportunities to learn (OTL): The TEDS-M framework considers opportunities to learn as the content of TEPs that future teachers study. The questionnaire investigates the OTL in mathematics content, mathematical pedagogy, and general pedagogy. Courses on mathematical content have been shown to be a quality indicator of a TEP; however, they are only the foundation for mathematics teachers (Blömeke, 2012). Therefore, the framework also includes courses on professional preparation for teaching specific to mathematics and in general. In addition, it considers the teaching methods that the participants experience and the opportunities they have to plan and teach classes. These three elements clearly have an impact on the outcomes of teacher education (Blömeke, 2012).

Mathematical content knowledge (MCK): This corresponds to what Shulman (1986) refers to as subject matter content knowledge. To evaluate future teachers´ mathematical knowledge, it was important in the TEDS-M framework to define what knowledge was considered necessary for lower secondary mathematics teachers in different countries. The TEDS-M study uses the same framework of content and cognitive domain as the Trends in International Mathematics and Science Study (TIMSS) data for lower secondary teaching. The TIMSS framework, which is used for designing tasks to be taught at this level, includes four categories of content domain: number, algebra, geometry, and data (see Tatto et al., 2008, p. 36). The tasks fall within topics of mathematics that are taught in lower secondary school, but topics from higher level (upper secondary and university) are also included (see Table 1).

Table 1. Content knowledge domains in the TEDS-M test

Numbers	Geometry	Algebra	Data
Whole numbers Fractions and decimals Number sentences Patterns and relationships Integers Ratios, proportions, and percent Irrational numbers Number theory	Geometric shapes Geometric measurement Location and movement	Patterns Algebraic expressions Equations/formulas and functions Calculus and analysis Linear algebra and abstract algebra	Data organization and representation Data reading and interpretation Chance

Source: TEDS-M Conceptual Framework (Tatoo et al., 2008)

The cognitive domains also follow the TIMMS framework, covering three main components: knowing, applying, and reasoning. Tables 1 and 2present the topics and skills included in the content and cognitive domain, respectively.

Table 2. Cognitive domains in the TEDS-M test

Knowing	Applying	Reasoning
Recall, recognize, compute, retrieve, measure, classify/ order	Select, represent, model, implement, solve routine problems	Analyze, generalize, synthesize/integrate, justify, solve non-routine problems

Source: TEDS-M Conceptual Framework (Tatoo et al., 2008)

Mathematical pedagogical content knowledge (MPCK): For the TEDS-M framework, the MPCK includes all the knowledge about teaching and learning mathematics. It also includes what Shulman called "curricular knowledge", namely the order of topics and the connections between them, as well as the curricular requirements (Blömeke, 2012). Hence, this construct focuses on "the temporal dimension of teaching, moving from what mathematics to teach, to planning to teach it, to carrying out instruction" (Senk et al., 2008, p. 5). In the TEDS-M framework, the MPCK has three sub-domains: mathematics curricular knowledge, knowledge of planning mathematics, and knowledge of enacting mathematics. The descriptions of each are presented in Table 3. In the MPCK test items, the first two sub-domains are combined.

Table 3. Mathematical pedagogical content knowledge sub-domains

Mathematics curricular knowledge	Knowledge of planning for mathematics teaching and learning	Enacting mathematics for mathematics teaching and learning
-Establishing appropriate learning goals -Knowing different assessment formats -Selecting possible pathways and seeing connections within the curriculum -Identifying the key ideas in learning programs -Knowledge of mathematics curriculum	-Planning or selecting appropriate activities -Choosing assessment formats -Predicting typical students' responses, including misconceptions -Planning appropriate methods for representing mathematical ideas -Linking didactic methods and instructional designs -Identifying different approaches for solving mathematical problems -Planning mathematics lessons	-Analyzing or evaluating students' mathematical solutions or arguments -Analyzing the content of students' questions -Diagnosing typical students' responses, including misconceptions -Explaining or representing mathematical concepts or procedures -Generating fruitful questions -Responding to unexpected mathematical issues -Providing appropriate feedback

Source: TEDS-M Conceptual Framework (Tatoo et al., 2008)

In light of our interest in the opportunities to learn that have an impact on teacher outcomes and the questionnaire for investigating future teachers' mathematical content and pedagogical knowledge, this research aims to answer the following questions:

1. What are the opportunities to learn offered in Costa Rican mathematics teacher education programs?
    a) How are the OTL distributed in the knowledge areas?
2. How do trainee mathematics teachers perform in the teaching items in the area of mathematical knowledge?
    a) How do they perform in mathematical content knowledge and cognitive domains items?

b) How do they perform in mathematical pedagogical content knowledge and sub-domains items?

3. How are learning opportunities and the performance of future teachers related in the Costa Rican context?

Methods

Context

In Costa Rica, eight universities offer a major in teaching mathematics. Graduates from these majors can teach grades seven to 11 or become university teachers of basic math courses for non-mathematics majors. The teacher education programs (TEPs) in Costa Rica are delivered concurrently (Tatoo et al, 2008), meaning that they comprise mathematics courses, general education courses, and mathematics education courses in the same program. Neither the government nor the ministry of education sets any standards or stipulations for the universities to design their TEPs. Hence, each university designs its program according to what it believes teachers need to learn and for the context. Consequently, the TEPs vary between universities in content, duration, and quality (Román & Lentini, 2018). A TEP leading to a bachelor's degree at a private university takes two and a half years, whereas programs at public universities take four years for a bachelor's degree and five years for a licenciate. Recruitment in Costa Rica does not take account of these differences, the only requirement being that an applicant must have a teaching degree. The ministry of public education is the principal hiring entity, and in the hiring process the teachers are not interviewed or assessed in content knowledge or pedagogical skills (Román & Lentini, 2018). Nor is there a mechanism to filter students entering education careers. Teaching is generally considered tiring and the workload is very high. However, in some cases, teachers are willing to pay the price in exchange for the job security offered by public positions.

Sample

The subjects in this study were Costa Rican pre-service mathematics teachers. There are eight institutions in the country that prepare mathematics teachers, and all were invited to participate. Five public universities agreed to be part of the study; however, one offered only distance learning, so the data collection was not possible with that population. Hence, the sample consisted of 80 pre-service mathematics teachers from four public universities in Costa Rica. The participants were at the end of their studies, in either the fourth or the fifth year of their teaching program. The average age of the future teachers was 23.8 years (SD = 2.89), and 44 (N = 80) of them were male. The questionnaire was administered in seven groups, as described in Table 4. Participation was voluntary, and the participants were informed that their performance on the test would be not considered in their grades. The data were collected in the autumn of 2019, with a pencil-and-paper questionnaire. The students had a maximum of three hours to do the questionnaire.

Table 4. Distribution of the participants by university

University	Number of groups	Number of participants
A	2	24
B	1	8
C	2	19
D	2	29
Total	**7**	**80**

Instrument

The instrument for collecting the data was the questionnaire used in the TEDS-M study from the IEA. With permission from the IEA, the first author translated the documents into Spanish. After the translation and contextualization of the questionnaire, two mathematics education researchers and one university mathematics teacher were asked to proofread the questionnaire in order to check the language and comprehensibility of the items, as well as the suitability of the context. Improvements for the questionnaire were made based on the comments. All the collaborators were Spanish speakers and outsiders to the research. The international reliability of the TEDS-M questionnaire scales ranges from 0.78 to 0.97, and the items have been internationally tried and examined by expert panels (Tatoo et al., 2008).

The questionnaire consisted of three parts: students' background, opportunities to learn (OTL), and the test of mathematics content knowledge (MCK) and mathematical pedagogical content knowledge (MPCK). A further question was added at the end of the questionnaire, asking the students which knowledge area they thought was necessary to make their TEP more relevant. The first part of the questionnaire included questions regarding the participants' background, for instance, parents' education level and students' previous mathematics level. The second part had questions about the OTL the participants had in their teaching programs, with answers ranked on Likert scales. This section included questions about the courses they had, how they took part in them, and what skills they learned related to their profession. In the section on OTL, there were three types of questions, as described in Table 5. The OTL covered tertiary level and school mathematics topics, as well as topics of general pedagogy and mathematics education pedagogy. In addition, questions were asked about the teaching practices the participants experienced at university and about the opportunities they had to engage in activities for improving and reflecting on their practice, or learning how to deal with and value diversity in the classroom. Finally, it involved questions about the participants' practical experience in school and their perception of the program's coherence.

Table 5. Types of questions to be rated on Likert scales (statements from TEDS-M questionnaire)

Type 1	Consider the following topics in university-level mathematics. Please indicate whether you have ever studied each topic. G. Set theory (Studied, not studied)
Type 2	In the mathematics education courses that you have taken or are currently taking in your teacher preparation program, how frequently did you do/engage in activities that gave you the opportunity to do the following? L. Write mathematical proofs (Never, rarely, occasionally, often)
Type 3	To what extent do you agree or disagree with the following statements about the field experience you had in your teacher preparation program? D. I learned the same criteria or standards for good teaching in my courses and my field experience (Disagree, slightly disagree, slightly agree, agree)

The last part of the questionnaire consisted of mathematics tasks to assess the participants' mathematical content knowledge and mathematical pedagogical content knowledge. The students were presented with 13 tasks, divided into 31 items, on different mathematics topics. There were 22 items relating to MCK and nine relating to MPCK. Those tasks corresponded to the released items of the TEDS-M test (see Brese & Tatto, 2012) and concentrated on the mathematics taught at secondary level in the Costa Rican context and some topics taught at university level. The item distribution is shown in Table 6.

Table 6. Distribution of the TEDS- M released items used in the test

Content domains	Mathematical content knowledge Cognitive domains			Mathematical pedagogical content knowledge Sub-domains	
	Applying	Knowing	Reasoning	Implementing Teaching &Learning	Curriculum & planning
Numbers	-	4	4	3	-
Geometry	4	2	-	-	-
Algebra	5	-	2	1	4
Data	1	-	-	1	-

To complete the entire questionnaire, the participants were allowed one session, taking as much time they considered necessary up to a maximum of three hours. The TEDS-M international study indicates that the participants should have 90 minutes for completing the questionnaire, but, owing to sample limitations, we decided to give more time to reduce the likelihood of missing data. The questions did not vary within tests as in the original study; in this case, all the participants answered the same tasks. Therefore, as there are some differences in application and format between this study and the one carried out by the IEA, comparisons should be made with caution.

Analysis

The data were analyzed using quantitative methods, such as descriptive statistics and non-parametric statistical tests. Before the analysis, the data were cleaned and the missing data for the latent variables in the Likert scales were handled using median imputation. For the school practice scales, the cases of the participants who had not yet taken part in that practice experience were not counted. The scales with type 1 questions were analyzed by adding the studied topics and computing the percentage of studied topics in each OTL category. This means that, of the 19 tertiary-level mathematics topics presented, we calculated the percentage of studied topics the students reported. The comparison between categories and TEPs was easier to make by percentage. Type 2 and 3 group scales were computed using the mean. The tasks in part three were evaluated according to the coding provided in the TEDS-M supplement 4 (Brese & Tatoo, 2012), and the results were obtained by computing the percentage of correct answers out of the totality of the tasks, by MKC and MPCK, by knowledge area, and by cognitive domain.

Results

In this section, we present the results relating to the mathematical knowledge for teaching that is intended and achieved by future teachers in Costa Rica at the end of their TEPs. First, we describe the OTL they were exposed to; next, we present the results of the future teachers' performance on the MCK and MPCK tasks; and finally, we discuss the correlations found between the OTL and the test results.

Opportunities to learn (OTL)

Investigating the OTL to which the future teachers are exposed during their TEPs is crucial for analyzing the quality of their training, as well as for understanding their performance on the test. The participants were asked to report on their OTL in categories that would be assumed to enhance their mathematical knowledge for teaching (Tatto et al., 2012). The categories were: 1) tertiary-level mathematics, 2) school-level mathematics, 3) general pedagogy, 4) mathematics education pedagogy (academic content and teaching methods), 5) teaching diverse students, 6) reflecting and improving practice, 7) learning through school-based experience, and 8) the coherence of the TEP. The results for each category are described in the following section.

1 Opportunities to learn tertiary-level mathematics topics: In this category, the questions are type 1. In total there are 19 tertiary-level topics, divided into six knowledge areas: geometry (4), discrete structures and logic (6), continuity and functions (5), probability and statistics (2), topology (1), and theory of real and/or complex functions (1). Each statement refers to a general topic and presents examples of what could be included in that topic, enabling the participants to choose. For instance, one statement is "Linear algebra (e.g., vector spaces, matrices, dimensions, eigenvalues, eigenvectors)." The results show that the participants had studied on average 78.4% of the tertiary-level mathematics topics presented, with some variability among universities. Table 7 indicates the mean number and average percentage of tertiary-level math topics per university, showing that University C covered the largest number of courses in this area and University D the fewest. Thus, the TEPs vary and each institution has a different view of the number of mathematics topics required for mathematics teachers, considering the limited time for training.

Table 7. Mean number and average percentage of the 19 topics studied in tertiary-level mathematics, by university

University	Number of future teachers	Mean number of topics	Average % of topics
A	24	15.1	79.4
B	8	15.1	79.4
C	19	16.4	86.3
D	29	13.7	72.1
Composite results		**14.9**	**78.4**

The OTL can also be analyzed by mathematics content. Here, the results show that future teachers studied a higher number of structure- and logic-related courses (5.2 out of 6), followed by the continuity and functions area (3.9 out of 5). The areas with fewer topics were geometry (2.9 out of 4) and probability and statistics (1.9 out of 2). The focus on the areas of structure and logic and continuity and functions is consistent with the Costa Rican mathematics school curriculum for secondary level (MEP, 2012), in which algebra and relations occupy more time and topics. However, in the second topic of importance in the school curriculum, statistics and probability, future teachers studied only two topics. On the other hand, future teachers were trained in more areas than are strictly necessary for teaching in secondary schools, such as calculus, logic, differential equations, and number theory. The knowledge in these areas supports teaching in secondary schools and gives teachers the tools they need if they decide to teach in another environment, such as a university or institute, as stated in the professional profile of some TEPs.

2 Opportunities to learn school mathematics topics: The questions in this category are type 1 and ask the participants whether they have studied the seven school-level math topics. These topics are divided into two content areas, the numbers, measurement, and geometry area (3 questions) and the functions, probability, and calculus area (4 questions). The topics described as school-level are topics that the future teachers will teach in secondary school. The results for this category showed that 93% (SD = 12.7) of the given topics were studied in the TEPs. This indicates that the participants from all the TEPs studied at least six of the seven topics. The variation between universities was very small; nevertheless, University A and University C reported higher numbers (6.6 and 6.7, respectively) than University B and University D (both 6.3).

3 Opportunities to learn general pedagogy: Questions in this category explore whether the courses the participants studied had covered the following eight topics: the history of education and education systems, philosophy of education, sociology of education, educational psychology, theories of schooling, methods of educational research, assessment and measurement, and knowledge of teaching. The results presented in Table 8 show that the participants studied on average 5.8 of the eight topics related to educational pedagogy. The figures for Universities A, B, and C were very similar, but a smaller number of topics was covered at University D.

Table 8. Mean number and average percentage of the eight topics studied in general pedagogy, by university

University	Number of future teachers	Mean number of topics	Average % of topics
A	24	6.2	77
B	8	6.3	78
C	19	6.1	76
D	29	5.3	66
Composite results		**5.8**	**73**

4 Opportunities to learn mathematics education pedagogy topics: There are two areas of interest in this category: the academic content and the teaching methods experienced. Eight topics are related to academic content, namely foundations of mathematics, the context of mathematics education, development of mathematics ability and thinking, mathematics instruction, development of teaching plans, mathematics teaching, mathematics standards and curriculum, and affective issues in mathematics. The results presented in Table 9 show that the TEPs covered on average 5.6 out of the eight topics, although there was high variability among the universities. For example, while Universities A and B covered approximately four topics, University C covered 5.9 and University D covered 6.8 of the topics. This again highlights the variation among TEPs in Costa Rica.

Table 9. Mean number and average percentage of the eight topics studied in mathematics pedagogy, by university

University	Number of future teachers	Mean number of topics	Average % of topics
A	24	4.4	55
B	8	4.0	50
C	19	5.9	74
D	29	6.8	85
Composite result		**5.6**	**70**

At this stage, it is important to note that the TEP of Universities A and B is the same but is delivered on different campuses. Nevertheless, both universities showed a deficit in this area. In addition, it should be noted that, at University D, mathematics education is the only teaching major they offer; therefore, the courses are subject-focused, whereas, in the other universities, some education courses are shared with other subject majors. This could explain why University D offers significantly more topics in this area than the other universities.

The teaching methods experienced were measured with type 2 questions using Likert scale rankings for the answers. The questions asked how often the participants had the opportunity to practice or learn a specific activity. In this section, the statements included the teaching methods typically employed in university programs, such as “listen to a lecture,” and those specific to mathematics programs, such as “write mathematical proofs.” The section also covers the methods important for teaching practice in areas such as instruction and assessment and those that support the role of teacher as researcher. Figure 1 shows the average frequency with which participants reportedly experienced the activities mentioned. Overall, 63% of the participants read papers related to mathematics education occasionally or frequently and 67% had the opportunity to ask questions, participate in discussions, make presentations, or teach class sessions, all forms of active participation in their classes. Moreover, future teachers had the chance to practice their problem-solving skills occasionally (32%) or frequently (34%). These results demonstrate that the class activities were varied, and the time spent balanced between class participation, class reading and solving problems.

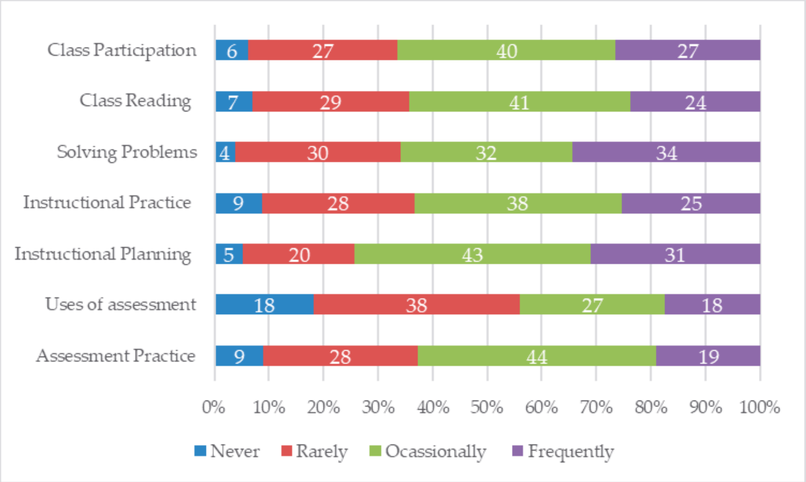

Figure 1. Frequency of experiencing teaching methods (N of students = 80).

Most of the students reported that they occasionally or frequently had the opportunity to learn or practice assessment and instruction activities, except those related to the uses of assessment, which was rarely practiced (38%). This means they did not have enough practice in using feedback to enhance their students' (and parents') learning or their own teaching. However, as shown in Figure 1, the participants reported having occasionally (44%) or frequently (19%) practiced their assessment skills in practical tasks such as devising exams and evaluating the attainment of learning goals.

Asked to what extent the future teachers took part in instruction planning activities, 74% said they had done so occasionally or more often. This suggests that the TEPs are giving the participants good opportunities to learn to plan and design classes, taking into consideration time, motivation, and learning difficulties. The practice of instruction covers the topics of integrating mathematical ideas, showing different procedures for solving tasks, and giving explanations. For this section, the participants reported having had the opportunity to practice it occasionally (38%) or frequently (25%). Experimenting with different teaching methods provided the future teachers with a strong body of resources to support good teaching practice.

5 OTL about teaching diverse students: This scale measures how frequently the students were trained in teaching students with diverse needs, notably those from racial, cultural, and linguistic minorities, those with learning or physical disabilities or behavioral problems, and the gifted and talented. Most of the future teachers (67%) were rarely or never exposed to these topics in their TEPs, suggesting that they lack the necessary knowledge for teaching such students. Only 23% of the participants reported occasionally or frequently learning strategies for teaching pupils from a minority cultural background. The TEPs in Costa Rica that cover diversity usually focus on pupils with learning and physical disabilities. In their responses to statements, the future teachers reported occasionally (43%) or frequently (39%) learning about these topics. The infrequency with which diversity issues are discussed in the TEPs is undoubtedly worrying.

6 OTL for reflecting and improving practice: In their teaching practice, teachers must have the time and tools to reflect on their practices and continuously try to find ways of improving them. This category explores how often the participants had the opportunity to learn how to do so. Among the activities covered are the development of strategies to reflect on their teaching effectiveness and professional knowledge, and to identify their learning needs. In this regard, 60% of the participants answered that they never or rarely had the opportunity to learn such strategies. Also, they were asked how often they participated in activities for learning to enhance their practice, such as "developing and testing new teaching practices." The results suggest that 55% of the future teachers never or rarely participated in such activities in their TEP.

7 OTL through school-based experience: The questionnaire also investigates the experiences the participants had of practice teaching in schools. Although all the TEPs involved in this research included school-based practice, only 73.8% (N = 80) of the participants had already done so. Asked how often the knowledge about teaching they learned at university was applied in their practice, future teachers' answers included occasionally (28.2%) and frequently (26.5%). They were also asked about the extent to which the role of supervisor in giving feedback complied with the university's goals. Most of the students agreed (74.2%) that the comments they got from their supervisor helped them to improve their teaching methods, their understanding of their pupils and the curriculum, and their mathematics content knowledge. Hence, the feedback was positive in terms of improving their teaching performance. Concerning the supervisor's reinforcement of the university's goals for practice, 78.3% of the participants affirmed that the actions and knowledge they gained during their teaching practice aligned with what they had learned during their university course.

8 OTL in a coherent program: Participants were asked about the connections between the courses they studied and if they seemed to have been organized logically and functionally, allowing the participant to learn what they needed to learn to become effective teachers, and met the expected standards. Slight agreement with the statement that the program was coherent was reported by 40% of the students (N = 80). Greater agreement was shown by the participants from University D, of whom 86% reported slight or total agreement (see Figure 2). By contrast, 48% of participants from University A disagreed or slightly disagreed with the statement about the coherence of their program.

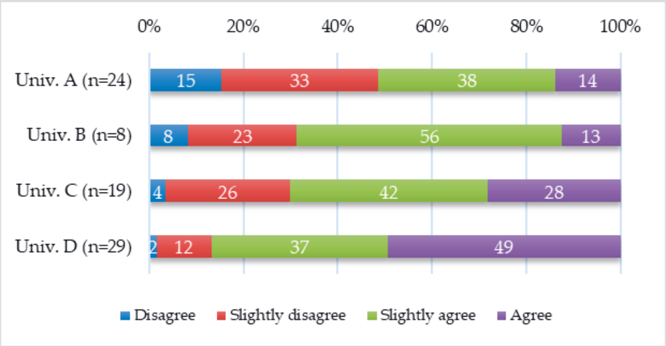

Figure 2. Participants' agreement with the statement that their program was coherent.

In order to gain more precise information about how the future teachers felt about their TEP, they were asked to select which, if any, area of their TEP needed to be supplemented with more courses. There were five options, from which they could choose more than one. Most of the participants responded that more courses were needed in mathematics education, as shown in Figure 3. A majority also considered that their TEPs should offer more opportunities for practice teaching.

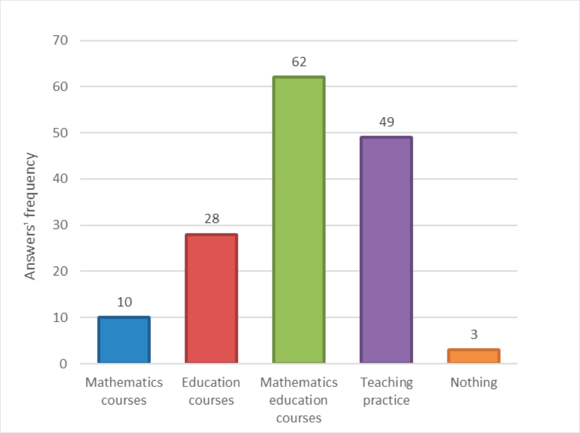

Figure 3. Courses that trainee teachers felt should be added to their TEP.

Mathematical content knowledge (MCK) and mathematical pedagogical content knowledge (MPCK) tasks
In addition to responding to the questionnaire, the participants were asked to undertake 13 mathematical tasks relating to knowledge for teaching mathematics. The tasks were grouped according to the theoretical constructs of MCK and MPCK and assessed mathematics content knowledge, cognitive domains, and skills. In total, there were 31 questions, 22 about MCK and nine about MPCK. The MCK tasks were categorized by content knowledge and cognitive domain, and the MPCK were classified by content area and teaching knowledge.

Considering the two constructs, the results showed that the participants were able to answer 66.9% of the MCK questions and 79.2% of the MPCK correctly. The results are analyzed separately in the following.

Mathematical content knowledge tasks: There were four content knowledge domains involved: numbers, geometry, algebra, and data. The task of each content knowledge domain was classified according to the cognitive domains of applying, knowing, and reasoning, as shown in Table 3. In geometry, participants answered 63.8% of the questions correctly, with a better performance in the applying (66.4%) domain than in the knowing domain (58.8%). In algebra, 67.0% of the items were answered correctly, although there were differences in performance between exercises in applying knowledge (70.3%) and reasoning (58.8%). Regarding the numbers domain, the participants answered 66.9% of the eight exercises on the knowing and reasoning domains correctly, with performances of 68.1% and 65.6%, respectively. Finally, the data exercise in the applying domain was solved correctly by 85% of the participants; however, as it consisted of only one question, the number is not representative. In all, the results demonstrate that the students' performance was better in the algebra cognitive domain, excluding the single data exercise. Moreover, they did better on the exercises in the applying domain, with 70.1% correct answers, compared with 65% in knowing and 63.3% in reasoning (see Table 10). According to the framework used in the TEDS-M study, these results suggest that trainee teachers are better at selecting, representing, modeling, implementing, and solving routine problems than in the knowledge skills of recall, recognize, compute, measure, or order and in the reasoning skills of analyze, generalize, integrate, justify, and solve non-routine problems.

Table 10. Percentage of correct answers in the mathematical content knowledge exercises

MCK	Numbers %	Geometry%	Algebra %	Data %	**General average %**
Applying	-	66.2	70.2	85	**70.1**
Knowing	68.1	58.8	-	-	**65**
Reasoning	65.6	-	58.8	-	**63.3**
General average	**66.9**	**63.8**	**67**	**85**	

Mathematical pedagogical content knowledge tasks: The exercises in MPCK assess teaching skills, as well as abilities in curriculum and planning. The participants were asked to solve questions about algebra, numbers, and data. It was found that 79.2% of the students correctly solved the numbers exercises, 76.3% the data task, and 79.8% the algebra tasks. In the teaching and learning domain, 76.8% of the participants answered correctly, and 82.2% gave valid answers to the curriculum and planning tasks. Although there was no big difference between the results, the content knowledge area of algebra received the highest number of correct answers. With regard to teaching knowledge, the participants showed better performance in the curriculum and planning domain.

Correlation between the topics studied and the test results

Considering OTL tertiary and school-level mathematics topics and topics in mathematics education pedagogy in the TEPs, as well as the results of the exercises, we ran statistical tests to determine if there was a correlation between OTL offered in the TEPs and the students' performance in the exercises.

First, we analyzed if there were performance differences among universities. A Kruskal–Wallis H test showed that there was a statistically significant difference in the distribution of the total correct answers between universities, ($\chi 2(3) = 17.079, p \leq 0.001$). Further analysis showed that the distribution of MPCK correct answers was the same across universities, but there was a significant difference in the correct answers on MCK items, ($\chi 2(3) = 17.084, p \leq 0.001$). The plot in Figure 4 presents the distribution of the percentage of correct answers for each university. In both constructs, participants from University A performed better, with half of the participants obtaining 80% or more correct answers. Participants from Universities B and D had the lowest performance in the MCK tasks, more than half of the participants answering less than 60 % of the tasks correctly.

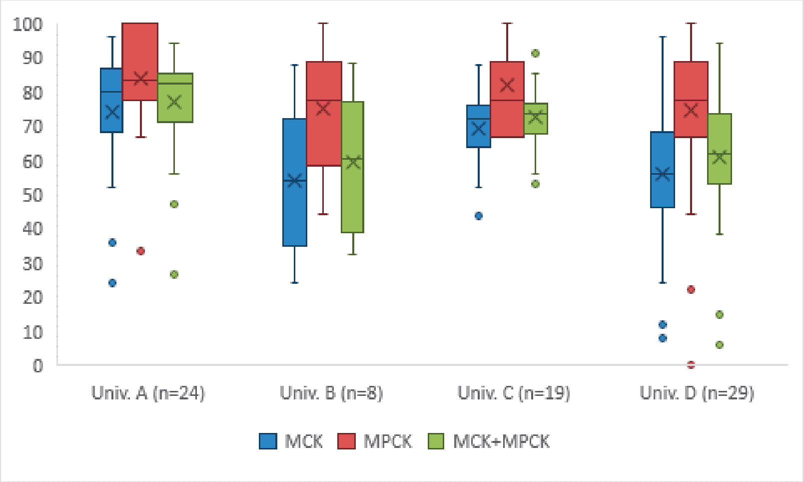

Figure 4. Future teachers´ performance on the MCK and MPCK items, by university.

Considering the results of previous studies linking performance on the test with the number of topics studied (e.g., Schmidt et al., 2011b; Qian & Youngs, 2016), we then analyzed the weight that was given to each area in the TEPs. The distribution was analyzed in the areas of general education pedagogy, mathematics, and mathematics education pedagogy. School mathematics was included in mathematics education pedagogy. In general, the TEPs in Costa Rica covered 33 of the 42 topics in the questionnaire. The mathematics topics represented 45%, mathematics education pedagogy 37%, and general education pedagogy only 18%. Table 11 shows how the topics were distributed in each area in the four universities involved. University D covered more topics related to mathematics education, but fewer related to mathematics and education pedagogy. The results for Universities A and B were consistent, given that they share the same TEP but taught at different campuses. On the other hand, University C had the highest number of courses in mathematics and offered a large number of courses in mathematics education pedagogy, general pedagogy being the area with less emphasis.

Table 11. Mean number and percentage of topics studied, by university

University	Mathematics		Mathematics education pedagogy		General education pedagogy	
	Mean	%	Mean	%	Mean	%
Univ. A	15.1	47	11	34	6.2	19
Univ. B	15.1	48	10.3	32	6.3	20
Univ. C	16.4	47	12.7	36	6.1	17
Univ. D	13.7	43	13.1	41	5.3	16
Composite results	**14.9**	**45**	**12.1**	**37**	**5.8**	**18**

From the performance in the MCK and MPCK items and the details of topics studied in each TEP, shown in Table 11, it may be seen that the performance do not seem to be associated with the extent to which the future teachers have studied topics related to MCK or MPCK. For instance, although Universities A and C have fewer topics of mathematics education pedagogy in their program than University D, the participants from the first two had a better performance in the MPCK test. Similarly, students from University A gained the best results in the MCK items, despite studying the same number of mathematics topics as University B and fewer than University C. This assumption is supported by a Spearman's analysis of correlation assessing the relationship between the numbers of MCK correct answers and the number of topics studied, which found no significant correlation between these categories. The same analysis conducted on the number of MPCK correct answers and the MPCK topics studied again found no significant correlation.

Similarly, we examined if there were correlations between the knowledge areas of the mathematical topics (geometry, discrete structures and logic, continuity and functions, and probability and statistics) and the MCK grade. Spearman's analysis of correlation found that only the probability and statistics area had a positive correlation ($rs = 0.235, N = 80, p \leq 0.05$) with the MCK grade. The other three areas did not show any significant correlation. Using the Spearman's test again, we also explored whether there was correlation between MPCK knowledge and the OTL of the teaching methods experienced; no significant correlation was found.

Discussion

As the literature confirms, opportunities to learn and the quality of TEPs have an important effect on students' performance. This study aimed to identify the structure of four Costa Rican mathematics teacher education programs, exploring the opportunities to learn in the courses and teaching experiences offered to 80 future teachers in the last year of their TEP. Using MCK and MPCK items, we studied the performance of the participants in mathematical knowledge. The OTL and the test results allowed us to investigate possible correlations and analyze the possible consequences of large variations in the quality and offer of TEPs. As an additional feature, we used the TEDS-M international results to compare Costa Rican TEPs with those designed for lower secondary and higher secondary (Groups 5 and 6; see Tatoo et al., 2012) teaching in countries rated top-achieving or A+ (i.e., Taiwan, Russian Federation, Singapore, and Poland, according to Smith et al., 2011a), and the countries closer to Costa Rican region, the United States and Chile.

The OTL were used to describe the structure of the TEPs. The courses taken in the three areas of mathematics, mathematics education pedagogy, and general education pedagogy were considered. The results showed that in the TEPs the emphasis was on tertiary-level mathematics courses, which represented 45% of all the topics studied. Of the 19 topics mentioned, Costa Rican future teachers studied 14.9. According to the TEDS-M results, the TEPs in Costa Rica thus cover more tertiary-level mathematics topics than Chile with 10.3 (Tatto et al., 2012) and the United States with 9.5, while A+ countries offer on average 17.1 (Schmidt et al., 2011a). Thus, on average, future teachers in Costa Rica have access to only two fewer mathematics courses than the top-achieving countries.

The TEPs in Costa Rica dedicate 18% of the topics to general education pedagogy, which makes it the area with the fewest topics. For this area, the programs include on average 5.8 topics, in contrast with 7 in Chile (Tatto et al., 2012), 6.7 in the US, and 6.6 in the A+ countries (Schmidt et al., 2011a). This places the TEPs in Costa Rica toward the bottom of the list of countries mentioned for number of topics in this area. On the other hand, the TEPs allocate 37% of the courses to the mathematics education pedagogy area, covering 12.1 related topics. The number of topics covered by the A+ countries is 11.8, by the US 11.3, and by Chile 9.4 (Tatto et al., 2012). Thus, in Costa Rica, the TEPs focus on the mathematics pedagogy topics more than the other countries. In conclusion, the TEPs in Costa Rica devote 45% of their courses to mathematics, 37% to mathematics pedagogy, and 18% to general pedagogy. By contrast, the two top-achieving countries, Taiwan and Russia, dedicate approximately 50%, 30%, and 20% to the three areas, respectively, the figures for the US being 40%, 30%, and 30%, respectively (Schmidt et al., 2011b). Hence, the percentages by which mathematics and general pedagogy in the Costa Rican TEPs undershoot the top-achieving countries are allocated to mathematics education pedagogy, which suggests that the future teachers have more OTL in this area than the rest.

Interestingly, when the participants were asked in which area they thought there should be more courses in their TEPs, the majority suggested mathematics education. These results align with the findings of a similar study in Finland, in which practicing teachers demanded more courses on "teaching mathematics, students' learning difficulties, and how to differentiate mathematics teaching" (Koponen et al., 2016, p. 165). The need for more mathematics pedagogy courses may respond to the historical dissociation between courses with mathematical and pedagogical content in Costa Rican universities (Alfaro et al., 2013). However, this disconnect also occurs in other contexts (e.g., Koponen et al., 2016) and risks impacting on the quality of mathematics education because it is translated into teaching practice. Another possible reason why the future teachers feel they need more support in mathematics education topics could be the quality and specificity of the courses in this area. Further research will be needed to determine the facts of this issue.

Another significant finding related to the structure of TEPs is the variability in the distribution of the topics in the three areas in each university. For instance, Universities C and D include on average two more topics in mathematics education pedagogy than the others, whereas the difference between these two universities in the mathematics topics is around three. Therefore, the training offered varies from one university to the next, and, according to Blömeke (2012), this responds to the fact that each institution has its vision of what teachers must know and do to be good teachers, as well as the organization of teacher training. Moreover, the outcomes also show variability in the distribution of the correct answers supplied by the future teachers, which, in the case of participants from Universities B and D, amounted to only 50% of the MCK items. Hence, considering the lack of national strategies to assess the quality of teachers before and during service (Román & Lentini, 2018), policymakers and the government, being the main entity for teacher recruitment, should review the variations among the TEPs to ensure that all programs provide appropriate and high-quality training. As has been mentioned, the quality of teacher education influences the quality of education that pupils receive (Hill, et al., 2005).

The analysis of the OTL related to the teaching methods experienced revealed some strengths and weaknesses of the TEPs. For instance, a positive result is that future teachers were found to participate actively in their classes, doing a variety of activities such as solving problems and reading research material on teaching mathematics. Moreover, they had a chance to practice teaching and engage in lesson planning. However, most of the participants claimed to have few opportunities to learn about the use of assessment or the skills they need to become critical teachers, for instance reflection and improvement of practice and awareness of the gaps in their knowledge. Moreover, the results show that the participants had very few opportunities to learn about teaching mathematics to diverse students. That finding coincides with the international outcomes of the TEDS-M study (Tatoo et al., 2012), in which the teachers also reported an apparent lack of knowledge in this area. This deficit and the fact that the participants appear to have few opportunities for reflecting on and enhancing their practice are matters that university authorities and policymakers should concern themselves with, as these are important competencies for teaching mathematics.

Regarding performance on the MCK and MPCK test, the participants had acceptable results, answering correctly approximately 64% of the MCK tasks and 79% of the MPCK tasks. If we compare the correct answers by item, the Costa Rican participants performed above the international average in TEDS-M in most items of both constructs (Brese & Tatoo, 2012). This finding may seem encouraging, but it does not tell us much about the participants' specific teaching strengths or weaknesses, so it must be taken with caution. Nevertheless, it can be observed that algebra was the content area in which the participants performed better, applying was the cognitive domain with more correct answers and curriculum and planning the MPCK sub-domain with better performance, although only 70% of correct answers were supplied in the first two areas. Thus, further analysis is required to examine the specific knowledge and skills of Costa Rican future teachers, in order to identify strategies for developing and improving their weaker skills during the TEPs.

It is interesting to analyze the extent to which the opportunities to learn in the TEPs and the participants' performance on the MCK and MPCK are associated. Previous research has stated that "MCK is associated with the number and content of mathematics content courses taken" (Qian & Youngs, 2016, p. 374). However, the results from the present participants showed that there was no significant correlation between the MCK test results and the number of topics on mathematics studied. The results in the test differed between universities, although they offered similar numbers of topics. For instance, Universities A and C covered 15.1 and 16.4 mathematics content topics, respectively, but the future teachers from University A performed significantly better than those from University C. Likewise, University C students' performance was similar to that of students from University D, although the latter studied fewer topics, 13.7. There was also no evidence of a correlation between the content domains studied (i.e., discrete structures and logic) and the MCK results. The same phenomenon pertained for the MPCK construct.

There is no evidence that either the number of mathematics education pedagogy courses or the OTL teaching methods were correlated with the performance of future teachers on the MPCK questions. This also runs counter to previous results, in which the number of mathematics education pedagogy courses correlated with the teachers' level in the MPCK (Qian & Youngs, 2016). These results raise a crucial question: if the number or type of content studied in the TEP courses is not related to the participants' performance in MCK and MPCK, then what determines the success or failure in these areas of teacher knowledge? Would that be explained by the quality of the courses or by the teacher educators? The answers to these questions are beyond the scope of this article. However, it can be concluded that improving TEPs requires more than just adding or removing courses.

The findings of this article suggest that more studies should be conducted to analyze the quality of the courses in the TEPs. Although there is, in Costa Rica, a national accreditation system for majors, and two of the universities involved in this study have their study plans accredited, this process does not efficiently evaluate the quality or relevance of the programs offered (Alfaro et al., 2013). In addition, the answers to the MCK and MPCK items should be studied in more detail to gain an overview of the teachers' knowledge and aptitudes for teaching mathematics. Further, research is needed to study future teachers' beliefs about the teaching and learning of mathematics and thereby acquire a full picture of the skills and aptitudes that define competence for teaching (Blömeke & Kaiser, 2014).

Attention must also be drawn to the absence of private universities from these studies and to emphasize the importance of including them. Previous studies have shown that private universities' study programs include fewer mathematics courses and the pedagogical knowledge offered is weaker than that offered by public universities (Alfaro et al., 2013). Unfortunately, such institutions showed little or no interest in participating in the present study, which made the required data unavailable for the investigation. Add to this the government's lack of hiring strategies that include assessment of the quality of teacher competencies and the increasing possibility of poorly trained teachers reaching the classroom, and the quality of mathematics education is compromised.

Acknowledgements

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II

Mathematical Beliefs Held by Costa Rican Pre-Service Teachers and Teacher Educators

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Article

Mathematical Beliefs Held by Costa Rican Pre-Service Teachers and Teacher Educators

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Abstract: Beliefs have been conceived as a hidden variable in mathematics education. It is important to know teachers' beliefs as they can inform the way that teachers teach mathematics, make decisions in the classroom, and form opinions about the abilities of students. In Costa Rica, studies about beliefs have been conducted with in-service teachers, but there is no research on pre-service teachers and the beliefs they bring to the classroom from their teacher education programs (TEPs). This research aims to describe the beliefs held by 76 pre-service teachers and 19 teacher educators from four Costa Rican public universities, using the Teacher Education and Development Study in Mathematics (TEDS-M) questionnaire. The results suggest that both pre-service teachers and teacher educators believe in a constructivist orientation focused on the learner. Both groups support the view of mathematics as a process of inquiry and active learning and agree that mathematical skills are not fixed or associated with gender or culture. In the literature, the beliefs manifested by the participants are associated with positive results regarding student outcomes and teaching practices. Therefore, policymakers should be concerned with providing environments that allow and encourage teachers to continue with these belief orientations when they start teaching.

Keywords: mathematics nature beliefs; mathematics teaching beliefs; mathematical abilities beliefs; pre-service teachers; teacher educators; TEDS-M

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1. Introduction

The mathematical performance of Costa Rican high school students has been low in national and international tests, resulting in efforts to improve mathematics teaching and learning in the country. The ministry of public education introduced a new school curriculum in 2012, with a problem-solving approach that supports a constructivist national education policy. With this, it is expected that students have a more active and independent role in the process of learning [1]. However, the study "The State of Education" [2] shows that the curriculum has not been implemented correctly and that traditional teaching practices remain dominant, involving the transmission of knowledge from teachers to learners and a teacher-centered classroom. As one of the main reasons for this issue, the study highlights the gaps in the initial training of in-service teachers that have not been remedied with the offer of professional development. In this sense, it suggests that universities should revise their teacher education programs (TEPs) and the ministry of education, as a hiring entity, should define a national framework of qualifications for the education major. These reviews should take into account, in addition to the mathematical and pedagogical contents of the TEPs, the beliefs that teachers have about mathematics, including the teaching and learning of mathematics, since both elements influence the teaching processes.

Over time, researchers in the field of teacher education have begun to consider the study of teachers' beliefs as essential because the way that teachers conceive the world can influence their instructional practice [3–6]. Beliefs about mathematics and the teaching and learning of mathematics might define how teachers interact with students in the classroom and how they perceive and develop students' skills [4,7,8]. Furthermore, the way that teachers approach the content, the methodological choices they make, and the assessment practices they use may also be affected by their beliefs [8–10]. In other words, the assumption is that "teachers' beliefs influence how they interact with students in the classroom, thus affecting the quality of their instruction and, in turn, students' learning outcomes" [7] (p. 254).

Studies have been carried out to investigate the relationships between teachers' belief orientations and student performance [7,11,12], and the coherence between these orientations and teaching practice [5,6,8,13,14]. The results have shown a positive relationship between teachers who have constructivist points of view and the performance of students, for example, in solving verbal problems [12], and a negative relationship between teachers with transmission beliefs and student performance [7]. According to Voss and colleagues [7], when teachers have a constructivist orientation, they offer more opportunities for the cognitive activation of their students; therefore, pupils experience gains in achievement. However, some studies have shown that teachers' beliefs are not always consistent with their instructional practices [8,14]. For example, Raymond's study points out the phenomenon of new teachers holding constructivist teaching beliefs but engaging in transmission teaching in practice [15]. This pattern described by Raymond is evidenced in the studies developed in Costa Rica with in-service teachers and their students. The teachers' responses to the questionnaires are inclined towards constructivist views of mathematics, in which students are expected to be active participants. Nevertheless, students express that teachers are traditional in their teaching methodologies [16,17].

Although many of these studies have been developed with in-service teachers, including the ones in Costa Rica [16,17], it is clear that studying the beliefs of pre-service teachers is of the utmost importance as their beliefs can define their future approaches in practice [9]. Studying the beliefs of mathematics teachers before they start teaching allows us to know their vision before they are "consumed by the system," a phenomenon that many in-service Costa Rican teachers experience in the course of their teaching practice. Pre-service teachers' beliefs may be influenced by their previous experiences in school, by the social context in which they are immersed, and by the teaching they experienced in their TEPs [15]. Teacher educators play a crucial role in pre-service teachers' training as they are the ones who "design and develop the structure and contents of teacher preparation, and also are those who directly execute instruction to future teachers" [18] (p. 256). In this sense, the beliefs of teacher educators manifest in the university courses can influence pre-service mathematics teachers' beliefs and practices. Therefore, knowing teacher educators' beliefs is important to attain a more informed overview of the factors that shape future teachers' beliefs.

Considering the above, this study aims to describe the beliefs expressed by the pre-service teachers and teacher educators who are part of Costa Rican TEPs, to offer policymakers and university authorities inputs that can illuminate decision-making in modifying or updating the TEPs. To achieve this aim, we used the Teacher Education and Development Study in Mathematics (TEDS-M) questionnaire. The TEDS-M study, conducted with data from 17 countries, includes research on the beliefs of pre-service teachers and those of their educators about the nature of mathematics, teaching and learning, and the abilities of students [10].

2. Theoretical Framework

2.1. Understanding the Concept of Belief

Teachers' beliefs are considered an element of cognition that affects teaching and has been used to explain the nature of teachers' instruction [5]. Defining this concept is not an easy task; teachers' beliefs have been considered "a 'messy construct' with different interpretations and meanings" [5] (p. 365) that is interchangeable with terms such as conceptions, opinions, attitudes, and knowledge. Many researchers, including Pajares in his seminal work, have attempted to provide a clear definition of beliefs and particularly to differentiate between beliefs and knowledge [4,19]. A characteristic that makes the difference between the concepts evident is the existential feature of beliefs that defines them as personal truths (subjective), while knowledge is considered an objective social construct shared by the general public [6,20]. As stated before, there is no consensus on the definition of beliefs. Some attempts to define the concept have used approaches from the fields of psychology and cognition. Therefore, some authors define beliefs as subjective mental constructs to which a person gives value and which are relatively stable [21,22]. Other explanations consider beliefs to be psychologically held understandings or premises about the world that the subject assumes true [7,23]; these understandings are shaped by cultural influences [9]. In the field of mathematics education, Schoenfeld considers beliefs "as an individual's understandings and feelings that shape the ways that the individual conceptualizes and engages in mathematical behavior" [24] (p. 358). With these notions of beliefs in mind, it makes sense that teachers' beliefs shape their practice and inform their decision-making on the knowledge that is important to teach, teaching methods, and the goals to reach in the class [4,5]. Some authors even claim that beliefs can be seen as the bridge between teacher knowledge and actual teaching [8,9,25,26].

In the field of mathematics education, the original rationale for investigating teachers' beliefs comes from the idea that beliefs can explain how mathematics is taught and learned [21]. In other words, studying the beliefs of mathematics teachers can provide "insight into the way teachers understand and carry out their job" [27] (p. 43). Beliefs in mathematics education have been categorized in different ways [28,29]. Specifically, as mentioned by Voss et al. [7], teachers' beliefs can be grouped into three levels of belief systems. One level includes the beliefs that teachers have about themselves, their role as teachers, and their teaching abilities. Beliefs about the immediate context of teaching and learning are part of another level, which includes beliefs about the teaching and learning of mathematics and of knowledge of mathematics. Finally, there is a level that includes beliefs about the policies of educational systems and the social context. The literature on mathematics education focuses on teachers' beliefs regarding the nature of mathematics and teaching and learning [5]. These elements, together with the teachers' perceptions of the students' mathematical abilities, are those contemplated in the TEDS-M study's structure of beliefs [10], which we used in this study.

2.2. Beliefs Areas of Mathematics Teachers

The TEDS-M study considered three areas of teachers' beliefs, as presented in Figure 1. These are (a) beliefs about the nature of mathematics; (b) beliefs about teaching and learning; and (c) beliefs about students' mathematical abilities [10]. These categories, in turn, are divided into Likert scales.

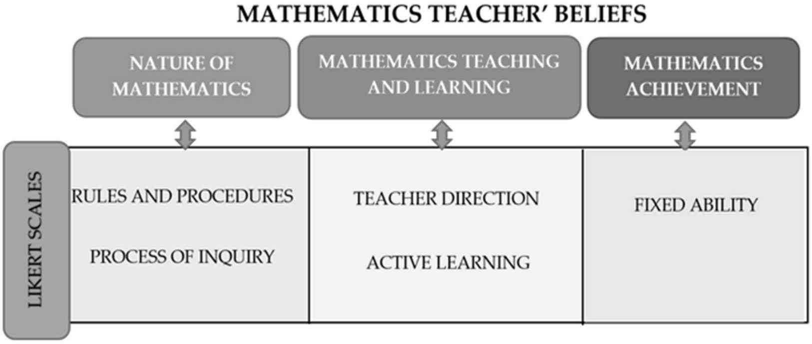

Figure 1. Teacher Education and Development Study in Mathematics (TEDS -M) study structure of beliefs.

The first area considers beliefs about the nature of mathematics that are derived from the epistemological study of beliefs; that is, it includes the nature of knowledge and the nature of knowledge. Blömeke and Kaiser [30] mention different ways in which these beliefs have been classified. One of the approaches presents three fundamental views: instrumentalist, Platonist, and problem-solving. These three views coincide, respectively, with the traditionalist, formalist, and constructivist conceptions of another of the categorizations. However, as they state, the TEDS-M study utilizes the approach developed by Grigutsch and colleagues [31], which has two fundamental beliefs regarding the nature of mathematics: the static view and the dynamic view. Considering mathematics as a static system means seeing it as an unalterable unified entity [9], as a set of rules and procedures [30]. In this regard, the framework of the TEDS-M study defined the scale of rules and procedures for investigating pre-service teachers' and teacher educators' beliefs. Agreeing on this scale will mean to conceive mathematics as a set of rules that need to be memorized and applied in the solution procedures. In contrast, conceptualizing mathematics as a dynamic process implies conceiving the subject as something that is in a constant process of change and revision, which also requires the activation of creativity to generate new knowledge or solution paths [9]. The scale designated for this view is named process of inquiry.

In the area of beliefs about teaching and learning, the literature has suggested different approaches [7]. The one that informs the TEDS-M framework developed by Peterson and colleagues. From this approach, we can extract two major categories: transmission and constructivist [11]. In the transmission view, the teacher acquires the role of possessor and transmitter of information and knowledge, while the student has the passive role of a receiver that must obey the teacher's instructions [30]. The scale of teacher direction can be associated with this category. On the other hand, the constructivist category gives the student greater responsibility in the process of knowledge and meaning construction. In this vision, the teacher must mediate in the creation of environments that promote the active participation and engagement of students in learning [7,30]. The scale of active learning is related to this category.

The third area of beliefs considered in the TEDS-M study concerns teachers' beliefs about students' abilities to learn mathematics, including topics about whether gender and culture influence the learning of mathematics. In this sense, the study only includes a scale called fixed ability, which studies whether teachers conceive the ability to learn mathematics as something stable that cannot be changed, despite efforts to improve, or as a body of skills that can be built through the learning process [18].

The general results of the TEDS-M study show that the most common pattern across countries was to endorse strongly the statements viewing mathematics as a process of inquiry that requires active student learning. The views of mathematics as a set of rules and procedures that requires the direction of a teacher to learn received less support. The view of mathematics as a fixed ability was strongly rejected by most of the participant countries [10].

2.3. *Teacher Profiles According to the Beliefs Held*

The previous description of the main categories in each of the areas of beliefs can make them seem opposite and incompatible; it could be assumed that a teacher has the beliefs of one category or another. However, as demonstrated by Voss et al. [7], "constructivist and transmissive beliefs are not two ends of a one-dimensional continuum and are not mutually exclusive categories ... they are two distinct, negatively correlated dimensions" (p. 257). Furthermore, considering belief systems as psychological constructs, it is important to bear in mind that they do not have a logical order; thus, in some cases, they may be contradictory or inconsistent [6]. Therefore, a teacher can see mathematics as a set of rules and procedures and also think of it as a process of inquiry. From this perspective, Wang and Hsieh [18] used the TEDS-M results to identify teacher profiles according to their beliefs in each area. In this way, they labeled as comprehensive the class of teachers who supported the idea of mathematics as a set of rules and procedures and, at the same time, considered mathematics to be a process of inquiry. The class of teachers who only supported the latter opinion was called the inquiry preferred profile. The teachers who endorsed both the belief that the study of mathematics requires active learning and the idea that teacher direction is still needed were categorized in the comprehensive class for the teaching and learning area. Those who supported the idea of active learning and not that of teacher direction were classified as active learning preferred. Finally, for the area of mathematical abilities, teachers who conceived mathematical abilities as fixed, as resulting from natural talent, and as differing according to gender or culture were placed in the entity-view-endorsed class. The class for the teachers disagreeing with this view was named incremental-view-endorsed.

2.4. Research Questions

As has been stated, studying the beliefs of pre-service teachers is useful to get an idea of what their teaching practice might be like, for example, regarding instruction methods or how they address the learning needs of students. Along the same lines, studying the beliefs of teacher educators can illuminate the reasons why pre-service teachers formed their beliefs in a certain way. Therefore, studies on the subject are important to update and improve TEPs. However, in Costa Rica, the issue of the mathematical beliefs of these groups has not yet been investigated. We aim to collect information about both groups of interest and investigate what they believe about the nature of mathematics, mathematics teaching and learning, and mathematical abilities. In addition, informed by the literature, we consider it important to study how the beliefs held by the participants relate to other factors, such as academic background or TEPs. Hence, we pose the following research questions:

1. What are the beliefs of Costa Rican pre-service teachers and teacher educators about the nature of mathematics, mathematics teaching and learning, and mathematical abilities?
2. What factors are related to the beliefs of Costa Rican pre-service teachers?
    (a) How are school performance and the beliefs of pre-service teachers related?
    (b) How are TEPs and the beliefs of pre-service teachers related?
    (c) How are the performance on the TEDS-M test and the beliefs of pre-service teachers related?
3. What factors are related to the beliefs of Costa Rican teacher educators?
    (a) How are academic background and the beliefs of teacher educators related?
    (b) How are years of experience and the beliefs of teacher educators related?
    (c) How are special preparation for teaching and the beliefs of teacher educators related?

Answering these research questions will meet our goal of providing a description of the beliefs of the Costa Rican pre-service teachers and teacher educators and factors that can influence them.

3. Materials and Methods

3.1. Mathematics Teacher Education in Costa Rica

To become a secondary school mathematics teacher in Costa Rica, one can major in mathematics teaching at a public or private university. With a degree in this major, it is possible to teach from grade 7 to 11 in high school, and sometimes, when needed, the basic mathematics courses at universities. Currently, there are eight universities that offer the specialty, of which four public institutions agreed to participate in this research.

In the TEPs of these public institutions, four years are necessary to obtain a bachelor's certificate and one more year for a licentiate degree. The TEPs involve tertiary mathematics, mathematics education, and general pedagogy courses. The courses are not taken in separate blocks but are distributed in such a way that students attend courses in each area every semester. The area that receives the most emphasis in the four programs in tertiary mathematics, while general education pedagogy receives the least. The general distribution of the courses in the four TEPs is similar in terms of the proportions dedicated to each area, 45% to tertiary mathematics courses, 37% to mathematics pedagogy and 18% to general pedagogy. However, there is variability between the number of topics covered by each university. For example, Universities C and D offer more mathematics education pedagogy topics than the rest, but the former offers more courses in tertiary mathematics than D [32]. In addition to these variations in their academic offerings, TEPs have differences in terms of their focus and resources. The program at Universities A and B contains the same courses; however, as they are developed in different locations, University A serves more students and has more teacher educators. On the other hand, University D has a program focused on teaching mathematics using technological environments, which is also the only one in the area of education at that institution. The other three universities have courses in their curriculum that they share with students from other education majors.

The training of mathematics teachers who participate in this study is carried out mainly by mathematicians and education professionals. Lately, teacher educators specializing in mathematics education have also joined. Nevertheless, very few of them have special training for preparing future mathematics teachers.

3.2. Participants

For the selection of the sample, a convenience sampling was used; that is, the selection criterion was based on the disposition of the institutions to collaborate, taking into account that the eight universities that train mathematics teachers were contacted, and only four participated. Likewise, the participants, both teacher trainers and teachers in training are the ones who agreed to fill out the survey. Their sample consists of two groups of participants: the pre-service teachers and the teacher educators. The first group is composed of 76 pre-service mathematics teachers from four public universities in Costa Rica who were in the last year of their TEPs. This sample represents the total of students in these conditions from these institutions. In this sample 43 (N = 76) students are male and 33 are female, with ages ranging from 20 years to 33 years (M = 23.8, SD = 2.9). Nineteen teacher educators from the four institutions answered the questionnaire that was sent to them by mail, nine of them are women, and 10 are men. They have between two and 20 years of experience in preparing mathematics teachers (M = 9.3, SD = 5.2). There are six interim professors (who are hired for one or more semesters) and 13 tenured professors (with permanent contracts). Regarding the academic level, there are six professors who hold a PhD: three in mathematics, two in education, and one in mathematics education. There are 10 instructors holding a master's degree: one in mathematics, four in education, and five in mathematics education. Finally, 13 have a licentiate degree: one in education and 12 in mathematics education. The distribution of both groups of participants by the university is shown in Table 1. The sample from University B is smaller than the others since it is a campus that serves less population. Therefore, when the results are presented by the university, they must be interpreted with caution.

Table 1. Participant distribution by university

University	Pre-Service Teachers	Teacher Educators
A	23	7
B	8	3
C	19	5
D	26	4
Total	76	19

Note: the letters A, B, C, and D are used as pseudonyms for universities.

3.3. Data Collection

For data collection, we used the TEDS-M questionnaire of the International Association for the Evaluation of Educational Achievement (IEA). With the permission of the IEA, the questionnaire was translated into Spanish by the first author. In the same way, the necessary adjustments were made so that the questionnaire was contextualized to the reality of Costa Rica. Both the translation and the contextualization were reviewed by three Costa Rican professors of mathematics education who were outsiders to the project. The full questionnaire for pre-service teachers has four parts: background information, opportunities to learn, a test to evaluate mathematical content knowledge and mathematical pedagogical content knowledge, and beliefs. The second and third parts have been analyzed in a previous study [32]; in this study, we focus on the beliefs part. In this regard, both groups of respondents had the same questionnaire. The pre-service teachers answered the questionnaire on paper, while the teacher educators completed it online for their convenience. Participation in this study was voluntary, and both groups were informed that their responses would remain anonymous.

The beliefs are investigated using Likert scales, and the questionnaire has three sections: beliefs about the nature of mathematics, beliefs about learning mathematics, and beliefs about mathematics achievement. The first section is divided into two scales, one with a statement that assesses the belief that math is a set of rules and procedures and another one that considers math as a process of inquiry. For the section on beliefs about learning mathematics, there is a scale considering whether mathematics should be learned by following the teacher's direction and another in which learning requires active involvement. The last section has one scale focused on the view of mathematics as a fixed ability. There were six response options ranging from "strongly disagree" to "strongly agree." The reliability of the Likert scales was calculated in the TEDS-M study using Cronbach's alpha coefficient, which ranged between 0.86 and 0.93, and the items have been examined by expert panels [33].

The background questions were different for pre-service teachers and teacher educators. Pre-service teachers were asked about the grades they usually received in high school, ranging from "generally below average" to "always at the top" of their year. The teacher educators were asked about their academic background in mathematics, their mathematics education, whether they received preparation for training pre-service teachers, and their teaching experience in high school and university.

3.4. Analysis

The data collected in the questionnaire were quantitative. Once all the questionnaires had been gathered, the data of the Likert scales from both groups were coded and cleaned. The original number of pre-service teachers was 80. However, four of them had high percentages of missing data, so a list-wise deletion was applied, leaving a final sample of 76. In the case of the teacher educators, for the few cases of missing data presented, a median imputation was applied. For the analysis, we used quantitative methods. First, we used descriptive statistics to show the level of agreement of the participants regarding the Likert scales. Here we applied two measures: the percentage of endorsement of the scale from each group and the average level of agreement to each statement per group. For computing the percentage of endorsement of each scale, we followed the TEDS-M methodology [10]. Hence, we considered that responses 1 to 4 ("strongly disagree" to "slightly agree") do not support the statement and responses 5 to 6 ("agree" and "strongly agree") endorse it. The proportion of answers endorsing the scales represents the groups' support for the beliefs. The average level of agreement corresponds to the mean of the respondents' answers to each statement. Second, we performed nonparametric tests to analyze correlations and data distributions. We chose nonparametric tests due to the size of the sample.

4. Results

4.1. Nature of Mathematics

The area of the nature of mathematics has two belief scales: mathematics as a set of rules and procedures and mathematics as a process of inquiry.

Regarding the first scale, the results presented in Figure 2 show that only 35.6% (n = 76) of the pre-service teachers endorse this view, which means that the majority do not see mathematics as only rules to memorize and procedures to follow. When analyzing the data by the university, University B shows 62.5% agreement and University D 42.3%. This indicates variations among the participants from the different TEPs in this regard; however, the chi-squared test of independence showed that there was no significant association between the TEP of belonging and the endorsement to this belief, $X^2(3,\ N = 76) = 6.4,\ p > 0.05$. The teacher educators also disagree with this view, with only 26.4% (n = 19) of them supporting it. In the same way, the proportion of teacher educators supporting the rules and procedures beliefs is not associated with the university where they teach, $X^2(3,\ N = 19) = 2.4,\ p\ > 0.05$. Here, it is interesting to note that none of the teacher educators from University B endorsed this view, despite the fact that the majority of their students did. This may be because the students forged those beliefs from experiences prior to their TEP, and the instruction during the program did not modify these beliefs. It could also mean that teacher educators differ in their espoused and enacted beliefs. The average level of agreement of pre-service teachers is 4.05 (SE = 0.1), and that of teacher educators is 3.56 (SE = 0.26).

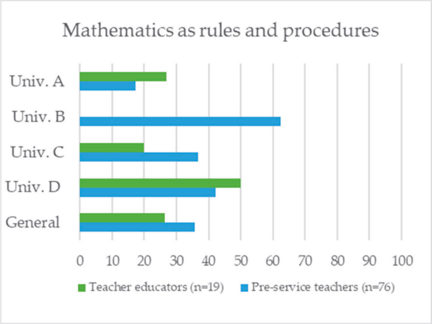

Figure 2. Percentage of endorsement of the view of mathematics as a set of rules and procedures.

The second scale studies the beliefs of the participants regarding the view of mathematics as a process of inquiry. On this topic, the endorsement was almost unanimous in both groups: 92.2% (n = 76) of the pre-service teachers and 89.5% (n = 19) of the teacher educators. The average levels of agreement are 5.23 (SE = 0.08) and 5.39 (SE = 0.15), respectively. In Universities A and C, the support was slightly higher from the students than from the educators, as observed in Figure 3. Nevertheless, as shown by the chi-squared test of independence, the view of mathematics as a process of creativity and discovery is shared in all the TEPs, being $X^2(3,\ N = 76) = 1.4,\ p > 0.05$ the results for pre-service teachers and $X^2(3,\ N = 19) = 1.4,\ p\ > 0.05$ the ones for teacher educators.

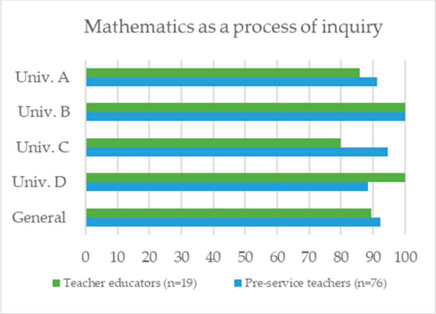

Figure 3. Percentage of endorsement of the view of mathematics as a process of inquiry.

Following Wang and Hsieh [18], the profile of both groups of participants can be defined as inquiry preferred, where the view of mathematics as a set of rules is not endorsed, but the one that considers mathematics as a process of inquiry and creativity is. Yet, if we analyze the groups by the university, the pre-service teachers from University B follow a comprehensive profile because the majority of them support both views—mathematics as rules and procedures, and as a process of inquiry.

In Figure 4, it is possible to observe the participants' responses for each Likert statement regarding the nature of mathematics. Considering four as the neutral point, we can see from the means that the respondents agreed with three statements of the rules and procedures scale: 1B, 1D, and 1E. Hence, while both groups of participants disagree with limiting the meaning of mathematics to a set of rules that must be learned, remembered, and followed to solve problems, they recognize that definitions, formulas, and strategies for solving problems are necessary to do mathematics. On this scale, the level of agreement of the teacher educators was lower than or equal to that of the pre-service teachers for all the statements, but the differences are not large.

Figure 4. Average level of agreement of the participants on the scales of the nature of mathematics. Note: statements are taken from TEDS-M questionnaire.

Regarding the process of inquiry view, as mentioned before, it is strongly supported. The statement that shows the lowest mean (4.63 for teacher educators and 4.84 for pre-service teachers) is 1H, which states that "In mathematics many things can be discovered and tried out by oneself." This suggests that there is still doubt about doing mathematics without a guide. The level of agreement of both groups shows a similar pattern; however, teacher educators seem to be a little more skeptical about discovering things in mathematics than pre-service teachers.

4.2. Teaching and Learning Mathematics

When studying beliefs, it is also important to include the thoughts of the participants about how mathematics is taught and learned. In this sense, two scales were studied, one which considers that the teaching and learning of mathematics need strong direction from the teacher and the other in which more importance is given to the active learning of students. The participants' position in this category is strong. Neither group agreed with the teacher direction view (0% endorsement). The average level of agreement of teacher educators is 1.9 (SE = 0.17), and that of pre-service teachers is 2.1 (SE = 0.75). This reflects a consensus on the idea that in mathematics teaching and learning, the teacher should not be the protagonist. Both groups endorse the idea that the approach to learning mathematics should be an active one. The support for this view is solid, with 94.7% of the teacher educators (n = 19) agreeing, with an average level of agreement of 5.52 (SE = 0.12), and 96.1% of the pre-service teachers (n = 76), with an average of 5.4 (SE = 0.56) for the level of agreement. Overall, the results show a homogeneous agreement regarding the beliefs about teaching and learning mathematics from both groups, without significant differences between the belief endorsement and the university neither by the pre-service teachers, $X^2(3,\ N = 76) = 6.0,\ p > 0.05$, nor by teacher educators, $X^2(3,\ N = 19) = 3.9,\ p > 0.05$. It seems that participants from all TEPs follow an active-learning-preferred profile [18]. This means that both teacher educators and pre-service teachers agree that students should be actively engaged in the learning process rather than being mere recipients of teacher instructions. Thus, they agree that students should engage in activities that allow them to discover and test their own strategies for solving exercises, understand why strategies work, and decide why some are better than others.

The level of agreement with each statement of the teaching and learning mathematics scale is shown in Figure 5. The beliefs of both groups follow a similar pattern. In the teacher direction scale, the statement that had more support (2.58 from teacher educators and 2.75 from pre-service teachers) was 2E, which states that pupils learn best by following teachers' explanations. This result is consistent with the one regarding the process of inquiry scale suggesting the need for a guide, instead of the idea of students discovering by themselves. In the scale of active learning, the statement that had slightly less support (5.37 from teacher educators and 4.93 from pre-service teachers) in both groups was 2 M—"Teachers should encourage pupils to find their own solutions to mathematical problems even if they are inefficient." The use of the word inefficient likely influenced the participants' decision. Both groups of respondents totally disagreed with statement 2C, which states that understanding the problem and its solutions is not as important as getting the correct answer. This result is consistent with the inquiry's preferred view on the nature of mathematics and the active learning position.

4.3. Mathematical Abilities

In this category, we analyze whether the participants perceive mathematical ability as a fixed ability that depends on natural talent and has categorical differences depending on gender or culture, or whether it is something that can grow and change. The computed scales for this category show that there is no support for the former perspective in either group: 0% of the participants agree. The average level of agreement of both groups is low, considering the 1–6 scale; for pre-service teachers, it is 2.4 (SE = 0.08) and for teacher educators 2.1 (SD = 0.16). In Wang and Hsieh's terms [18], this corresponds to an incremental-view-endorsed. Although the general disagreement with this vision is strong, the participants were less firm in disapproving of the statements 3 F ("Mathematical ability is something that remains relatively fixed throughout a person's life"), which had a mean of 2.45 from teacher educators and 3.17 from pre-service teachers, and 3G ("Some people are good at mathematics and some aren't"), which had an average level of agreement of 3.53 in both groups. In this way, the participants show that they are totally against the idea that mathematical skills are linked to cultural or gender aspects, or even that they are due solely to natural talent. They only slightly disagreed that math skills can be improved over time and that there are people who are good at math and others who are not.

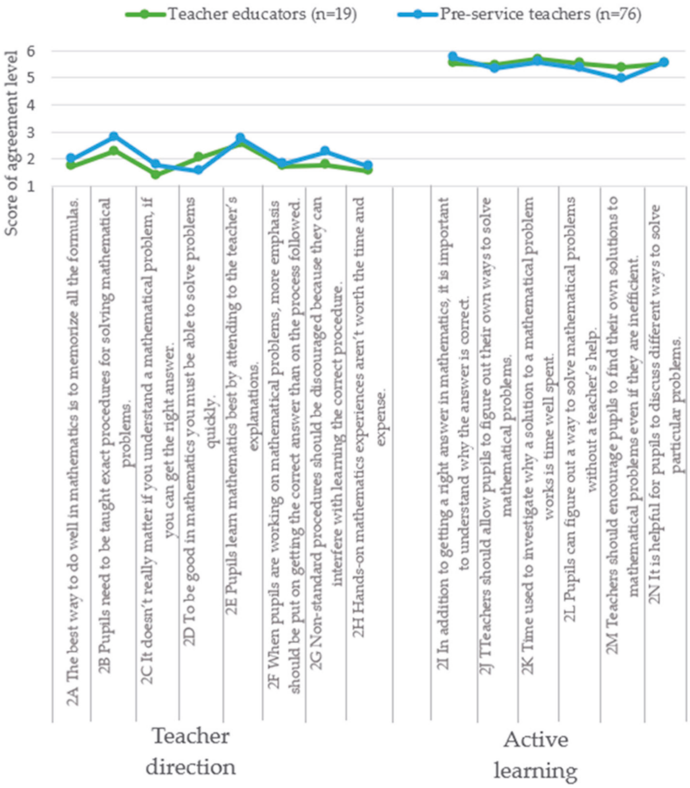

Figure 5. Average level of agreement of the participants on the scales of mathematics teaching and learning. Note: statements are taken from TEDS-M questionnaire.

In Figure 6, it is possible to observe that the level of agreement with the statements of this scale is very similar in both groups of participants. That is, teacher educators and pre-service teachers hold similar beliefs about mathematical abilities.

4.4. Relations between Pre-Service Teachers' Beliefs and Their Background and Performance in the Mathematical Knowledge Test

According to the literature, beliefs about mathematics are strongly influenced and formed by participants' experiences [34], and in the case of pre-service teachers, TEPs also have a moderate influence [15]. Therefore, we studied whether the beliefs of pre-service mathematics teachers are related to variables such as their TEPs and their high school grades. In addition, we studied whether there is a relationship between their beliefs in the different areas and their performance in the TEDS-M test about their mathematical knowledge (presented in a previous study [32]).

Regarding the association between TEPs and the participants' beliefs, we considered the variable distributions. The Kruskal–Wallis H test found that the distribution of the five beliefs scales was the same among the four universities, suggesting that the pre-service teachers' beliefs are homogeneous despite the differences between TEPs and the teacher educators in each university.

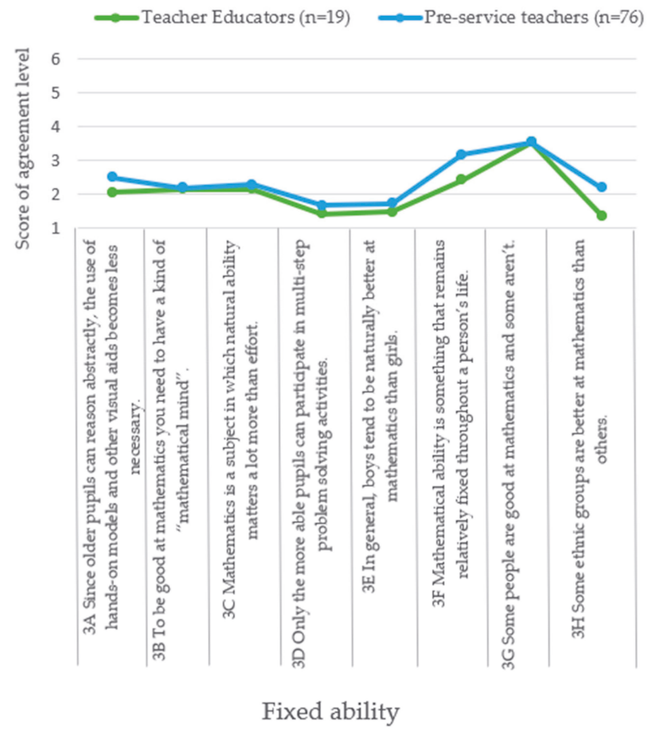

Figure 6. Average level of agreement of the participants on the scales of mathematical abilities. Note: statements are taken from TEDS-M questionnaire.

The analysis of correlations between the belief areas and the pre-service teachers' high school grades showed that there were no significant correlations between the pre-service teachers' high school grades and their beliefs about the nature of mathematics or teaching and learning. The only area that showed a correlation, according to Spearman's analysis of correlation, was the belief that math is a fixed ability. There was a small negative correlation ($r_s = 0.27$, $n = 76$, $p \leq 0.05$) between those two variables. This suggests that respondents who had higher grades in high school gave less support to the idea that mathematical abilities do not change and are just for some people. With these results, and considering their high school grades as a variable of their school experience, it is not possible to know much about the relationship between the participants' experience and the beliefs they have about mathematics. Still, we can mention the fact that having a good academic performance in high school led the pre-service teachers to see mathematical ability as something reachable for all kinds of learners.

Finally, we did Spearman's analysis to explore the relationships between the belief scales and the participants' mathematical knowledge. When considering the whole sample, we found correlations between only two variables—the test scores and the belief in mathematics as rules and procedures—which had a small negative correlation ($r_s = -0.24$, $n = 76$, $p \leq 0.05$). This means that the participants with better performance in the test show less agreement with that view of mathematics. Mathematical knowledge and the other belief scales were not associated with our data. However, when performing analyses with the pre-service teachers by a university, we observed other relations. For instance, there was a medium positive correlation ($r_s = 0.46$, $n = 23$, $p \leq 0.05$) between the mathematical knowledge of the participants from University A and the belief in math as a fixed ability. The same happened with participants from University B, but the association, in this case, was large ($r_s = -0.71$, n = 8, $p \leq 0.05$). Here, the participants with higher scores in the test tend to have a higher level of agreement with this belief; nevertheless, they reject the idea of math as a fixed ability, with a mean level of agreement of 2.4 on a scale from 1 to 6. The mathematical knowledge evidenced in the test of the participants of University B was negatively related to the beliefs scale of rules and procedures ($r_s = -0.84$, n = 8, $p \leq 0.05$), which coincides with the general result. Finally, the data from the participants of University C showed a medium positive association ($r_s = 0.41$, n = 19, $p \leq 0.05$) between mathematical knowledge and the scale of the process of inquiry. That is, the better participants performed in the test of mathematical knowledge, the more they agreed with considering mathematics as a process of inquiry. It is important to highlight that these analyses were conducted to explore the relations suggested in the literature [10], but there is no intention to draw definite conclusions.

4.5. Relations between Teacher Educators' Beliefs and Their Academic Background, Years of Experience, and Special Preparation for Teaching

Considering academic background, years of teaching, and special preparation for teaching as part of the experience of teacher educators, we examined whether those factors were associated with their beliefs. For this purpose, we ran Spearman's analysis. First, we investigated whether teacher educators' academic background was related to the way they perceive the nature of mathematics, learning, and achievement. For this variable, the teacher educators were asked for their highest qualification in mathematics, mathematics education, and education. The possible degrees were bachelor's, licentiate, master's, or doctorate. The results evidenced that participants with a higher degree in mathematics education more strongly endorsed the belief that active learning is required to learn mathematics ($r_s = 0.48$, n = 19, $p \leq 0.05$). It seems that the more preparation they have in mathematics education, the clearer the need to implement that kind of teaching. The other two academic areas were not associated with any of the beliefs.

To examine the association between years of experience and areas of belief, we conducted Spearman's analysis for years of high school teaching, years of university teaching, and years of math teacher preparation. The results showed no relationships between these variables and the beliefs. In addition, we studied whether receiving special preparation for training teachers was related to the teacher educators' beliefs, and the findings did not show a significant relationship. Finally, the outcomes of the Kruskal–Wallis H test found that the distribution of the five belief scales was the same among the teacher educators from the four universities, as with the pre-service teachers. Here it is important to keep in mind that the sample of teacher educators was small (n = 19).

5. Discussion

As reported by the literature, the study of teachers' beliefs about a subject is essential to understand their practice. The beliefs that a teacher has about the nature of mathematics, for example, will influence his way of interpreting mathematical knowledge as static or dynamic, and consequently, will also inform his way of teaching [4,30]. In this study, we investigate three areas of beliefs of 76 pre-service teachers and 19 teacher educators from four universities in Costa Rica in order to provide a description of the beliefs they manifest and how they are related to other variables. In the following, we will discuss our findings in two parts; the description of participants' beliefs orientations on one hand, and the association of those beliefs with other variables, on the other.

5.1. Descriptive Analysis of Pre-Service Teachers and Teacher Educators Beliefs

Our first research question was about the participants' beliefs regarding the nature of mathematics, teaching and learning mathematics, and mathematical abilities. In this sense, for the area of beliefs of the nature of mathematics, the outcomes showed that both groups of participants do not support the static view of mathematics as a set of formulas, definitions, and procedures that must be memorized and applied. Instead, they agreed that mathematics is a dynamic process of search and discovery, in which it is important to know and study mathematical concepts, as well as to have the teacher's guidance. Thus, both Costa Rican pre-service teachers and teacher educators are classified in the inquiry-preferred profile, which is also the one manifested by pre-service teachers and teacher educators from Germany, Switzerland, Russia, and Norway. However, pre-service teachers from countries closer to Costa Rica, such as the United States and Chile, have a preference for the comprehensive profile [18].

It is also shown in the results about the teaching and learning beliefs that both groups of research subjects completely reject the transmissive view in which learning mathematics is about memorizing formulas, repeating exact procedures, and getting the correct answer. Instead, they fully support the idea that learning mathematics requires the active participation of students, related to a constructivist conception. Based on that, participants are classified in the active-learning-preferred profile, which is also more prevalent internationally. Wang and Hsieh's study [18], which used data from 15 TEDS-M countries, shows that prospective teachers from 11 countries and teacher educators from eight countries also belong to the active-learning-preferred profile.

By combining the findings on the first two belief areas, we can highlight two points. First, despite the fact that it is theoretically possible to support the ideas of both orientations [7], the beliefs of the Costa Rican participants were maintained in only one of them. Second, it can be inferred that the beliefs of pre-service teachers and teacher educators in Costa Rica manifest the characteristics of a dynamic constructivist orientation [8] and also share the common pattern of supporting views described in the TEDS-M study [10]. Finding that Costa Rican teachers in training and teacher educators manifest this orientation of beliefs adds information to the scarce knowledge on this subject in Latin American countries, where most studies on teachers' beliefs consider teachers in service.

In the Costa Rican context, the results of this study show consistency between the beliefs of the respondents and teachers in service who declared earlier that they had constructivist beliefs [16,17]. Previous studies about the relationship between teacher beliefs and student outcomes have shown that this belief orientation is "positively related to instructional quality and student learning outcomes" [7] (p. 264), so one would expect to observe teaching practices that encourage active student participation, resulting in Costa Rican students' better performance in mathematics. However, the classroom observations described in the study "The State of Education" reveal traditional teaching methods [2], and high school students even categorize their mathematics classes as focused on learning algorithms [17]. Therefore, in-service teachers are implementing teaching strategies that are negatively related to positive outcomes [7]. These facts suggest the existence of an inconsistency between the beliefs expressed by teachers and what they do in class. To address this, it is necessary to determine the factors that make teachers modify their practice against their beliefs and the time it takes for them to be "consumed by the system". This is an important issue for future research.

In the last area of beliefs related to mathematical abilities, the participants' positions were against statements suggesting that gender or culture make a difference in learning skills. However, there was a lower level of disagreement with the statements about mathematical skills being fixed and a natural talent. This phenomenon is also observed in the international study, where the belief in mathematical skills as a natural talent obtained the highest average score (around 3.5), even in the group of countries that have an incremental-view-endorsement, such as Germany, Switzerland, Chile, Taiwan, and Singapore [18]. Attention should be paid to this result because it could incentivize supporting students who are good at math but neglecting those who do not appear to have the skills.

Overall, the beliefs of both pre-service teachers and teacher educators were quite homogeneous between universities. In other words, the evidence from this study suggests that universities, as institutions, do not influence the beliefs of teacher educators. Similarly, the results do not show a significant association between TEPs, or the teacher educators who are part of each program, and the beliefs of pre-service teachers. This fact is supported by Tatto et al. [10], who claims that "there is little conclusive evidence that beliefs can be effectively influenced by teacher preparation" (p. 153).

Nevertheless, this should not discourage the efforts to know, shape, and change the beliefs of pre-service teachers during their training process. Identifying the beliefs that pre-service teachers have from their school experience with the teaching and learning of mathematics is essential to create strategies that redirect them, if necessary, and avoid them promoting negative attitudes towards mathematics in students. Many actions can be taken in TEPs to promote the desired constructivist orientations, for this, the existing positive and negative beliefs must be made visible, and the less pronounced positive beliefs must be cultivated [7].

5.2. Variables Associated with Participant Beliefs

The second research question aimed to identify correlations between the beliefs of pre-service teachers and other variables. We found that respondents who achieved better academic results in high school agree less with the idea that math skills are fixed. This result could be explained by participants believing that if they managed to get good grades, everyone can. However, when we analyzed the performance of the participants in a test related to mathematical knowledge for teaching, the data of the subjects from two universities showed the opposite behavior, it means. This suggests that their beliefs when they think of themselves as teachers differ from their beliefs when they see themselves as students.

Furthermore, the results obtained from the exploration of the relationships between the performance of the pre-service teachers in the TEDS-M test on mathematical knowledge and the belief scales did not show the correlations presented in the findings of the TEDS-M study [10]. Results in the literature suggest positive correlations between student performance and the belief that mathematics is a process of inquiry and that learning requires active participation. Negative relationships between student outcomes and views of mathematics as a set of rules and procedures, teacher guidance being required to learn mathematics, and mathematics as a fixed skill are also mentioned [10]. However, the results with the Costa Rican pre-service teachers only evidenced a negative correlation between the results of the mathematical knowledge test and the belief that mathematics is a set of rules and procedures. These data must be interpreted with caution because the sample size is small, and there is a lack of participants from the other four TEPs.

Finally, the third question addresses the association of the teacher educators' beliefs and variables such as the TEPs, their teaching experience and their academic background. Our results showed that training in mathematics education makes a difference in beliefs about the teaching and learning of mathematics.

6. Conclusions

This project represents the first investigation to address the beliefs of pre-service mathematics teachers in Costa Rica, as well as those of their educators. Several studies have been carried out, but only with teachers in service. In this research, we found that our participants hold the belief that the nature of mathematics is dynamic, and its learning must follow a constructivist orientation. That profile is associated with the principles defined in the math curriculum of the Costa Rican Ministry of Education [1], which in turn are associated with better student results and strategies of teaching [7].

At first, this seems an encouraging result. However, there appears to be a phenomenon that does not allow novice teachers' practice to be consistent with their beliefs. This may be associated with the complex characteristics of the classroom context. This situation must be considered by education policymakers in order to provide school environments that are conducive to student-centered education, where learning can be built through discovery and investigation. This requires a manageable number of students, the necessary resources, and teachers being able to have agency in their actions. Moreover, Universities should concentrate on providing tools to pre-service mathematics teachers that allow them to overcome the factors that induce them to fall into traditional teaching methodologies, which, according to our results, do not coincide with their beliefs.

Notwithstanding the relatively limited sample, the knowledge offered by this work allows us to create an image of how the beliefs of our population are oriented. In addition to the size of the sample, there is an absence of participants from private universities. It would be valuable to complement our result with the information of the beliefs of the teachers in the training of these universities, since it is known that the programs differ in duration and contents studied, with respect to the TEPs of public universities [35].

Further research should be undertaken to investigate what happens to the beliefs of pre-service teachers in the transition from their TEPs to the classroom. It would also be interesting to perform a longitudinal study to analyze whether, how, and by which factors pre-service teachers' beliefs are modified during their TEPs. In doing so, the professed beliefs and those demonstrated in practice must be considered. This information will provide insights about which practices help to shape one or the other belief orientation.

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Costa Rican Preservice Mathematics Teachers' Readiness to Teach

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ABSTRACT

Mathematics teachers' knowledge for teaching mathematics has been broadly studied in recent years, and many theoretical frameworks and instruments have been created to measure and improve knowledge and competencies to teach mathematics. The knowledge gained from these studies has been crucial in understanding and determining what mathematics teachers should learn. In Costa Rica, there is a lack of regulations regarding the training that mathematics teachers receive and the knowledge and competencies they acquire in the different teacher education programs. This study investigates the knowledge for teaching mathematics of Costa Rican preservice teachers using the Teacher Education and Development Study in Mathematics (TEDS-M) instrument to identify strengths and weaknesses in their training. A mixed-method analysis of the responses of 79 participants revealed that they were well prepared for cognitive application skills but showed weaknesses in the development of reasoning skills. Additionally, the solutions highlighted significant deficiencies in participants' monitoring of their own work and in the ability to provide feedback on students' work. We hope that our findings could inform universities and policy makers to improve the quality of teacher education programs.

Keywords: knowledge for teaching mathematics, professional competences, preservice teachers, MCK, MPCK, TEDS-M

INTRODUCTION

The quality of mathematics teaching offered in Costa Rican secondary schools has been questioned in recent years (Programa Estado de la Nación [PEN], 2019). One of the reasons for this is the poor performance demonstrated by students in national and international tests. However, the latest reports (PEN, 2019; Román & Lentini, 2018) have pointed out major weaknesses in Costa Rican teacher policies ranging from teacher training, particularly the lack of qualification frameworks and quality standards, to their hiring and performance assessment. In Costa Rica, there are eight public and private universities that offer programs for becoming a mathematics teacher, which vary in content, duration, and quality. Nevertheless, as stated by Román and Lentini (2018), "there is no reference framework that guides teacher training programs around minimum skills and common goals" (p. 22), and there is little control over the implications of the differences in the quality of teacher preparation.

Most studies conducted in Costa Rica with mathematics teachers have focused on in-service teachers. A very representative one is the diagnosis conducted by the Ministry of Public Education (MEP) in 2010 with the participation of 1,733 in-service mathematics teachers, which revealed worrying results (MEP, 2011). In this study, the in-service mathematics teachers from public institutions solved multiple choice items about topics studied in the secondary school curriculum; the items were part of the national test for students in last high school year. The results revealed that 43.4% (N=1,733) of the participants performed below the average, evidencing heterogeneous results and differences in mathematical knowledge. Moreover, the mathematics teachers who graduated from public universities obtained a better score than the ones from private institutions, suggesting differences in the teacher education programs (TEPs). Other studies (e.g., Alfaro et al., 2013; Chaves, 2003) have revealed deficiencies in TEPs, such as the lack of connections between mathematics and education courses. Thus, it is necessary to investigate the contents, quality of training, and effectiveness of different mathematics TEPs in Costa Rica, which are reflected in the knowledge for teaching mathematics that preservice teachers have at the end of their university training. This need is consistent with the research interest in the mathematics education community regarding the content and acquisition of knowledge a teacher must have to teach mathematics (Carrillo, 2011).

Many studies have been researching the professional knowledge or competencies necessary for teaching mathematics (Blömeke & Delaney, 2012; Kaiser et al., 2017). Studies have focused on identifying and distinguishing categories of mathematics knowledge with the intention of finding ways to develop it (Ball et al., 2008; Carrillo et al., 2018) or to measure it (e.g., Tatoo et al., 2008). Some studies for measuring mathematics teachers' professional knowledge include the German Cognitive Activation in the

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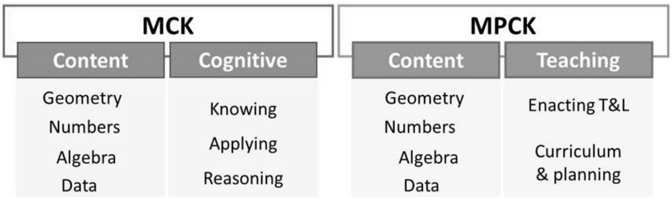


Figure 1. Subdomains of mathematical content knowledge and mathematical pedagogical content knowledge facets in the TEDS-M framework (Tatoo et al., 2008)

Classroom (COACTIV) project (Kunter et al., 2013), the Learning Mathematics for Teaching Project (LMT) at the University of Michigan (Hill et al., 2008), and the international comparative study of the International Association for the Evaluation of Educational Achievement (IEA) Teacher Education and Development Study in Mathematics (TEDS-M). Each study has its own theoretical framework, but all of them have been built on using the categories defined by Shulman (1986) in his seminal work on teacher content knowledge (Kaarstein, 2014). The three studies mentioned above also differ in context, type of participants (COACTIV and LMT with in-service teachers and TEDS-M with preservice teachers), expert members, and project goals. However, all of them have offered insights into the categories of content knowledge needed for teaching mathematics. As Kaarstein (2014) stated in her comparison study, all three frameworks coincide in including "(i) knowledge about content and student; (ii) knowledge about content and teaching/instruction; (iii) knowledge about planning for teaching the content; and (…) (iv) curricular knowledge" (p. 40). The frameworks and results from these studies provide a good starting point for investigating mathematics teacher's knowledge.

In this study, we used the TEDS-M instrument, since it assesses the preparation of secondary school mathematics teachers in training, as well as the level and depth of teaching knowledge related to mathematics and the mathematical knowledge that they achieve at the end of their TEP (Tatoo et al., 2008). We aimed to describe future Costa Rican secondary school teachers' knowledge for teaching mathematics, fill the knowledge gap on this subject, and determine the strengths and weaknesses of the existing TEPs. This information may be useful for policy makers to modify their teacher policies and for universities to update their TEPs, with the ultimate goal of improving the quality of mathematics education in the country.

MATHEMATICS TEACHERS' PROFESSIONAL KNOWLEDGE AND COMPETENCES

The study of the knowledge and competencies necessary for teaching mathematics, as well as the means and tools to develop them, is very complex; thus, there is no single way to define or operationalize them. As Hoover et al. (2016) stated, it does not exist a "theoretically grounded, well defined, and shared conception" (p. 3) of mathematical knowledge for teaching. However, many frameworks developed on this issue take into consideration the categories of content knowledge for teaching defined by Shulman (1986). According to Shulman (1986), content knowledge for teaching is one of the domains in which teacher knowledge can be divided; others could be, for instance, knowledge of "individual differences among students, of generic methods of classroom organization and management, of the history and philosophy of education, and of school finance and administration" (p. 10). Shulman (1986) proposed dividing content knowledge for teaching into three categories: content knowledge, pedagogical content knowledge, and curricular knowledge. Frameworks such as the Mathematics Knowledge for Teaching developed by Ball et al. (2008), the Professional Knowledge of Secondary School Mathematics Teachers (Baumert & Kunter, 2013), the knowledge for teaching mathematics (Tatoo et al., 2008), and more recently the Mathematics Teacher's Specialised Knowledge model by Carrillo et al. (2018) have their grounds on Shulman's (1986) categories.

The knowledge for teaching mathematics (Tatoo et al., 2008) is the framework of the TEDS-M. The study aims to survey the effectiveness of TEPs around the world, and for that the expert team developed a framework inspired by the teacher education standards of the participant countries and previous works (e.g., Ball et al., 2008; Schmidt et al., 2007). On the basis of the teaching knowledge categories of Shulman (1986), the TEDS-M study includes the facets of mathematical content knowledge (MCK), mathematical pedagogical content knowledge (MPCK), and curricular knowledge, which are divided into different subdomains.

The MCK has the contents subdomains of algebra, numbers, data, and geometry and the cognitive subdomains of *knowing*, *applying*, and *reasoning* (**Figure 1**), that were defined following the Trends in International Mathematics and Science Study (TIMMS) conceptualization (Tatoo et al., 2008). The *knowing* subdomain includes the skills of lower order thinking level (Thompson, 2008) such as recalling definitions and properties, carry out algorithmic procedures, recognize mathematical objects, and classify them according to their properties. In the *applying* subdomain, participants are expected to apply the knowledge from the knowing subdomain (Hsieh et al., 2014), which includes the skills to select appropriate solution strategies or methods to solve routine problems and use different representations of mathematical objects depending on the context. The most demanding tasks are under the *reasoning* subdomain, which requires the participants to analyze situations and provide justifications for given statements or their own work, make generalizations and solving nonroutine problems are also part of this subdomain (Tatoo et al., 2008).

The framework of the MPCK was built under the assumption that teacher competencies are linked to classroom situations, and therefore to decide which features of teaching mathematics were fundamental, the TEDS-M team designed the MPCK problems based on the standards of national teacher training programs (Blömeke & Delaney, 2012). According to Kaiser et al.

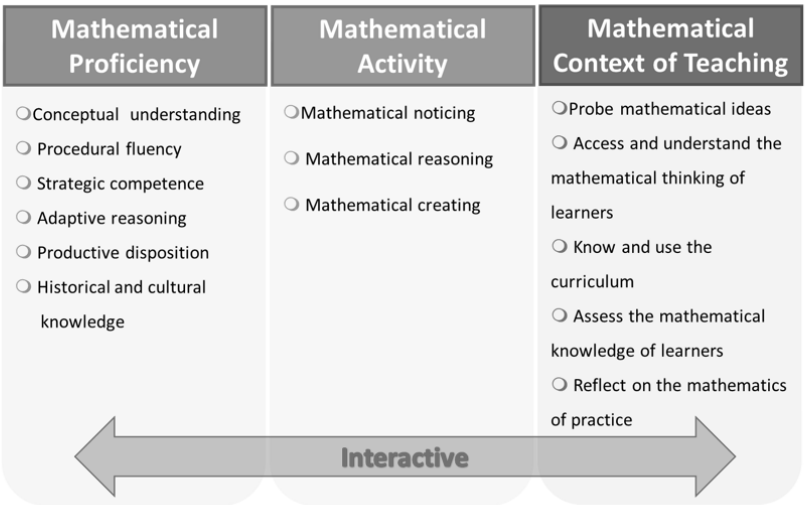


Figure 2. Framework of mathematical understanding for secondary teaching (Kilpatrick et al., 2015)

(2017) reaching a consensus about the MPCK that meets the context differences was a big challenge and therefore there are less items on this facet than in MCK. In the TEDS-M framework, the MPCK and the curricular knowledge facets were merged, and three subdomains were defined (**Figure 1**). In the first subdomain, the framework includes the elements of mathematical curricular knowledge, such as setting appropriate learning goals, knowledge of different assessment formats, and comprehensive knowledge of the curriculum. This subdomain was combined with planning knowledge for the teaching and learning of mathematics, a subdomain related to the “preactive” actions that teachers must perform before teaching, including planning classroom activities, predicting the typical responses and misconceptions of students, and linking teaching methods and instructional design. The third subdomain, enacting mathematics for teaching and learning, encompasses the interactive elements of the teacher’s role. For example, it includes the teachers’ analysis and evaluation of students’ solutions to mathematical exercises, the process of explaining the mathematical concepts and procedures, their ability to generate fruitful questions and the provision of feedback (Tatoo et al., 2008).

Similar to other frameworks (e.g., Ball et al., 2008; Carrillo et al., 2018), the TEDS-M framework attempts to separate the MCK and MPCK categories. However, these categories are not mutually exclusive (Döhrmann et al., 2012, Kaarstein, 2014), and MPCK generally requires MCK (Potari & da Ponte, 2017). Kilpatrick et al. (2015) consider that mathematics teaching cannot be conceived as a matter of knowing the mathematics and knowing how to teach; instead, it is a more complex process. Considering this, they developed the Mathematical Understanding for Secondary Teaching (MUST) framework.

The MUST framework was developed out of the analysis of classroom episodes of prospective and practicing secondary teachers as well as teacher educators at tertiary level. Kilpatrick et al. (2015) acknowledge that the mathematical understanding required for teaching mathematics in secondary schools is different from that required in other professions. Notably, this framework conceptualizes mathematical understanding instead of mathematical knowledge making it more flexible given that

> “knowledge may be seen as static and something that cannot be directly observed, whereas understanding can be viewed as the use of the knowledge one has (…) Also, because of its nature, a teacher’s understanding grows and deepens on the course of his or her career” (Kilpatrick et al., 2015, p. 10).

The MUST framework presents elements of mathematical understanding useful for secondary teachers from three perspectives: *mathematical proficiency*, *mathematical activity,* and *mathematical context of teaching* (**Figure 2**). These three perspectives allow observing different aspects and characteristics of a specific phenomenon, which means that they are interactive. According to Kilpatrick et al. (2015) the mathematical understanding of secondary teachers can be characterized by understanding the overall mathematics capacities relevant for teaching, competencies to enact the actions typical to the teaching job, and settings in which they will use their mathematical capacities and practice those actions.

The mathematical proficiency perspective is multifaceted and includes aspects of mathematical knowledge and skills required by mathematics teachers, such as those defined for students’ mathematical proficiency in earlier studies (Kilpatrick et al., 2001). However, the perspectives of the MUST framework demonstrate the in-depth knowledge that teachers must have of high school mathematics, but also of mathematics learned before and those to be learned later. To foster students’ mathematical proficiency, teachers need to comprehensively understand this. **Table 1** lists some skills corresponding to the aspects of the mathematical proficiency perspective relevant for this study.

The aspect of *productive disposition* refers to noticing the importance of mathematical activity and being able to recognize it and use it in situations outside the classroom. On the other hand, having historical and cultural knowledge of mathematics

Table 1. Aspects of mathematical proficiency considered in this study (original source: Kilpatrick et al., 2015)

Conceptual understanding (Knowing why)	-Understand and use mathematical concepts in various contexts -Monitor own's and students' work -Understand, identify, and use connections in mathematics -Formulate proofs -Remember and reconstruct methods
Procedural fluency (Knowing how and when)	- Quickly recall and accurately execute procedures and algorithms
Strategic competence (Knowing heuristics)	-Select strategies for solving problems -Have a flexible approach -Generate, evaluate, and implement problem-solving strategies -Know various solution strategies
Adaptive reasoning	- Provide valid explanations and justifications

improves the conceptual understanding of teachers in such a way that they can be aware of the epistemology of mathematical ideas and design the best way to teach about them.

Kilpatrick et al. (2015) conceive the perspective of the mathematical activity as the actions that are developed with the mathematical objects: notice, reason, and create. Within these three actions, they found specific elements. Mathematical noticing involves recognizing the structures of mathematical systems, symbols, and arguments, as well as noticing the connections within and outside of mathematics. Mathematical reasoning includes the skills of justifying and demonstrating, conjecturing, generalizing, restricting, and expanding; the teacher should always keep in mind the teaching activity and design classes using these actions such that students have a better understanding. Finally, the mathematical creation aspect includes the skills to represent, define, and manipulate mathematical objects in the most appropriate way according to the learning situation (Kilpatrick et al., 2015). As future teachers were not asked to demonstrate these skills in the TEDS-M test, they were not analyzed in the solutions.

The third perspective, the mathematical context of teaching, is about bringing to action the knowledge of the mathematical proficiency perspective and the skills of the mathematical activity into the classroom, focused on helping the students develop their mathematical understanding. The aspects of this perspective are exploring mathematical ideas, accessing and understanding the students' mathematical thinking using appropriate questions or analyzing their discourse, knowing and using the curriculum to plan the classes, assessing the mathematical knowledge of learners, determining their level of understanding, and reflecting on the mathematics in one's practice (Kilpatrick et al., 2015). All these aspects are directly related to the teaching activity in the classroom; thus, in the questionnaire responses, we could identify only one of them.

Using the knowledge for teaching mathematics and the MUST frameworks, we analyzed future Costa Rican teachers' responses to the TEDS-M questionnaire. The first framework provided a general overview of the participants' cognitive and teaching-related skills from the MCK and MPCK perspectives. On the other hand, the MUST framework allowed us to gain insights into the mathematical understandings and professional competences evidenced by the participants, providing the opportunity to identify the weaknesses and strengths in their TEPs.

Research Questions

This study evaluated the knowledge for teaching mathematics of future Costa Rican secondary teachers, considering their cognitive and teaching-related skills as well as their mathematical understanding for teaching. To do this, we used the MUST framework in addition to the TEDS-M theoretical framework to provide a more detailed and in-depth analysis. With this work, we hoped to answer the following questions:

1. What is the performance shown by Costa Rican preservice mathematics teachers in the TEDS-M questionnaire?
    a. What is their performance in the *knowing*, *applying*, and *reasoning* subdomains?
    b. What is their performance in the enacting and curriculum and planning skills?
2. How are the mathematical understanding competences shown in Costa Rican preservice mathematics teachers' responses to the TEDS-M questionnaire?

METHODOLOGY

The present study includes qualitative and quantitative research methods; thus, it follows the characteristics of a mixed-methods research. In this research method the qualitative strand complements the weaknesses of the quantitative part and vice versa, leading to a "best data explanation and best understanding for the studied research phenomena" (Maarouf, 2019, p. 3). Considering that, we chose the mixed-method research because it allows us to answer the research questions and approach the research problem from a more complete point of view.

Research Context

Secondary math teachers in Costa Rica are trained in programs that include content in mathematics, pedagogy, and mathematics pedagogy. Depending on the university, a teacher education program leading to a bachelor's degree can take from two and a half years in private institutions to four years in public institutions. Thus, the programs between universities vary in duration, but also in the contents covered and the quality of instruction (Alfaro et al., 2013). Differences in TEPs are not monitored

Table 2. Participant distribution by university

University	Number of groups	Number of participants
A	2	24
B	1	8
C	2	19
D	2	28
Total	**7**	**79**

Table 3. Distribution of the released TEDS-M items used in the questionnaire

Content subdomains	MCK cognitive subdomains			MPCK teaching competencies	
	Knowing	Applying	Reasoning	Enacting teaching& learning	Curriculum & planning
Algebra	-	5	2	1	4
Numbers	4	-	4	3	-
Geometry	2	4	-		-
Data	-	1	-	1	-

during or after teacher training. The main hiring entity, which is the Costa Rican Ministry of Education, does not conduct interviews or evaluations of content knowledge or pedagogical skills for future teachers (Román & Lentini, 2018). In this context, it is challenging to have a picture of teachers' knowledge and teaching skills when they finish the programs and go to the classrooms to teach.

Participants and Data Collection

The sample of this study is formed by 79 participants who were preservice mathematics teachers enrolled in a TEPs in Costa Rica. We considered only the preservice teachers in their last year of the bachelor or licentiate degree. Most participants were men (44 of 79), and the average age was about 24 years. The preservice teachers were informed that participation in the study was voluntary, that their performance on the questionnaire would not affect their grades and that the data would be treated confidentially. For selecting the sample, the eight Costa Rican universities, public and private, with active mathematics TEPs were invited to participate. However, our sample was reduced to the preservice teachers of only four TEPs, due to lack of interest in participating or logistical aspects. There are two TEPs from the same university, nevertheless, as they are from different campuses, we considered them to be two universities. This is because, although the study program was the same, the teacher educators, learning opportunities, and number of students were different. Therefore, we counted four universities—coded as A, B, C, and D. The questionnaire was administered to seven groups, and the distribution of the participants by university can be seen in **Table 2**. The role of the researcher in data collection, was the one of instrument implementer. The instrument was completed on paper and pencil, and the participants had 3 hours to complete the four parts. In this article, we only discuss Part 3. The data were coded according to TEDS-M guidelines (Brese & Tatto, 2012).

Instrument

The questionnaire used for data collection presents 13 exercises that are subdivided into 31 items. The items correspond to those released by the TEDS-M study (Brese & Tatto, 2012), except item MFC703 which was excluded due to problems about the intelligibility of the task. The items are classified in the domains of MCK and MPCK, and in the subdomains: content (*algebra*, *numbers*, *geometry*, and *data*), cognitive (*knowing*, *applying,* and *reasoning*) and teaching-related skills (*enacting*, and *curriculum, and planning*). There were three types of formats for items: constructed response seven items, multiple choice two items, and complex multiple choice 22 items (**Table 3**).

The items were translated from English to Spanish by the author, and both the translation and the contextualization, were validated by three Costa Rican mathematics educators not related to the study. IEA granted the respective permissions to use the items in this study, which already were validated by international experts. We did not intend to duplicate TEDS-M study; there were many differences between that and our study—for instance, the time for solving the test—that must be considered before comparing with TEDS-M data.

Data Analysis

We performed quantitative and qualitative analysis of the preservice teachers' responses to the TEDS-M items. For quantitative analysis, responses were coded using the TEDS-M keys and scoring guide (Brese & Tatto, 2012), and statistical tests were performed. In addition, using the participants' grade in all the items, that is, the MCK and MPCK, we divided them into quartiles. Q1 has the participants with scores below 58.8, the ones with grades from 58.8 and below 73.53 were in Q2. Q3 is composed by the preservice teachers with scores between 73.53, included, and 79.41. Finally, Q4 has all the participants with grades equal or over 79.41. The qualitative section was made using direct content analysis (Hsieh & Shannon, 2005)—that is, an analysis guided by the theory of the MUST framework as a lens to study the solutions of the participants. For the content analysis, we read the solutions of the 13 exercises in the 79 questionnaires and identified the following: solution strategies, content knowledge and procedural difficulties, as well as other characteristic of importance, such as the additional drawings and annotations made in the tasks that were not of complex response. Next, following the definitions and skills associated with each perspective of the MUST, the annotations were related to the relevant aspects. Thereafter, in each aspect, the annotations were analyzed and merged according to their characteristics. Consequently, four aspects for the mathematical proficiency perspective and one for the mathematical context perspective were evident from the solutions

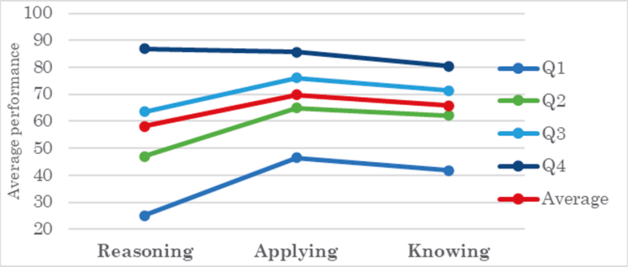


Figure 3. Preservice teachers' (N=79) performance patterns in cognitive subdomains by quartiles

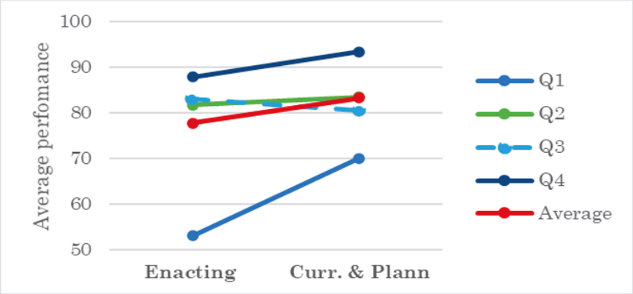


Figure 4. Preservice teachers' (N=79) performance patterns in teaching-related skills subdomains by quartiles

RESULTS

First, we determined the quantitative results of the cognitive subdomains and the teaching-related skills, using descriptive statistics and statistical tests, with which we intend to answer research questions 1a and 1b. For the second research question, we wrote the results of the qualitative analysis of the participants' solutions according to the categories of the MUST framework. Considering the structure of the questionnaire some aspects of mathematical understanding of secondary education cannot be observed, such as the aspects related to the mathematical activity perspective.

Preservice Teachers' Performance on Cognitive and Teaching-Related Skills Subdomains

As detailed in **Table 3**, there are 22 MCK items in the questionnaire across *reasoning* (n=6), *applying* (n=10), and *knowing* (n=6) categories, according to the TEDS-M categorization. For the nine MPCK teaching-related skills items, there are 5 that correspond to the *enacting* category and 4 to the *curriculum and planning* one. Considering the average performance in the questionnaire, including MCK and MPCK parts, we divided the participants into quartiles, as explained before, to observe the patterns by subdomain, with the aim to gain insights into the Costa Rican preservice teachers' strengths and weaknesses in the cognitive areas. In addition, significant differences in the performance were calculated by university.

Overall, the participants had a better performance in the *applying* items, followed by the *knowing* ones (**Figure 3**). The cognitive domain with lower rates was *reasoning*. Considering the competences assessed by each cognitive domain, as defined in the theoretical framework, it makes sense that *reasoning* tasks were more demanding that the *applying* ones, especially because they required to deal with nonroutine problems and writing proofs. However, following the same logic, the *knowing* items must be easier to solve than the *applying* ones, but this was not the case with our participants.

The general pattern of performance is repeated in all quartiles except the fourth, where the participants performed slightly better in the *reasoning* subdomain than in the *applying* subdomain, which suggests that they acquired the skills in the higher cognitive levels. On the other hand, the difference between the *reasoning* and the other two subdomains, in quartiles one and two is large, implies little engagement of these groups in the more demanding tasks.

The Wilcoxon signed-rank test revealed that the results of the *applying* items were significantly higher than the *reasoning* ones, ($Z=-3.45$, $p<0.05$). Similarly, the outcomes of the *knowing* items were significantly higher than the *reasoning* ($Z=-2.4$, $p<0.05$). The difference between *applying* and *knowing* was not statistically significant. The performance in *reasoning* subdomain was significantly lower than that in the other two.

Regarding the MPCK subdomains the participants performed better in the *curriculum and planning* area. This pattern, as observed in **Figure 4**, is the same in the all the quartiles except in Q3, where the *enacting* subdomain is slightly higher. In Q1, the results show a difference of 17 points between the variables, a greater gap than in the remaining quartiles. However, the Wilcoxon signed-rank test revealed that the differences between the *enacting* and *curriculum and planning* areas were not statistically significant. Notably, the MPCK competences are difficult to assess in a paper and pencil questionnaire, because that knowledge is

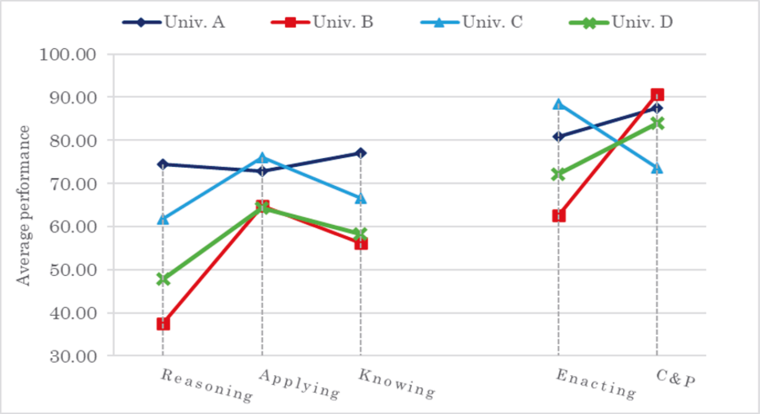


Figure 5. Preservice teachers' (N=79) average performance patterns in cognitive and teaching-related skills subdomains by universities

directly related with the practice of teaching. Moreover, as mentioned by Döhrmann et al. (2012), the TEDS-M study had a stronger focus on reporting about MCK facets than about the MPCK ones; therefore, there are fewer items measuring the latter. Nevertheless, from an overall perspective, the participants performed better in the *curriculum and planning* items that assessed if they could identify the previous knowledge required to teach certain topic; than in the ones in where they were asked to think about secondary students' possible difficulties, assessing solutions or understanding the reasons for mistakes.

To identify whether the pattern was the same across all TEPs, we performed a statistical test to study whether there were statistically significant differences between performance in subdomains and universities. A Kruskal–Wallis H test revealed that the distribution of the *applying*, and *curriculum and planning* subdomains between universities were not significantly different. Nevertheless, the same test revealed that the results distributions of the *knowing* subdomain ($\chi^2(3, N=79)=9.093, p\leq0.05$), the *reasoning* subdomain ($\chi^2(3, N=79)=17.242, p\leq0.001$), and the *enacting* subdomain ($\chi^2(3, N=79)=9.821, p\leq0.05$), were significantly different among universities. In **Figure 5** it is possible to observe it in the average performance of the cognitive and teaching-related skills subdomains, by university.

Regarding the cognitive subdomains, the average performance by universities follows the same pattern as the performance by quartiles (**Figure 5**), with *applying* the best category, followed by *knowing* and *reasoning*. Only Univ. A presented a different pattern, with better results in *knowing*, next in *reasoning* and then in *applying*, holding the best performance of all universities in the first two subdomains. The Wilcoxon signed-rank test revealed that the only significant differences between subdomains within the same university are among the *applying* and *reasoning* subdomains in universities B($Z=-2.1, p<0.05$) and D ($Z=-3.27, p<0.001$). With those results one can infer that the mathematics TEPs in Costa Rica have a focus on *applying* competencies and need to work more on reaching higher cognitive levels as *reasoning*.

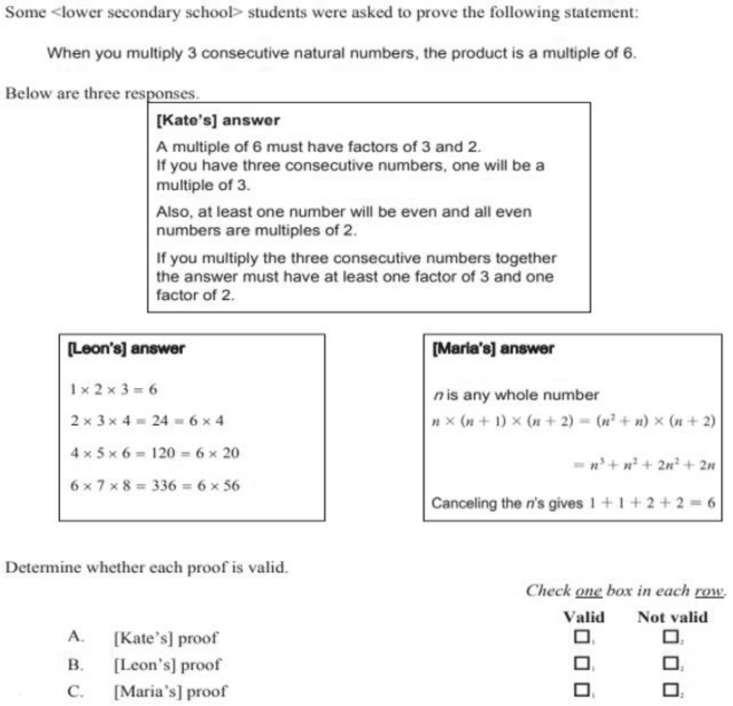
Considering the teaching-related skill subdomains, the pattern is also the same as the one in the analysis by quartiles, better results in the *curriculum and planning* subdomain than in the *enacting* one, except for Univ. C. The Wilcoxon Signed-rank test proved that the participants from Univ. C performed significantly better in the *enacting* subdomain than in the *curriculum and planning* one ($Z=-2.02, p<0.05$). The same test revealed that in the case of Univ. B, the *curriculum and planning* performance was significantly better than the *enacting* subdomain ($Z=-2.384, p<0.05$).

The results in this section have revealed that the Costa Rican preservice teachers are good at solving items in the *applying* cognitive subdomain, and that they are better at the *curriculum and planning* tasks that in the enacting ones. At the same time, it has indicated that the participants have a weakness in the *reasoning* items. Moreover, the outcomes indicate the fact that there are significant differences in the performance between TEPs, specifically the *reasoning*, the *knowing* and the *enacting* subdomain. These can be interpreted as differences in the quality of the programs offered in Costa Rica.

The following section will allow us to perform a more detailed analysis of the respondents' understandings through the qualitative analysis of their answers using the MUST framework.

Analysis of Preservice Teachers' Solutions Using MUST Framework

As described in section "mathematics teachers' professional knowledge and competences," the framework mathematical understanding for secondary teachers has three perspectives. We will analyze the preservice teachers' responses trying to identify the aspects of each perspective and providing a description on how the aspects are present. First, we will comment on the mathematical understanding perspective with the aspects conceptual understanding, procedural fluency, adaptive reasoning, and strategic competence. Then we will discuss one aspect of the mathematical context for teaching perspective namely assess the mathematical knowledge of learners.



Some <lower secondary school> students were asked to prove the following statement:

When you multiply 3 consecutive natural numbers, the product is a multiple of 6.

Below are three responses.

[Kate's] answer

A multiple of 6 must have factors of 3 and 2.
If you have three consecutive numbers, one will be a multiple of 3.

Also, at least one number will be even and all even numbers are multiples of 2.

If you multiply the three consecutive numbers together the answer must have at least one factor of 3 and one factor of 2.

[Leon's] answer

$1 \times 2 \times 3 = 6$

$2 \times 3 \times 4 = 24 = 6 \times 4$

$4 \times 5 \times 6 = 120 = 6 \times 20$

$6 \times 7 \times 8 = 336 = 6 \times 56$

[Maria's] answer

n is any whole number

$n \times (n+1) \times (n+2) = (n^2+n) \times (n+2)$

$= n^3 + n^2 + 2n^2 + 2n$

Canceling the n's gives $1 + 1 + 2 + 2 = 6$

Determine whether each proof is valid.

Check one box in each row.

		Valid	Not valid
A.	[Kate's] proof	☐	☐
B.	[Leon's] proof	☐	☐
C.	[Maria's] proof	☐	☐

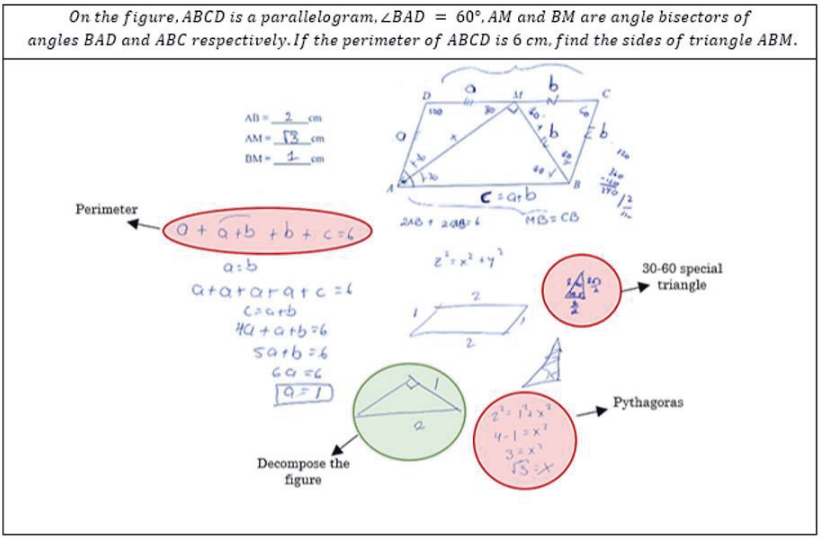
Figure 6. TEDS-M item 709 (Brese & Tatoo, 2012)

Conceptual understanding

The aspect of conceptual understanding is the most evident in the solutions of the participants, four skills were tracked from this category. They will be discussed in the following paragraphs.

Understand and use mathematical concepts in various contexts: The skill of understand and use mathematical concepts in various contexts in a proper way, was observed in different situations. First, in the use of mathematical properties and definitions. In Exercise 704, a geometry task that requires to determine the lengths of the segments of a parallelogram, the participants used the parallelogram properties to investigate the lengths of the rhomboid's segments and for measuring the angles. Similarly, they needed to recur to the definition of an irrational number to decide if $\pi, \sqrt{2}$ and 22 divided by 7 belong to that set off numbers, in Exercise 610. For that, the preservice teachers demonstrated that they were clear about the first two numbers, however, the decision if "the result to divide 22 by 7" was irrational all the time was not easy for them. When reviewing their annotations in the questionnaire, we observed that they did the division but did not notice that there were repeating decimals, nor did they remember that, by definition, all the numbers $^{a}/_{b}$ $(a, b \in \mathbb{Z}, b \neq 0)$ could be known directly as rational numbers. Other example of this case was evident when the participants used as part of the proof of Exercise 814 the "zero-absorption property." The exercise consists of proving whether it is true that when operating two 4×4 matrices using an operation that multiplies input by input, the result is zero, at least one of the matrices must necessarily be the null matrix. Thus, the idea of using the zero-absorption property was correct, but it should be used in the multiplications of the inputs and not for the matrix operation, as many participants did.

Another way to demonstrate this skill was showing the fully understanding of a situation explaining it with their own words or being able to use general representations when doing operations. For the former case, in Exercise 804, one participant explained his choice for how many possible ways there are to choose 2 and 8 students out of 10 writing "it is the same to choose the 2 that stay or the 2 that leave" (P78), showing a deep understanding of the situation. For the latter, in Exercise 711, a proof about adding functions was included. Here, the participants needed to use the explicit form of a linear function to make computations even though the proof could be performed without them because the property must be true for all the linear functions with the given characteristics. In this sense the participants exhibited poor understanding of the situation and concepts in the task as well as poor abstraction skills. The last observed aspect in this skill is the understanding showed when assessing several solutions attempts to the same task, as in Exercises 802 and 709. In item 802, the participants exhibited their understanding of the concepts when writing the statement "If the square of any natural number is divided by 3, then the remainder is only 0 or 1," in a symbolic language easier to compute such as $\frac{n^2}{3}$ (P3, P8, P22, P25, P47, and P52) or $x^2 = \begin{cases} 3m \\ 3n+1 \end{cases}$ (P78). Similarly, Exercise 709 (**Figure 6**) tests their understanding by asking them to point out the specific reason why one of the given solutions attempts did not work. For instance, P12 pointed out that Leon's option "showed that it is true for the first n numbers; it means, conjecture," thus concluding that the attempt is invalid. For the same case, P25 wrote "there is no test for all; it is missing" justifying why the test



On the figure, ABCD is a parallelogram, ∠BAD = 60°, AM and BM are angle bisectors of angles BAD and ABC respectively. If the perimeter of ABCD is 6 cm, find the sides of triangle ABM.

Figure 7. TEDS-M Exercise 704 with an example of participants' solutions strategies (P23)

was incorrect. For reaching these conclusions, the participants must have a clear understanding of proves were the condition have to be true "for all" cases.

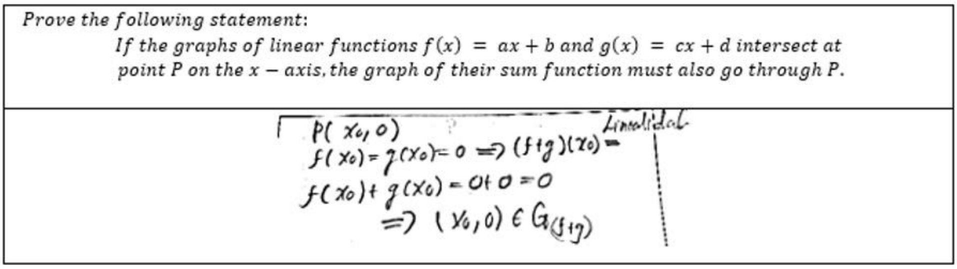
Monitor own's and student's work: This skill has two directions and requires the preservice teacher to practice the task of reviewing the correctness of the procedures, the most convenient word choice and that the results make sense in the context. Similarly, they must be able to monitor students' solutions. From the analysis of participants' solutions, it was possible to observe that they were strict on the items that required reviewing the possible solutions of the students to the exercises, identifying, and pointing out why the procedures were not valid, as we already mentioned P12 and P25 did. Nevertheless, when it comes to monitoring their own work, there is evidence of various flaws. For example, in giving values to the interior angles of a parallelogram that did not add up to 360 (P15). It is also evident in Exercise 604, a word problem solved by a system of equations, when the students obtain values for the variables that exceed the total indicated in the statement, but they did not notice it (P12 and P68).

There was also an issue with the words they chose to refer to mathematics objects. In Exercise 806, where they must describe why the students committed a mistake inferring information from a histogram, P75 wrote that they were thinking that "each graphic represented a country," when the correct was to say, "each bar represented a country." On the other hand, P63 and P69 attributed the error to the students answering that there were seven countries represented because it was the number of countries in Central America. This statement is erroneous for two reasons: first, this is not the number of countries in Central America, and second, because the title of the graph indicated that the sample was from Central America and South America. There are preservice teachers who are not paying attention to the way they carry out their procedures or how they express themselves about mathematical objects or give answers without checking their justifications. When it is considered that students learn from what teachers do and how they do it, these failures in teachers' monitoring of their own performance are not favorable for student learning.

Understand, identify, and use connections in mathematics: Considering the way mathematics is built, as well as understanding, verbalizing, identifying, and using connections between concepts and variables, are crucial skills for the teachers. Concerning this skill, we observed that participants were capable of posing and using connections in many cases, as in Exercise 704 (**Figure 7**), where many participants (i.e., P17, P19, P36, and P48) connected the results obtained from different formulas (perimeter, Pythagoras, trigonometrical identities) to find the solution to the problem. Nevertheless, in other cases, even though they could pose different and many equations from a statement, they became confused and were not able to connect them to reach the goal (i.e., P58, P72, P77, and P80). It seems that they were just writing equations without a clear purpose for them.

The connections are also made when linking the same mathematical object or situation in different languages. Here there were examples of associations between the functions and the situations they model, as in Exercise 710, where the participants conclude from the statement "the height h of a ball t seconds after it is thrown into the air" that it can be modeled by a quadratic function and not with an exponential function. Another example of that type of connection is in Exercise 610 when they must evaluate whether "the diagonal of a square with side of length 1" and "the result of dividing the circumference of a circle by its diameter" are always an irrational number or not. The participants made connections between the definition of an irrational number and the geometric formulas that allowed them to identify the number that each situation represented. In this case, many participants (i.e., P3, P12, P22, P39, and P47) made annotations of the formula for the circumference of the circle and drawings of a square with its diagonal and property of the 45-45 special triangle.

On the other hand, the participants exhibited flaws in making connections between natural or word language and symbolic language. In this sense, we observed that some preservice teachers interpreted the statement "Peter has 6 times as many marbles as David" in Exercise 604 A1, as P=6+D instead of P=6×D (i.e., P13, P49, and P70). This type of mistake is common in secondary



Prove the following statement:

If the graphs of linear functions $f(x) = ax + b$ and $g(x) = cx + d$ intersect at point P on the $x-$axis, the graph of their sum function must also go through P.

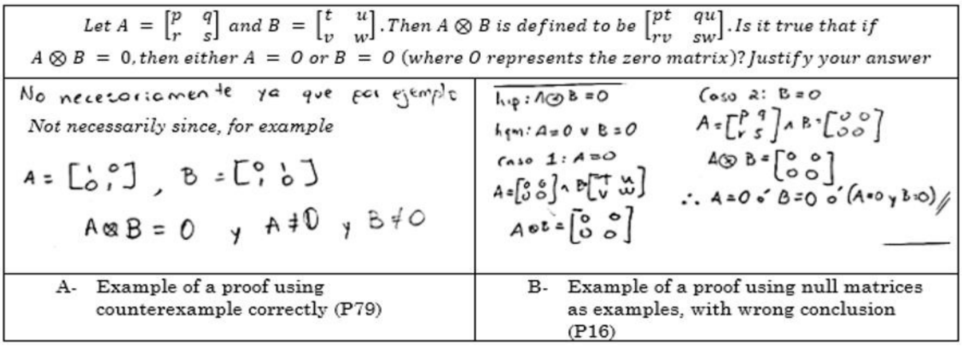
$P(x_0, 0)$ Linealidad

$f(x_0) = g(x_0) = 0 \Rightarrow (f+g)(x_0) =$

$f(x_0) + g(x_0) = 0 + 0 = 0$

$\Rightarrow (x_0, 0) \in G_{(f+g)}$

Figure 8. TEDS-M Exercise 711, example of a proof using the hypothesis efficiently (P80)



Let $A = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$ *and* $B = \begin{bmatrix} t & u \\ v & w \end{bmatrix}$. *Then* $A \otimes B$ *is defined to be* $\begin{bmatrix} pt & qu \\ rv & sw \end{bmatrix}$. *Is it true that if* $A \otimes B = 0$, *then either* $A = 0$ *or* $B = 0$ *(where 0 represents the zero matrix)? Justify your answer*

No necesariamente ya que por ejemplo

Not necessarily since, for example

$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$A \otimes B = 0$ y $A \neq 0$ y $B \neq 0$

A- Example of a proof using counterexample correctly (P79)

hip: $A \otimes B = 0$

htm: $A = 0 \vee B = 0$

Caso 1: $A = 0$

$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \wedge B = \begin{bmatrix} t & u \\ v & w \end{bmatrix}$

$A \otimes B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Caso 2: $B = 0$

$A = \begin{bmatrix} p & q \\ r & s \end{bmatrix} \wedge B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$A \otimes B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\therefore A = 0$ ó $B = 0$ ó $(A = 0$ y $B = 0)$ //

B- Example of a proof using null matrices as examples, with wrong conclusion (P16)

Figure 9. TEDS-M Exercise 814, participants' solutions examples

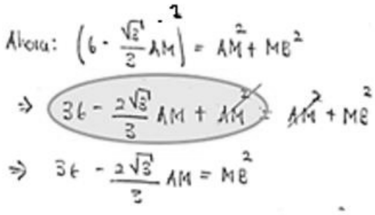
students; therefore, teachers are expected to not only able to avoid them but also implement strategies, so their students have a better understanding and do not make them.

Formulate proofs: The skill of formulating proofs requires a very well understanding of the mathematical knowledge, the connections between results, and the logic behind the proof. The questionnaire had two exercises that required the participants to formulate proofs and two where they had to decide if some attempts of proofs were valid or not. However, some participants (P36, P49, and P64) used the structure of a proof for organizing their thoughts in the Exercise 704 of the side lengths of the parallelogram. In Exercise 711, they had to prove that if adding two linear functions that intersect at a point P on the x-axis, the graph of the sum function would also pass through point P. The solutions to this exercise showed that the participants had difficulty using the hypothesis in a useful and efficient manner. Most participants who failed to provide a valid proof used the fact that f(p)=0 and g(p)=0 to equalize the functions (f(p)=g(p)), but the efficient strategy was to add them since the aim was to know if the point belonged to the sum function (f(p)+g(p) = 0). This caused their proofs to be long and difficult to follow; in some cases, the proofs did not even reach the goal (**Figure 8**). In the same exercise, three participants (P13, P21, and P63) tried to reason by contradiction using specific functions without verifying that their examples did not meet the conditions given in the statement; thus, the test attempt was invalid.

Another mistake regarding the proof strategies was noticed in Exercise 814, when proving that the only way that the result of operating two 4×4 matrices, with an operation that multiplies input by input, would be zero, is that one of the matrices is the zero matrix. The statement is false and requires the use of a counterexample to prove it, as P79 did (**Figure 9A**). However, as presented in **Figure 9B**, the reasoning of preservice teachers P7, P16, P26, P29, P49, and P71 was that as the statement was valid when using a zero matrix, then it must be true. Thus, they are not considering all the cases and are overgeneralizing. These examples highlight that some participants are not proficient in the ability to formulate proofs, and this is without considering the mathematical rigor but only the way the participants thought.

Procedural fluency

Procedural fluency is the most practiced aspect by students in schools, and it is related to making procedures correctly, flexibly, and efficiently. The aspects of this skill analyzed in future teachers' responses consist of quickly recalling and accurately executing procedures and algorithms. In different exercises, participants exhibited their proficiency recalling formulas and relations such as the perimeter and Pythagoras formula, the special triangle 30–60 relation, and the law of sines, as well as posing a system of equations using given relations in a written problem. Proficiency requires not only recalling the formulas but also using them in a correct context and without calculation mistakes. However, some participants made computation errors very similar to those made by high school students. For example, P22 failed to sum the algebraic expression "x+x+2x+2x=6⇔8x=6", and P55 made a mistake solving the special product of square of a difference (**Figure 10**). As mentioned, these errors may relate to the poor monitoring of their work.



Ahora: $\left(6 - \frac{\sqrt{3}}{2} AM\right)^2 = AM^2 + MB^2$

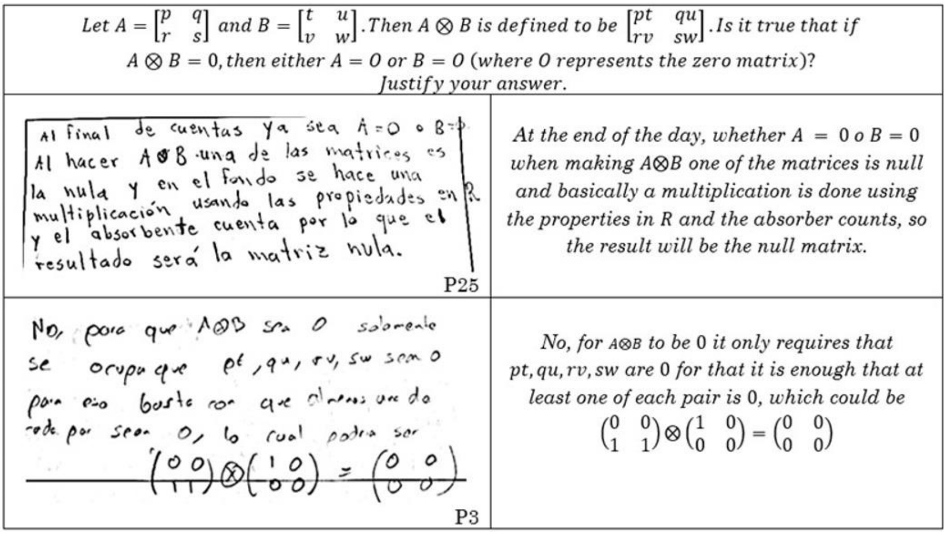
$\Rightarrow 36 - \frac{2\sqrt{3}}{3} AM + \cancel{AM^2} = \cancel{AM^2} + MB^2$

$\Rightarrow 36 - \frac{2\sqrt{3}}{3} AM = MB^2$

Figure 10. Example of participants' procedural fluency error (P55)

Table 4. Examples of participants' solutions to Exercise 604B

because the students have to "use the same variable for two different characteristics" P33
because "a variable x can be related at the same time to two other different situations" P47
because it "involves 2 times the same variable" P49
because it "refers to how much one has with respect to two people" P79



Let $A = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$ and $B = \begin{bmatrix} t & u \\ v & w \end{bmatrix}$. Then $A \otimes B$ is defined to be $\begin{bmatrix} pt & qu \\ rv & sw \end{bmatrix}$. Is it true that if $A \otimes B = 0$, then either $A = 0$ or $B = 0$ (where 0 represents the zero matrix)? Justify your answer.

Al final de cuentas ya sea A=0 o B=0 Al hacer A⊗B una de las matrices es la nula y en el fondo se hace una multiplicación usando las propiedades en R y el absorbente cuenta por lo que el resultado será la matriz nula. P25	At the end of the day, whether $A = 0$ o $B = 0$ when making $A \otimes B$ one of the matrices is null and basically a multiplication is done using the properties in R and the absorber counts, so the result will be the null matrix.
No, para que A⊗B sea 0 solamente se ocupa que pt, qu, rv, sw sean 0 para eso basta con que almenos uno de cada par sea 0, lo cual podría ser $\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ P3	No, for $A \otimes B$ to be 0 it only requires that pt, qu, rv, sw are 0 for that it is enough that at least one of each pair is 0, which could be $\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Figure 11. TEDS-M Exercise 814, participants' solutions examples

Adaptive reasoning

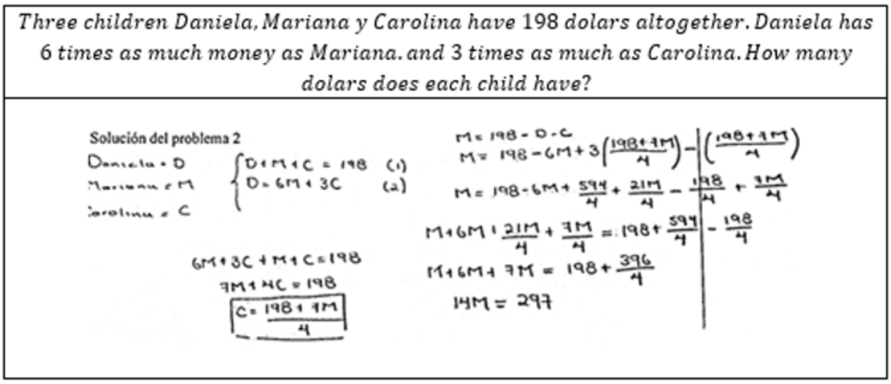
This skill is about providing explanations and justifications of mathematical decisions, either solving or evaluating the solution of a problem, and it is a very important part of mathematical thinking. The participants' solutions made it possible to observe this facet from two moments: when they had the role of teachers who had to explain the reasoning of students and when they were the ones who had to offer justifications for their procedures. The first situation occurred in two exercises. In Exercise 604 B, the preservice teachers had to explain the reasons why a word problem is more difficult for students than another. Although 73.4% (n=79) of the respondents provided valid reasons, such as the use of fractions or that making calculations with rational numbers is difficult for students, some of their explanations were not easy to understand (see examples in **Table 4**).

Other example of the skill of evaluating students' solutions, was observed when the preservice teachers made annotations such as "it is necessary to prove it for all [numbers]" P25 when referring to a proof that only tested for few numbers. Considering that the participants would be teachers who will have to provide feedback to students and parents and understand and diagnose the possible difficulties their students might face in an exercise, the word choice and the depth of their explanations must improve.

On the other hand, when they had to provide explanations of their own work, there were cases when they were not successful and resulted in proofs composed of equations without connections and other ones when they had it so clear that they could write it mostly in natural language, as the examples in **Figure 11** show, although the answer is not correct or complete.

Strategic competence

The strategic competence skill is related to the heuristics and their implementations for solving problems. It includes the skills of selecting strategies for solving problems, having a flexible approach, generating, evaluating, and implementing solving problem strategies, and knowing various solutions approaches. These four skills were the ones observed in the preservice teachers' work.



Three children Daniela, Mariana y Carolina have 198 dolars altogether. Daniela has 6 times as much money as Mariana. and 3 times as much as Carolina. How many dolars does each child have?

Solución del problema 2

Daniela = D
Mariana = M
Carolina = C

$$\begin{cases} D + M + C = 198 & (1) \\ D = 6M + 3C & (2) \end{cases}$$

$$6M + 3C + M + C = 198$$
$$7M + 4C = 198$$
$$\boxed{C = \frac{198 + 7M}{4}}$$

$$M = 198 - D - C$$
$$M = 198 - 6M + 3\left(\frac{198+7M}{4}\right) - \left(\frac{198+7M}{4}\right)$$
$$M = 198 - 6M + \frac{594}{4} + \frac{21M}{4} - \frac{198}{4} + \frac{7M}{4}$$
$$M + 6M + \frac{21M}{4} + \frac{7M}{4} = 198 + \frac{594}{4} - \frac{198}{4}$$
$$M + 6M + 7M = 198 + \frac{396}{4}$$
$$14M = 297$$

Figure 12. TEDS-M Exercise 604 A2 (contextualized), example of participants solution strategies (P1)

The ability to select strategies was shown, for example, in the parallelogram exercise where the participants chose convenient, practical, and valid strategies to find the side length. Using the fact that they had a right triangle formed by one side of the parallelogram and the two bisector lines, they chose strategies such as the Pythagoras formula or the trigonometric ratios (see **Figure 7**). They also exhibited this skill when choosing the proof strategy of the counterexample in Exercise 814 (see **Figure 11**), instead of trying to test many cases. When solving problems in mathematics, it is important to have a flexible approach, which means to be able to see the problem from different perspectives or solve an easier problem first. In this sense, some participants demonstrated this skill when dividing the parallelogram figure or extracting the triangles from it as a strategy to observe properties and relationships more clearly, as highlighted in **Figure 7**. Similarly, they could solve the problem in different steps, noticing that different strategies could solve different values.

However, when implementing the solution strategies, they were not always able to connect all the parts they generated, resulting in long chains of equations without a clear north. **Figure 12** is an example of this situation in which participants could not successfully connect their equations.

Assess the mathematical knowledge of learners

Understanding the way students are thinking in mathematics allows the teacher to know how they are interpreting the mathematical contents and how they are using them in their practices. Some evidence of this skill is revealed in the items that required preservice teachers to evaluate the students' work or analyze the reasons for the mistakes made. For instance, when they had to explain why the participants wrongly interpreted a histogram about the frequency of the adult female literacy rate in Central and South American countries, the respondents provided different reasons. It is possible to distinguish between the participants who explain the students' responses by pointing out that they only counted the bars (29 of 79) and those who make explicit the fact that the student assumed that each bar represented a country (32 of 79). Among the responses, P80, P52, and P23 stand out, referring to the fact that the student in the example did not pay attention to the axes of the histogram (y-axis was frequency and x-axis adult female literacy rate). P71 goes further, adding that probably the student thought "that each bar represented a country, and the highest rate would be the highest column." Another of the reasons mentioned by P63 and P69 is that the student could have thought about the number of countries in Central America, ignoring that the study also included South America. From these explanations, it is possible to observe that the future teachers tried to not only about think the most obvious possibilities but also explain typical errors such as not reading the axes or the title of the graph, or the confusions more related to the context as in the case of Central American countries.

In the case when item 604B asked for why one-word problem was more difficult for secondary students than the other, they mentioned points as the data were "less explicit" (P77) or that "students generally have trouble interpreting the relations of proportionality, regarding the translation from natural language to mathematical" (P80). These explanations are closer to typical errors faced by students when solving word problems than with the structure of the equations systems involved in the solution of the problem.

DISCUSSION

Given the lack of information on the knowledge for teaching mathematics and the professional competences of Costa Rican preservice teachers, we evaluated the cognitive and teaching-related skills covered in the TEDS-M questionnaire of 79 future mathematics teachers, as well as their mathematical understanding for teaching evidenced in the solutions to the items.

The quantitative analysis of the cognitive subdomain revealed that preservice teachers were better prepared to solve items in the *applying* domain than in the *knowing* and *reasoning* domains. However, their performance in the *reasoning* domain, the one of higher cognitive demand, was unsatisfactory. The reasoning domain was also the one with lowest performance in the international TEDS-M study (Hsieh et al., 2014). In the mathematics school curriculum, five crucial processes must be practiced in the classroom: reason and argue, pose, and solve problems, communicate, connect, and represent (MEP, 2012). Accordingly, the teacher has to be proficient not only in *applying* but also in *knowing* and specially *reasoning* to guide the students to achieve mathematical proficiency. The fact that the preservice teachers in the higher quartile had a different pattern of performance, achieving a higher score in *reasoning* than in the other subdomains, and the big gap between them and the other three quartiles, make evident that 75% of the respondents have been left behind in the development of their *reasoning* skills during their TEP.

Additionally, when observing the pattern of the results in the cognitive subdomains by university, it is possible to infer that the TEPs of universities B, C, and D have an approach focused on *applying*. The descending pattern *applying-knowing-reasoning* is also followed by Chile and the United States according to the TEDS-M results (Hsieh et al., 2014), so it may be associated with cultural factors. On the other hand, the pattern of university A descending from *knowing* to *reasoning* to applying is not consistent with any of the patterns found by Hsieh et al. (2014). Nonetheless, the results that the *reasoning* subdomain has the worst performance in the analysis by quartiles and by TEP should be a wake-up call to universities to urgently improve teacher preparation in *reasoning* skills. Notably, the *reasoning*, *knowing*, and *enacting* subdomains were significantly different in the TEPs. This result is consistent with the issue addressed by Román and Lentini (2018) about the heterogeneity of the TEPs and the lack of control of TEP quality (Alfaro et al., 2013). However, our results are an attempt to describe how the TEPs differ.

The qualitative analysis allows us to get closer to the participants' understandings and have a clearer panorama of their strengths and weaknesses. Regarding the positive aspects, we can highlight the understanding and correct use of properties, definitions, and representations to solve and prove mathematical tasks. As well, the participants could make numerous connections between them and, sometimes, give them good use. In addition, many contestants exhibited good performance in writing proofs. All of these skills are of daily use in the mathematics classroom. Other strengths shown by Costa Rican preservice teachers include proficiency recalling formulas, providing explanations of their work, and selecting valid strategies for solving problems.

More related to teaching skills, the respondents indicated that they could monitor students' work by revising attempts to solutions identifying the correct attempts and pointing out the mistakes in the incorrect ones, which shows conceptual understanding and understanding of students' way of thinking. On the same topic, they were also able to provide various explanations of the students' difficulties, including the most common ones on mathematical content and some more atypical ones related to beliefs or context. All these skills are expected from mathematics teachers as mentioned in several frameworks (e.g., Ball et al., 2008; Carrillo et al., 2018). Therefore, it is a good sign that Costa Rican future teachers have them. However, we also found weaknesses worth considering before drawing conclusions.

The analysis of participants' solutions revealed errors, from a very basic level such as computing mistakes, and others more complex in the structure of proofs, for instance issues with understanding the zero-absorption property, and consequently it was used incorrectly. Some mistakes are like the ones committed by students, such as not checking that the answer makes sense with the problem statement or doing incorrect translations from natural language to the symbolic language of an algebraic expression. The complex response items evidenced the participants' problem of posing numerous equations without having clear what they would be useful for and resulting in large chains of equations systems that sometimes did not lead to an answer. It is evident that the preservice teachers were not monitoring their work—an important moment when solving problems (Polya, 1945) and also had problems with what Schoenfeld (1985) calls control in problems solving process, which has to do with a correct and efficient use of the heuristics.

Other mistakes observed were related to reasoning by contradiction when the task did not allow it or overgeneralizing results after trying it with just one case, observed in other studies with preservice teachers (Demir et al., 2018). These reasoning actions are considered in the MUST framework as part of the mathematical activity perspective (Kilpatrick et al., 2015). Teachers have to come up with conjectures and generalizations spontaneously while providing explanations in the class or persuading students of an incorrect solution path. Therefore, future teachers must be proficient in these skills, first in their role as solvers to translate them into the teaching role. Last but not least is the participants' weak abilities to provide feedback, although they could identify the students' mistakes.

As mentioned above, they were not very good at explaining them. The word choice and the depth of the explanations were difficult to comprehend, and students and parents would have trouble gaining insights from them. The Costa Rican preservice teachers' gap regarding giving feedback was already pointed out in a previous article where 56% of they stressed they rarely or never had the opportunity to learn to do this type of assessment activities (Alfaro Víquez & Joutsenlahti, 2020). Many of the weaknesses noticed in qualitative analysis are coherent with the preservice teachers' performance in the reasoning subdomain.

In conclusion, we identify that the preservice mathematics teachers that participated in this study are proficient in the *applying* subdomain, which means they are good at solving routine problems, representing mathematic information, selecting solving strategies, and implementing instructions (Tatoo et al., 2008). However, they failed in the *reasoning* subdomain, which was required to analyze, generalize, integrate, justify, and solve nonroutine problems (Tatoo et al., 2008). The respondents also exhibited weaknesses in teaching-related skills, such as monitoring their own work or providing clear and useful feedback, and an alarming number of basic mistakes in computations, not verifying the answer or translating algebraic expressions from natural to symbolic language. Moreover, although they could pose numerous equations by using connections between contents, they were posed without having a clear use to reach the answer.

Our findings suggest that universities need to make adjustments in their TEPs to reinforce the problems identified. Actions for the remediation of these issues are important to address the teachers' knowledge and skill gaps before they start teaching (PEN, 2019). Similarly, the results on the heterogeneity in the performance of the universities require the action of the policy makers to establish quality standards for the training of mathematics teachers and to implement hiring policies that ensure the quality of the teachers who go to the classrooms.

This study had some limitations. First, we only assessed the knowledge for teaching mathematics from a cognitive perspective and leaves situated aspects aside. To have a complete picture of the knowledge required to teach mathematics and the professional competencies of preservice teachers, it is necessary to complement this study with others where the participants can be observed in practice. As Kaiser et al. (2017) stated, there is complex interaction between the knowledge-based and the situated competence facets, and to have a better understanding about teachers' professional knowledge, both facets need to be considered. Therefore, as teaching practice is crucial to becoming true experts in teaching mathematics, future studies could be performed on that topic. Second, it will be important to discuss these results with the teacher educators with the purpose of identifying the causes and proposing ways of improving mathematics teacher preparation in Costa Rica. Another interesting investigation will be to apply the TEDS-M questionnaire to in-service teachers and observe if the teaching experience makes a difference in the results. Finally, it is essential to obtain information from future teachers prepared in private universities, since studies have indicated that those TEPs cover fewer mathematics topics and offer weaker pedagogical knowledge than public universities (Alfaro et al., 2013).

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