UNDERGRADUATE STUDENTS' LEARNING OF LINEAR ALGEBRA THROUGH MATHEMATICAL MODELLING ROUTES

Abstract

Mathematical modelling has acquired relevance at all educational levels in the last decades since integrating this activity in instruction provides significant contexts for improving students' learning, including in linear algebra courses that have a notable presence in many undergraduate courses from different fields, including engineering and sciences. This paper reports a study aiming to characterize the distinct modelling routes performed by Costa Rican undergraduate students when solving a mathematical modelling task involving the concept of system of linear equations (SLE). In analysing those modelling routes, it was possible to identify their learning of linear algebra concepts and their modelling competencies as well as the associated difficulties that students faced. Data collection included participant observation, with audio recording of the students' discussions, their written work on the task and digital files of their work with technology. The results show that non-linear routes are associated with a greater mobilization of students' knowledge on SLE concepts and with their development of modelling competencies. The results also highlight the need to improve the students' competency of validating results, an important step that they did not take, and suggest the need to make technology relevant to the students' work on modelling tasks.

Keywords: linear algebra; systems of linear equations; mathematical modelling routes; undergraduate students.

INTRODUCTION

Linear algebra is a fundamental discipline in undergraduate education in many areas, including sciences, engineering, or economics. It involves the knowledge of important mathematical concepts such as systems of linear equations (SLE), which allow modelling a variety of real situations (Costa & Rossignoli, 2017). However, the learning of linear algebra concepts has proved to be difficult for many students, leading researchers to consider and explore innovative didactical approaches for the teaching of linear algebra, namely by fostering its connections with real situations, including those that are related with students' future professional activity (Bianchini, Lima, & Gómez, 2019).

The consistent body of research on the teaching and learning of mathematical modelling (Cai et al., 2014) clearly confirms its relevance in all educational levels, including tertiary education, namely in providing significant contexts for improving students' learning (Alsina, 2007). The understanding of everyday phenomena and the many real-world problems that can be modelled through a variety of mathematical models are important trends in current educational systems (Blum, 2015; Rosa & Orey, 2012).

Mathematical modelling, considered from a cognitive perspective, is the process of finding a solution to a real-world problem by using mathematical models and focuses on how the students' thinking leads to the construction of a model (Czocher, 2018) as a way to find answers to real life questions (Blum, 2015).

As the regular traditional classes of linear algebra in Costa Rican courses do not encourage students to work on real-world situations but instead only propose pure mathematics contexts (Sánchez, 2019), we developed a teaching experiment at the University of Costa Rica with undergraduate students of Engineering and Sciences, using mathematical modelling as an approach to the learning of linear algebra, which enables connections between concepts and real-world situations (Trigueros & Possani, 2013).

Typically, the mathematical modelling activity is described as a cyclic process involving several transitional phases between the real world and the mathematical world together with a set of sub-processes that mediate such transitions (Borromeo Ferri, 2018). However, as modelling is an iterative and idiosyncratic activity (Galbraith & Stillman, 2006), the students tend to go through some phases of the modelling cycle more than once and to avoid the transition to other phases during their work (Czocher 2018; Galbraith & Stillman, 2006); this results in different modelling trajectories known as mathematical modelling routes (Borromeo Ferri, 2018). Thus, assuming an educational and cognitive perspective (Kaiser & Sriraman, 2006), mathematical modelling is not only a pertinent approach to develop students' linear algebra concepts (Trigueros & Possani, 2013) but also an opportunity for teachers to identify the learning and modelling competencies developed by students, and the difficulties they face when working on mathematical modelling activities. The latter can be achieved by analysing their modelling routes, which show the phases that students go through, together with the processes performed while advancing throughout the cycle.

There are few studies that look at the modelling routes of undergraduate students aiming to understand how the mathematical modelling activity can be used as an effective learning context to promote and broaden their knowledge in linear algebra and the development of their modelling competences. To address this gap and the concerns stated above on the need to relate linear algebra concepts with real-world situations, this study aims to characterize the modelling routes performed by Costa Rican undergraduate students when solving a mathematical modelling task involving the concept of SLE. For this, we addressed the following research question: what are the characteristics of the modelling routes performed by students in solving the task and what knowledge of SLE and modelling competencies do they activate in those routes?

PREVIOUS STUDIES ON TEACHING AND LEARNING OF LINEAR ALGEBRA

Linear algebra is a field of mathematics whose applications range from pure mathematics to external areas such as engineering (Costa & Rossignoli, 2017), which explains why that field of knowledge is mandatory in numerous curricula in higher education courses related to science and technology. In the last decades, the research related to higher education has also been paying increasing attention to the learning of linear algebra topics, and several studies have been discussing pedagogical approaches aimed at improving the teaching and learning of linear algebra (e.g., Trigueros & Bianchini, 2016). Several authors have reviewed the literature stemming from empirical studies about linear algebra teaching, carried out in the last years (e.g., Bianchini et al., 2019; Stewart, Andrews-Larson, & Zandieh, 2019). Focusing on the Latin American context, studies such as that of Bianchini et al. (2019) offer a review of the research studies on the teaching and learning of linear algebra in undergraduate engineering programs; their findings however do not include the study of SLE, namely because the topic is not always considered to be fundamental in linear algebra but rather it is seen as a tool to solve linear algebra problems. The authors' analysis shows that the studies carried out have focused on promoting the learning of other topics, namely the concept of vector spaces and associated concepts, such as basis of a subspace (Parraguez, 2009), linear transformations (Silva, 2016), theorems on the matrix of a linear transformation (Roa-Fuentes & Parraguez, 2017), and eigenvalues and eigenvectors (Salgado & Trigueros, 2015), all considered as highly abstract for most undergraduate students who take a linear algebra course (Costa & Rossignoli, 2017).

On the other hand, Stewart et al. (2019) highlight extensive empirical studies on linear algebra, including literature reviews like the one by Bianchini et al. (2019), some of them introducing mathematical modelling contexts (Possani, Trigueros, Preciado, & Lozano, 2010). The authors indicate that mathematical modelling has been implemented by using specific theoretical lenses, such as Realistic Mathematics Education for the learning of abstract concepts, such as span and linear independence (e.g., Wawro et al., 2012). They also refer to the apparent need of further studies focusing on the learning of SLE, properties of linear transformations, orthogonality, and least squares, as well as on cross-cutting themes such as proof in linear algebra.

From the reviewed studies, Bianchini et al. (2019) reinforce the need to adopt pedagogical approaches involving real contexts to promote the learning of linear algebra in order to motivate students and show them the applicability of mathematics in their future professional life. In the same vein, Trigueros and Possani (2013) point out that mathematical modelling is a relevant learning environment that facilitates the consolidation of linear algebra concepts, allowing more effective learning if students are encouraged to work on problem situations in which they feel the need to use algebra concepts. In some of those studies, the pedagogical approaches also included the use of technology, namely mathematical software such as

Matlab or Maple (Nomura & Bianchini, 2009), Derive (Porras, Silverio, & Vargas, 2010), and Mathematica (Legorreta & Andalón, 2016), to address real problems in real contexts.

Investigations on the teaching and learning of linear algebra, outside the Latin American scope (e.g., Mallet, 2007), discuss specific didactical proposals for the learning of SLE by means of visual, algebraic, and tabular representations, using mathematical software that includes computer algebra systems, such as Maple, helping students to understand the meaning of SLE and of the solution set of a SLE. Mallet's results reveal that students are able to leverage the understanding gained on visual representations for working algebraically on SLE, by referring, for example, the solution set of a SLE with infinite solutions in terms of the infinite points on a line, intersection of three planes, or the intersection of two planes. However, they acknowledge students' difficulties in working on SLE with tabular representations, highlighting the need for further studies involving the use of tabular representations when working on the concept of SLE.

Zandieh and Andrews-Larson (2019) focus on the symbolizing processes that undergraduate students use when solving SLE, in an introductory linear algebra course, expanding their analysis to the use of matrices. The authors analyzed students' final examination papers by looking at two strategies: reduced row echelon form (RREF) strategies and other linear system solving strategies. The results show similarities and differences between the processes of solving SLE, with students using the RREF being able to represent a SLE as an augmented matrix, although having more difficulties in interpreting the information from the reduced matrix when solving systems involving planes compared to systems involving lines. In a similar way but focusing on the interpretation of a matrix equation $AX = B$ as linear combination, SLE, or transformation, Larson and Zandieh (2013) show different examples of tasks which encourage students to make sense of a matrix equation by assigning a symbolic and geometric interpretation to it. The results reveal that in each example there are students which interpret $AX = B$ as a SLE, which showed the pertinence of using matrix equations as tools to understand the students' ways of thinking about systems of equations.

In their work, Stewart et al. (2019) refer to the study developed by Possani, et al. (2010), which involved undergraduate students of Engineering, Social Sciences and Economy working on a modelling task related to a traffic flow situation, aiming at evaluating the possibility of introducing concepts related to SLE. The results revealed that the situation presented in the task was significant but cognitively demanding for the students, when compared with traditional practical exercises often performed in the linear algebra class. In addition, there were many opportunities for discussion in which students could reflect on the concepts of

SLE (augmented matrix, solution set of a SLE and inverse matrix), on which mathematical models were based and used to respond to the traffic flow problem situation. This study, like other investigations on the teaching and learning of SLE with mathematical modelling tasks, highlight students' difficulties in proposing adequate models to the problem situation, namely in establishing hypotheses that simplify the problem situation, identifying variables to create a mathematical model, understanding mathematical results, and using and interpreting parameters in mathematical solutions.

Moreover, investigations using mathematical modelling with technology, such as the study by Galbraith and Stillman (2006), show that the work on mathematical models for the study of linear equations is limited when the student does not have the mathematical and technical knowledge necessary for working on the task, which indicates that previous work with the students is necessary to maximize their learning within environments including modelling and technology.

From this standpoint, mathematical modeling is a pertinent didactical approach to promote competencies and pose different challenges to the students (Czocher, 2018), including skills to work with technology, as well as the application of relevant mathematical knowledge to solve a task, both being aspects that influence the students' modeling processes (Borromeo Ferri, 2018).

MATHEMATICAL MODELLING TASKS

Working on mathematical modelling tasks can make the linear algebra class a motivating but also a challenging learning environment for undergraduate students. According to Blum and Borromeo Ferri (2009), modelling tasks are highly demanding in terms of the competencies involved in the modelling cycle, but it is also possible that such competencies are acquired and mobilized by the students in the mathematics classroom, which means that modelling tasks offer many opportunities for learning (Blum & Borromeo Ferri, 2009; Kaiser & Sriraman, 2006).

Blum (2015) highlights four main reasons to justify the need for including modelling tasks in mathematics teaching: (1) *pragmatic justification*: to understand and master real-world situations; (2) *formative justification*: to foster mathematical modelling competencies; (3) *cultural justification*: to acquire a comprehensive sense of mathematics as a science and of its relation with the extra-mathematical world; and (4) *psychological justification*: to motivate the student's interest in mathematics and to promote understanding of mathematical content and its applications.

However, some limitations of this kind of tasks should be attended, namely the possible difficulties that the students may face in solving them. For example, the students often state that they have never worked with mathematical modelling tasks and therefore are not familiar with formulating hypotheses, exploring concepts, interpreting information, and validating mathematical solutions (Sokolowski, 2015). Also associated with the nature of the modelling tasks, the transition between phases in the modelling cycle is a possible difficulty (Galbraith & Stillman, 2006), being customary that many students have difficulties during the phase of understanding the problem situation (Blum, 2015). Validation of results, which is an important competency in solving modelling tasks, often results in a big challenge for the students as they do not always notice what is wrong in their models or results (Czocher, 2018).

Based on the findings of the previously mentioned studies, one may see some possibilities to overcome the identified obstacles in the learning of linear algebra: the use of modelling real situations, which offer opportunities to give meaning to concepts and procedures involved in them, and the development of relevant modelling competencies; moreover, the use of technological tools allows to combine the conceptual and procedural aspects in solving a modelling task.

MATHEMATICAL MODELLING ROUTES: BEYOND AN IDEAL MODELLING PATH

Depending on the goals of mathematical modelling in the classroom, different perspectives on the teaching and learning mathematical modelling have been identified over the last few years (Kaiser & Sriraman, 2006). From an educational perspective, modelling focuses on the teaching and learning processes, differentiating itself in didactic educational modelling (structuring and promoting learning processes) and conceptual educational modelling (introducing or consolidating concepts). Related with this educational perspective, the cognitive perspective seeks the identification and description of the cognitive processes and associated difficulties that are evidenced by the students when they go through the mathematical modelling cycle (Figure 1). So, an important target of studying the mathematical modelling activity is to identify the modelling path developed by the student through the *ideal* modelling cycle (Borromeo Ferri, 2018).

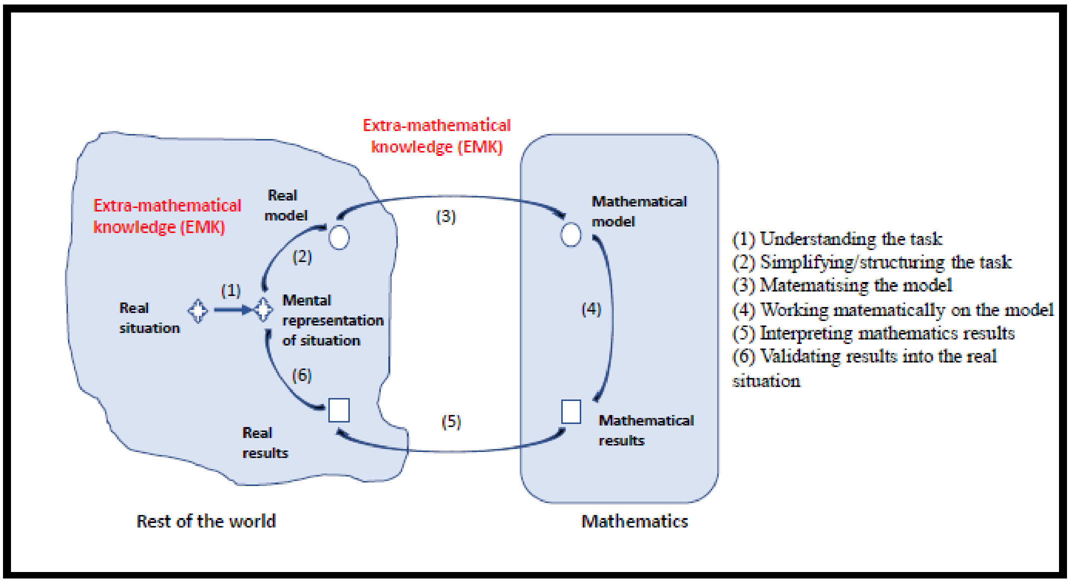


Figure 1: Modelling cycle from a cognitive perspective (Borromeo Ferri, 2018)

The *ideal* cycle (Borromeo Ferri, 2018) begins in the so-called "rest of the world", presenting a problem about *a real situation*. From then on, the student needs to make sense of the problem to create a *mental representation of the situation* that will allow the translation of the presented information. Next, the student must take decisions to *structure and simplify the problem*, and thus moves towards obtaining *a real model*, which constitutes an individual construction, that will require mathematization and leads to the *mathematical model*, through external representations (formulas, diagrams, tables, etc.) but also integrates extra-mathematical knowledge. With the elaboration of a mathematical model, the student enters the world called "mathematics", in which (s)he must use mathematical knowledge and, if needed or preferred, use any technological resource for working mathematically on the model to perform mathematical procedures until obtaining *mathematical results*. After that, students must interpret the mathematical results in the context of the real situation to get *real results*, which later must be validated in the context of the real situation and thus accepted or rejected. Depending on the validation activity, the modeller may choose to answer to the initial questions based on the results obtained or to start a new modelling cycle to improve the mathematical model, according to whether the actual results are considered adequate or not (Galbraith & Stillman, 2006).

It should be noted that the student will not always follow this ideal modelling cycle, but rather a trajectory that may reveal the transition through all or only some of the phases of the modelling cycle and may go backward to phases already transited. This is called the student's modelling route (Borromeo Ferri, 2007). According to the author,

> Modelling route is the individual modelling process on an internal and external level. The individual starts this process during a certain phase, according to their preferences, and then goes through different phases several times or only once, focussing on a certain phase or ignoring others. (p. 2083)

When the transitions between the sequential phases occur only once and following the direction of the ideal modelling cycle, we have a linear modelling route. On the other hand, when the transitions involving one or more phases take place several times, we consider it as a non-linear modelling route. Students need to learn the difference between a real model and a mathematical model to understand what mathematical modelling means, including that they can go through these and other phases several times (non-linear modelling route) to get an appropriate real solution for the task (Borromeo Ferri, 2018).

The research also shows that the student's mathematical knowledge and competencies are not enough to successfully complete a modelling activity, suggesting that the routes are influenced by three factors: (i) mathematical thinking styles (visual, analytical, integrated), (ii) mathematical competencies developed by the student, and (iii) extra-mathematical experiences and knowledge (Borromeo Ferri, 2007, 2018). In particular, "there is a strong connection between the conception of the modelling process and modelling competencies" (Maaß, 2006, p.114). The mathematical modelling competency, in this sense, refers to the abilities that are essential to appropriately carry out activities involved in the modelling process and is defined by Niss, Blum, and Galbraith (2007) as:

> the ability to identify relevant questions, variables, relations or assumptions in a given real world situation, to translate these into mathematics and to interpret and validate the solution of the resulting mathematical problem in relation to the given situation, as well as the ability to analyse or compare given models by investigating the assumptions being made, checking properties and scope of a given model. (p. 12)

In more detail, Maaß (2006) proposes a list of mathematical modelling competencies and offers a description for each of them. In Table 1, that list of modelling competencies is presented in columns 1 and 2, where column 3 contains the several cognitive processes taking place in the modelling activity, drawing on the cognitive view endorsed by Borromeo Ferri (2018). As such, Table 1 highlights the fact that developing competencies entails cognitive processes, and vice versa. The modelling competencies, in particular, influence the student's modelling route, which is often a nonlinear process (Galbraith & Stillman, 2006).

Table 1. Mathematical modelling competencies and cognitive processes

Modelling competencies	Competency to…	Cognitive processes
To understand the real problem and to set up a model based on reality	make assumptions about the problem and simplify the situation; recognize quantities that influence the situation, identify key variables; construct relations between the	(1) Understanding the task

	variables; look for available information and to differentiate between relevant and irrelevant information.	(2) Simplifying/ Structuring the task
To set up a mathematical model from the real model	mathematize relevant quantities and their relations: simplify relevant quantities and their relations if necessary and to reduce their number and complexity; choose appropriate mathematical notations and to represent situations graphically.	(3) Mathematising the model
To solve mathematical questions within this mathematical model	use heuristic strategies such as division of the problem into part problems, establishing relations to similar or analogue problems, rephrase the problem, view the problem in a different form, vary the quantities or the available data, etc.; use mathematical knowledge to solve the problem.	(4) Working mathematically on the model
To interpret mathematical results in a real situation	interpret mathematical results in extra-mathematical contexts; generalize solutions that were developed for a special situation; view solutions to a problem by using appropriate mathematical language and/or to communicate about the solutions.	(5) Interpreting mathematical results
To validate the solution	critically check and reflect on found solutions; review some parts of the model or again go through the modelling process if solutions do not fit the situation; reflect on other ways of solving the problem or if solutions can be developed differently; generally question the model.	(6) Validating results into the real situation

The modelling route may be supported by the use of technology, offering the student the option to use a computer model to obtain computer results, which occurs prior to obtaining mathematical results. The technology is especially relevant in building models that could not be created without technological resources and in simplifying and allowing new possibilities, such as: visualizing; exploring, organizing, or evaluating a large amount of data (Siller & Greefrath, 2010). Thus, the students must consider the need and relevance of using technology so that it becomes a helpful resource for working on mathematical modelling activities (Greefrath, Hertleif, & Siller, 2018).

METHODOLOGY

Context and participants

This study is part of a broader research based on a *teaching experiment,* which is considered a methodological approach aiming to understand the learning and reasoning of students when teaching and learning strategies are implemented to solve a specific problem that was taken as the target of the experiment (Bernabeu, Moreno, & Llinares, 2019). The study was carried out with 21 undergraduate students (13 boys and 8 girls) that were attending a linear algebra course at the University of Costa Rica, which involved a sequence of five modelling tasks combined with mathematical exercises, in both cases with the optional use of computer and educational software, such as Mathematica, GeoGebra and Excel. Each modelling task was proposed after the teaching of the concepts involved in it, through lectures and practice tasks involving only mathematical contexts. The students who participated in the study had no previous experience on applying concepts of linear algebra in real contexts. In the present article, we will focus on the first modelling task of the sequence, which aimed to activate and develop concepts associated with the topic of SLE, namely the concepts of SLE, augmented matrix and solution set of a SLE. Regarding the technological resources, they were expected to provide optional support to work on the task, so students were not required to use it, although mathematical software was available in the computer lab where the classes of linear algebra took place.

The researcher (first author) assumed the role of teacher during the five 70-minute classes of the teaching experiment where the students worked on the modelling tasks. The classes followed a three-phase structure: teacher presentation of the task to the students, explaining the real context of the task and the organization of the class work; students' autonomous work on the task, in 10 groups of 2 or 3 elements, involving moments of discussion between the students and between them and the teacher; and a final whole group discussion with the students, guided by the teacher, where they shared and discussed their models, and a synthesis of the solutions obtained in the class was produced.

The task "Preventive traffic" used in this study was adapted from the problem proposed in Possani et al. (2010) and aimed to activate concepts involving SLE in a context of car traffic. The context of the task is realistic as it presents a car traffic average flow in an area of the capital of Costa Rica, known by the students, where some fixed visual flow orientations obtained from Google Maps are indicated (Figure 2).

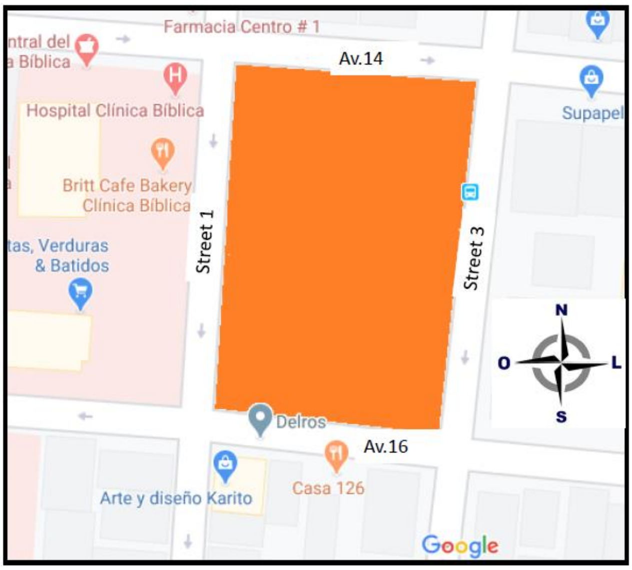
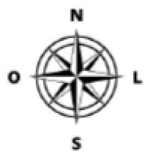


Figure 2: Task context representing the car traffic at four intersections

The task statement also includes information on the average number of vehicles, per hour, entering and leaving each of the four intersections and referred as Points 1 to 4 (Figure 3), although not all vehicle flows are provided for each intersection. The task questions (Figure 3), to be answered in the context of traffic flow, are oriented to exploring the possibility of closing routes but keeping the traffic circulating, and to analyse the minimum and maximum amount of traffic that will circulate on routes whose average flow is unknown.

Suppose that the traffic engineering center has done a study about the quantity of vehicles, per hour, which travel around a certain region. This information is available in the following table.

	Point 1 (intersection of 1st street and avenue 14)	*Point 2 (intersection of 1st street and avenue 16)*	*Point 3 (intersection of 3rd street and avenue 16)*	*Point 4 (intersection of 3rd street and avenue 14)*
Average number of vehicles passing per hour and direction of traffic	2000, North-south direction (entering intersection)	800, East-west direction (leaving intersection)	2500, North-south direction (leaving intersection)	1800, North-south direction (entering intersection)
	500, South-north direction (leaving intersection)	1500, North-south direction (leaving intersection)		1500, West-east direction (leaving intersection)
	500, East-west direction (leaving intersection)			

As a student of engineering or exact sciences, the traffic engineering center request your assistance. Please, help them to answer the following questions:

- What condition or conditions would be ideal for the traffic flow, through the paths and intersections that surround the region of the figure, so that the traffic keeps normal, without congestion?
- Imagine that the 14th avenue section is closed between Street 1 and Street 3. According to your model, would it profoundly affect vehicle traffic?

Figure 3: Data of car traffic at four intersections in task statement

Thus, the task invites the students to use their learning of SLE to formulate mathematical models that, after being analysed, would allow answering the questions proposed in the task, and at the same time it aims at promoting students' mathematical modelling competencies. It was expected that students used formal concepts of SLE (analytical representation and solution of a SLE, augmented matrix) to create an equation for each intersection, by using the known and unknown values of traffic flow, assuming that entries and exits in each intersection have to be equal for keeping the flow constant. As part of the simplifying/structuring process, students could decide whether to consider flows in one or both directions for each path, except for two segments of traffic: the street section above and the section on the left from point 1, according to the information provided in Figure 2 (known flows) and Figure 3 (flow directions between intersections). As the sketch presented in Figure 4 illustrates, the conditions imposed require students to consider the left and the up segment of traffic as two-way street sections. Moreover, if students choose one flow direction in some sections of the streets, they have to follow the directions of the arrows in Figure 2 and respect the flow values given in Figure 3.

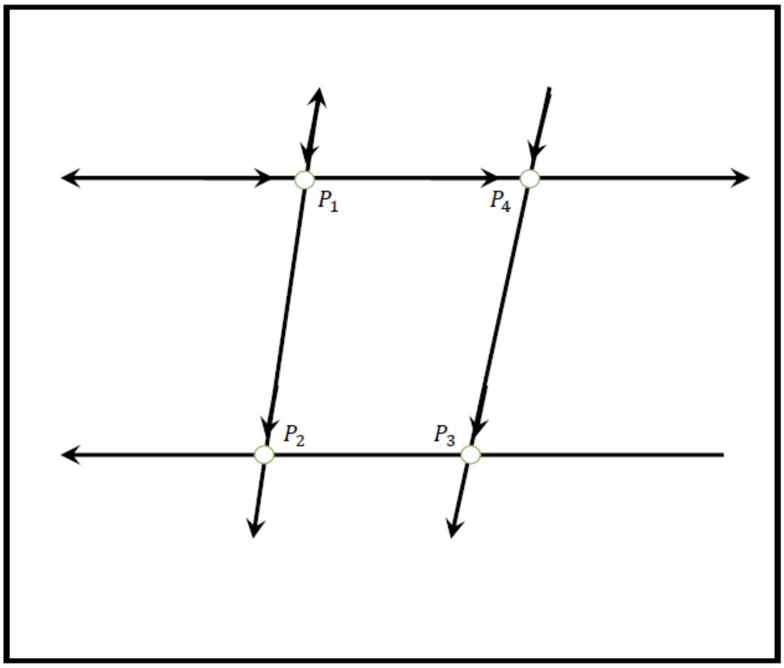


Figure 4: Sketch illustrating the conditions for the traffic flow given in the map and in the table

So, students had the possibility to formulate a real model with one-way or two-way street segments, but always considering the information provided in Figure 2 and Figure 3, suggesting a possible real context of traffic in Costa Rica. Then, students would have to analyse the SLE composed of equations formulated for each intersection and get the solution set by fixing some unknown traffic flow and determining conditions on the paths, always considering positive numbers for solutions in the real context. Real models with several two-way segments would require that students define several variables, associated with unknown traffic flows. It was expected that students would interpret solutions with negative values as impossible real situations and that they would consequently consider reformulate their real model.

Data collection and analysis

The study follows a qualitative and interpretative methodology (Cohen, Manion, & Mohinson, 2007). Data collection included the students' written work on the modelling tasks and the digital files created with the software *Mathematica* by some of the students, as well as participant observation, with audio recording of the discussions held in the groups. The descriptive and interpretative data analysis focuses on the algebraic concepts and modelling competencies that students used in solving the task, drawing on the modelling routes they performed.

For identifying such modelling routes, the modelling cycle proposed by Borromeo Ferri (2018) was used, according to the cognitive perspective presented in Figure 1, and the competencies as pre-established categories described in Table 1, associated with the cognitive modelling processes. The difficulties shown in the students' modelling processes and the knowledge on SLE they used were documented and interpretations for the reasons of such difficulties were proposed, in connection with their modelling routes. Divergent interpretations or doubts concerning the results of the analysis, independently performed by the first author, were discussed by all the authors until full agreement was reached.

In the next section, we characterize and discuss the modelling routes that represent the diversity of routes performed by the students in solving this task, including excerpts of their work to support the analysis. In the excerpts presented, the code Gx is used to designate the group x, in a total of ten groups of students. To ensure participants' confidentiality and anonymity, the students' names are fictitious.

RESULTS: UNDERGRADUATE STUDENTS' MODELLING ROUTES

In the "Preventive traffic" task, different linear and non-linear modelling routes were identified, revealing different scopes of the modelling processes carried out, some of them being complete and others only partial modelling cycles. There were those who ended up with the construction of a mathematical model (G1, G3, G9); those who finished with getting computer results (G5); those who achieved the phase of obtaining mathematical results, without interpreting those results (G2, G6, G10); those who achieved real results but incorrectly interpreted a mathematical solution (G8); and those who got to the initial real problem, exposing their final results, but also incorrectly interpreting results (G4, G7). In Table 2, we present the groups divided by those categories, along with the linear algebra concepts involved in their solutions, and their choice for the use of technology.

Table 2. Summary of the students' modelling processes on the task

Group (G)	Final modelling step	Learning of concepts involved	Type of route	Technology use
G1, G3, G9	Mathematical model	Formal concept of SLE	Linear	No
G2	Mathematical results	Formal concept of SLE, augmented matrix, solution set of a SLE	Non-linear	No
G4	Real situation	Formal concept of SLE, solution set of a SLE	Linear	No
G5	Computer results	Augmented matrix. Knowledge of *Mathematica*	Linear	Yes
G6	Mathematical results	Informal concept of SLE (equality between numerical quantities), solution of a SLE	Linear	No
G7	Real situation	Formal concept of SLE, augmented matrix, solution set of a SLE	Non-linear	No
G8	Real results	Formal concept of SLE, augmented matrix, solution set of a SLE	Linear	No
G10	Mathematical results	Formal concept of SLE, augmented matrix	Linear	No

From the table, we may observe that most students did not use technology to work on the task, except one group that resorted to the use of *Mathematica*. The task allowed the creation of diverse mathematical models where students mobilized knowledge on concepts of linear algebra, such as informal (non-analytical use) and formal (analytical use) concept of SLE, matrix associated with a SLE and solution set of a SLE.

Linear modelling routes

In the linear modelling routes, we observed that the students' modelling process did not go through any phase more than once. Within the linear modelling routes, we highlight the one schematized in Figure 5 that integrates the use of technology. This route was performed by Martim and Fabiana (G5), who relied on the *Mathematica* software to work on the mathematical model.

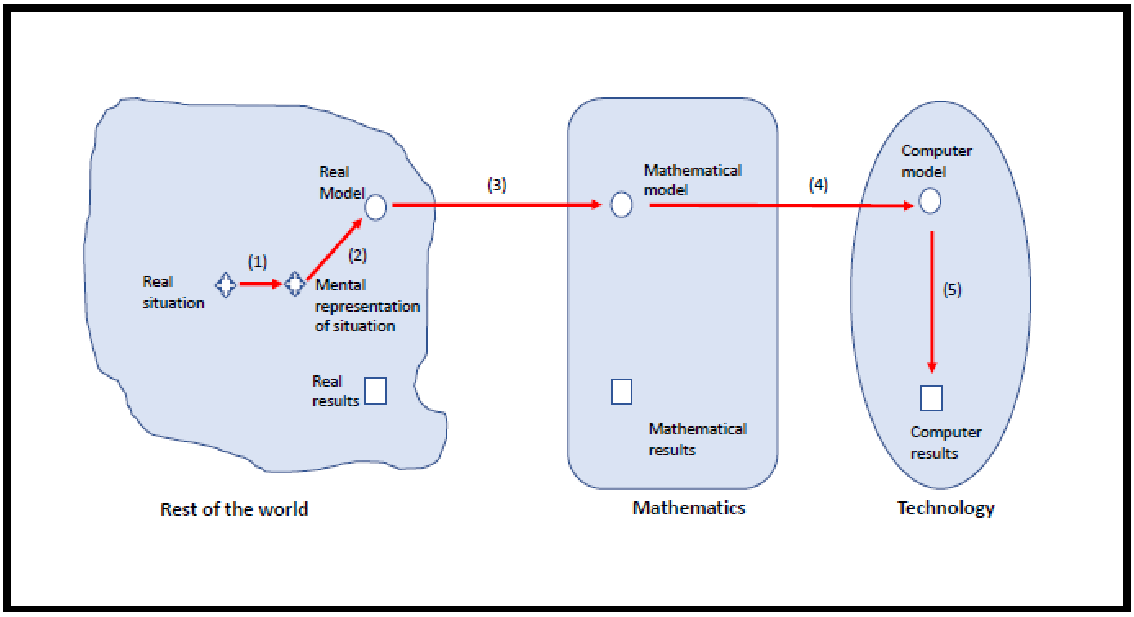


Figure 5: Modelling route of Martim and Fabiana (G5)

As shown in Figure 5, Martim and Fabiana (G5) ended their modelling process in the Technology world. These students and other five groups (G1, G3, G6, G9, G10) performed linear routes whose final modelling step was the construction of a mathematical model (G1, G3, G9) or some mathematical work on the model but without obtaining mathematical results corresponding to a solution set (G5, G6, G10). These six groups did not advance beyond the mathematical world, due to: having invested a lot of time in understanding the task, not having enough time to obtain a solution set after creating the mathematical model (G1, G3, G9, G10), as they state in their written report; having obtained a numerical solution for the task but without the verification of the results (G6); showing difficulty in connecting the real situation to the mathematical object of an augmented matrix and in defining the solution of a SLE (G5). In the case of Martim and Fabiana, they revealed competencies to work on the mathematical model using the technology, obtaining computational results (as shown in Figure 6), but they did not interpret those as mathematical results and thus were not able to find the solution set of the proposed model. Thus, the technology helped them with the calculations but not with interpreting them.

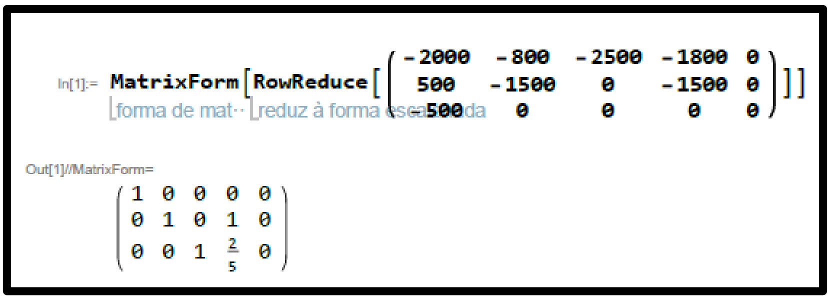


Figure 6: Mathematical work on *Mathematica* performed by Martim and Fabiana (G5)

These students used the values in the table provided (Figure 3), by drawing on their position in the table, to create the augmented matrix with which they worked with (Figure 6). They have incorrectly matched the empty cells in the table with zeros in the matrix instead of considering variables in the mathematical model. This shows that they were not adequately applying the concept of augmented matrix of a SLE. Moreover, the way of placing the values in the augmented matrix shows that they improperly associated the equations of the SLE with the columns of an augmented matrix, which contradicts the fact that the rows of the matrix associated to a SLE are the ones that should provide information about a particular equation.

In solving the task, Martim and Fabiana stated: “to solve this problem we thought of a homogeneous matrix, since we made all equations equal to zero so that there is no traffic jam”. By mentioning “we made all equations equal to zero”, the students reveal to recognize that the condition to keep the traffic flow constant is to keep the difference between the entry and exit flows at the same intersection equal to zero, which shows that they understood the problem. However, by associating this previous statement to the augmented matrix (Figure 6), there is evidence that these students did not mobilize the SLE concept since in their mathematical model they consider three equations, instead of the four corresponding to the total of intersections. This difficulty in creating real models was also seen in most of the groups that performed linear routes finishing in the mathematical world (G1, G3, G6, G9). To some extent, this was expected in this first task, as the students had never worked on linear algebra problems that involved real contexts. So, they were able to formulate mathematical models but with some inaccuracies resulting from the real model they formulated. For example, the route performed by Edite and Thiago (G6) (Figure 7) and their real model (Figure 8) revealed their competencies to make assumptions about the flow in the problem and to simplify the situation by using a trial-and-error process. However, this choice shows that there was no attempt to identify variables and, therefore, represents a non-algebraic model with some limitations.

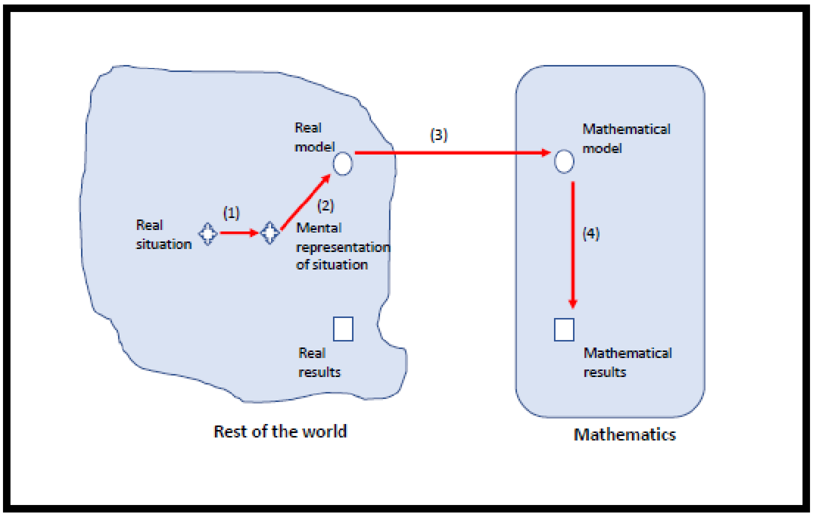


Figure 7: Modelling route of Edite and Thiago (G6)

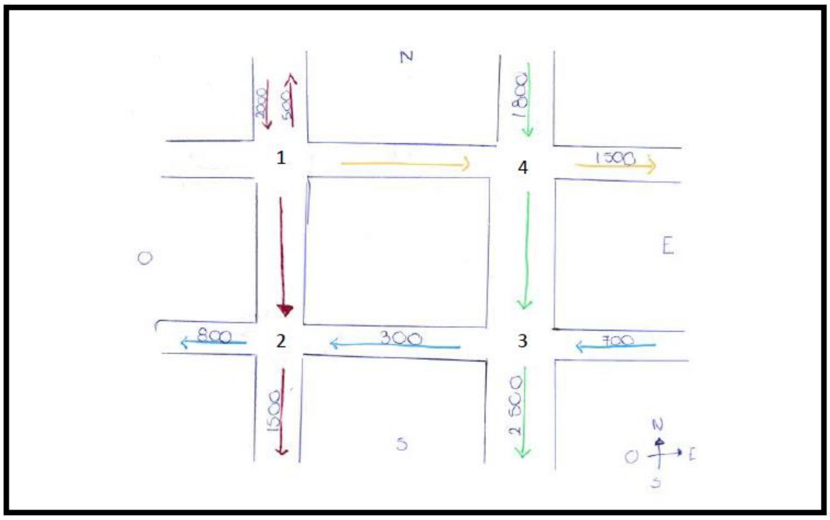


Figure 8: Real model of Edite and Thiago (G6)

Edite and Thiago tried to obtain a solution to the problem by experimenting possible numerical values for unknown flows and testing them to verify if they would meet the conditions imposed in some intersections, revealing a rudimentary notion of the concept of SLE. When they were asked by the teacher about their solution, while working on the task, Thiago replied:

> What we were thinking is that, from one point to another, for example here [from point 3 to point 2, in Figure 8], there are 300 vehicles crossing per hour. Then, we are looking for the values that complete the constant traffic flow, by making the differences between outgoing and incoming vehicular flows”.

From Thiago’s comment, we interpret that this group of students know the concept of SLE solution, since he refers to finding values that keep equal outgoing and incoming flows, simultaneously at the four intersections, which means to find a solution for a SLE. However, their solving process does not reveal the mobilization of the necessary knowledge to find the solution of the system, since they assume that by satisfying the equations of intersection 2, the equations of intersections 1, 3 and 4 will also be satisfied.

Furthermore, the choice of a single solution reveals that the concept of solution set of a SLE was not clear to these students, which is consistent with their incorrect idea that the problem would have a finite number of solutions. A similar finding was revealed in the approach of G8 that solved the task by using analytical procedures. As to the other groups, which also obtained analytical mathematical results and tried to understand their meaning in the real situation, all of them presented an infinite number of solutions to the problem, thus showing their understanding of the concept of solution set for the SLE in the given problem.

Regarding the linear modelling routes that terminate in the rest of the world, performed by two groups (G4, G8), we find the case of Artur and Hugo (G8) whose modelling route is shown in Figure 9.

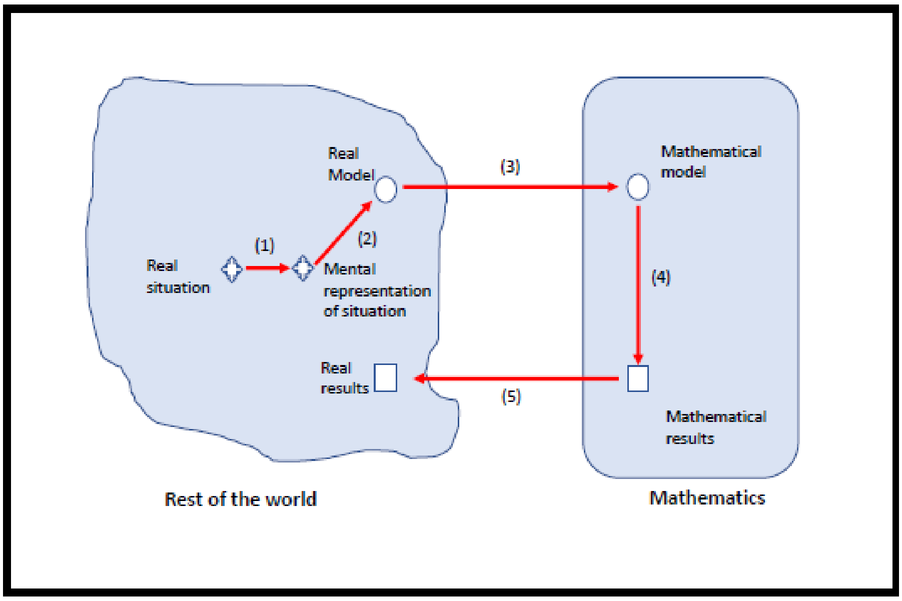


Figure 9: Modelling route of Artur and Hugo (G8)

The route shows the development of the first three processes of the modelling cycle, which are also developed by the other groups; nevertheless, this group reveals competencies to obtain and interpret mathematical results by communicating information about the mathematical solutions in terms of conditions for the *w* and *t* variables, thus obtaining also real results. Although these students have advanced in the process of interpreting mathematical results, their interpretation (Figure 10) is not completely correct, since they possibly consider the condition $w = t$ (observed in the first and third equations of the SLE) as not generating changes in the values of *x* and *z*, regardless of the *w* and *t* values, by deciding to consider the referred values "negligible".

$$\begin{cases} x - w + t = 3200 \\ y + w = 2300 \\ z - w + t = 2500 \\ v + t = 4500 \end{cases} \qquad \begin{cases} x = 3200 \\ y = 2300 \\ z = 2500 \\ v = 4500 \end{cases}$$

Nota: "w" y "t" son despreciables pues independientemente del valor que se les asigne, no va a variar en los valores de las demás variables

[Note: "*w*" and "*t*" are negligible, because regardless of the values assigned to them, the values of the other variables do not vary]

Figure 10: Interpretation of mathematical results by Artur and Hugo (G8)

Looking at the second and fourth equations (Figure 10), it is possible to observe that any change in the value of w and t implies changes in the values of y and v, so the statement of these students is not adequate in terms of analysing the solution set but it suggests a way of finding a particular solution for the problem situation, namely, when $w = t = 0$. Thus, Artur and Hugo, like one of the groups of students who performed non-linear routes (G7), activated the analytical concept of SLE, as well as the augmented matrix of a SLE. However, these students did not show the mathematical knowledge to realize the existence of infinite solutions of the SLE, judging by the single solution they were able to obtain for the SLE.

Non-linear modelling routes

In the non-linear modelling routes, we find the case of Fátima and Heitor (G7), whose route is schematized in Figure 11. Their work differs from the other groups because they were the only ones who answered all the questions of the task and went through all the phases of the modelling cycle, some of which carried out more than once. They have considered more than a real model and proposed two different solution processes, which was also the case observed in the group of Marcelino and Estela (G2).

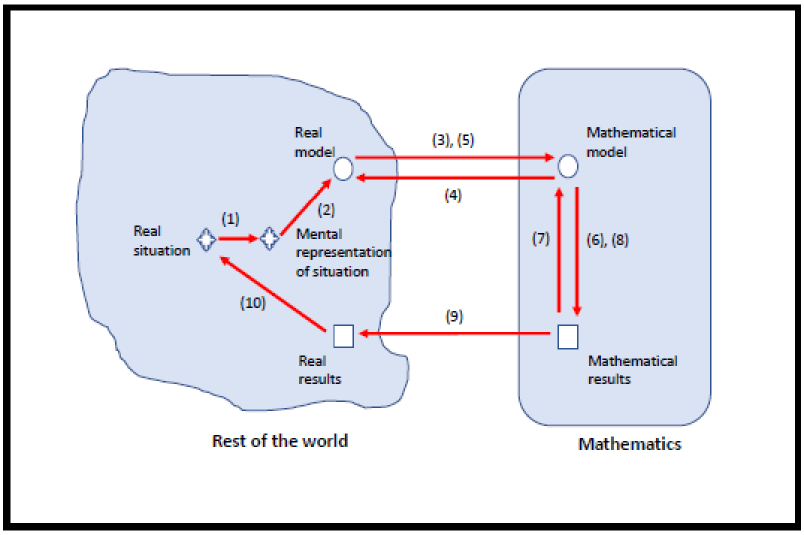


Figure 11: Modelling route of Fátima and Heitor (G7)

The route performed by Fátima and Heitor shows a transition between phases from the rest of the world to the mathematical world, advancing and returning to the previous stage in some moments of the modelling cycle, for example in the elaboration of the mathematical model, with the real model being modified at least one time, as they mentioned in their written work: "For the elaboration of the model, different conditions were analysed, some were inconsistent, and others complicated the problem a lot, so they were not considered".

From this comment and the mathematical model created by these students (Figure 12) we may interpret that they initially thought on models of two-way streets, and at the end they are able to identify relevant and irrelevant information (necessary conditions to avoid inconsistent flows) and reduce the complexity of the problem (discard options with many unknown flows). They decided to create models with only one-way streets for the unknown flows because the two-way model or any other similar "complicated the problem a lot" in terms of the variables to include in the mathematical model. Moreover, they recognized "inconsistent" conditions showing that they reflected on the context of the problem in terms of its solution (namely the impossible result of a negative flow of traffic). Thus, the transition between the mental representation of the situation and the elaboration of the mathematical model demonstrates the development of learning related to the concept of SLE and of the solution set of a SLE.

$$\begin{aligned}
&(1): w + 2000 = 500 + 500 + x + z \\
&\quad w - x - z = -1000 \\
&(2): x + y = 800 + 1500 \\
&\quad x + y = 2300 \\
&(3): q + w = y + 2500 \\
&\quad q + w - y = 2500 \\
&(4): z + 1800 = 1500 + q \\
&\quad z - q = -300
\end{aligned}$$

Figure 12: Mathematical model of Fátima and Heitor (G7)

After the mathematization process, Fátima and Heitor started to work on the mathematical model, having considered two options, as mentioned in their written work: “In order to simplify and obtain solutions from the constructed SLE, we thought to use substitution, but then we decided to use matrices and the Gauss-Jordan method”. From this affirmation is possible to interpret that students initially thought to solve the model without applying knowledge of linear algebra but ended up realizing that the presence of five variables in the model could make the calculations difficult. Therefore, they decided to restart the mathematical work, by moving from obtaining incomplete mathematical results to the initial mathematical model, this time working mathematically on the model using the Gauss-Jordan reduction method. Figure 13 shows students’ mathematical model reorganized and ready to be treated as a matrix, which shows that students recognised the concept of matrix associated to a SLE when they decided to use the Gauss-Jordan reduction method to work on the mathematical model in a correct way.

(1.1) Si el tramo "z" es cerrado, se obtiene:

$$\begin{aligned}
&\text{i)} \quad z - q = -300 \\
&\qquad 0 - q = -300 \\
&\qquad q = 300 \\
&\text{ii)} \quad x + q = 3050 \\
&\qquad x + 300 = 3050 \\
&\qquad x = 2750 \\
&\text{iii)} \quad y - q = -750 \\
&\qquad y - 300 = -750 \\
&\qquad y = -450
\end{aligned}$$

[If path z is closed, you get …]
Figure 13: Mathematical results for question 2,
presented byFátima and Heitor (G7)

Similar options and work on the mathematical model were observed in the group of Marcelino and Estela (G2), although they did not go beyond the mathematical results obtained. These students revealed competencies for dividing the problem into two parts (creating one-way and two-way models of flows), but they did not realize the need to verify their mathematical answer. After Fátima and Heitor reduced the matrix associated to the SLE as much as possible, they found the values for each unknown flow by defining the parameters of the solution set as belonging to $\mathbb{R}$, therefore not having interpreted the mathematical results in the context of the problem situation by observing that the traffic flow must have positive quantities. So, they revealed difficulties in interpreting their mathematical results in extra-mathematical contexts, which is evidenced by their decontextualized interpretation of the value $y = -450$ (Figure 13), corresponding to a mathematical result obtained when asked about what would happen if the path z was closed. They assumed this value as a possible measure of traffic flow, interpreting that the negative value means that "the flow behind must be increased", that is, a negative flow in one of the routes is a consequence of a positive flow in the rear route, to keep the flow constant. Fátima and Heitor, like all other students, did not validate their results, which led them to answer the problem situation transferring the mathematical results directly to the problem situation. So, their competencies for validating mathematical results were not evidenced.

CONCLUSIONS

In this study we have analysed, from a cognitive perspective, the modelling routes performed by Costa Rican undergraduate students when solving a mathematical modelling task involving concepts associated to systems of linear equations. The purpose of the study was to get insights into their learning of the linear algebra concepts and on their modelling competencies, when tackling a real-world problem. The results show a variety of students' modelling routes: those that performed partial or complete cycles; those who follow linear or non-linear routes; and those that used technology or not to build a mathematical model and work on that model.

The modelling task proved to be cognitively demanding for the students (Blum & Borromeo Ferri, 2009), but also suitable, both for working on linear algebra concepts such as the SLE, augmented matrix, and set solution and for developing modelling competencies such as creating adequate real models or interpreting and validating results (Possani et al., 2010). Stronger modelling competencies were evident in the groups that performed non-linear modelling routes and reflected about the mathematical procedures to work on the model and properly mobilized their knowledge about the concepts of SLE, associated matrix and solution set in $\mathbb{R}^n$. Those competencies were not revealed by most groups that performed linear routes, which reflects their difficulties in creating real models, mainly associated with a limited understanding of the context, and resulting in an inappropriate interpretation of the information provided, as also observed by Mallet (2007), and not so much with difficulties in understanding the task and its context. Moreover, in these groups there was no reflection on the concept of a solution set when mathematical models were built based on trial-and-error strategies, and their work with analytical models also revealed incorrect constructions of the matrix associated with a SLE, which may be associated with insufficient learning about the concept of augmented matrix and solution set, and about processes to find the solution set.

In the only modelling route supported by technology, the students' mastery of technological resources to work on the mathematical model did not increase their competencies to go through all the phases of the modelling cycle, since they stopped their route with the obtaining of computational results. They revealed difficulties in interpreting the mathematical results due to lack of knowledge about the concepts of SLE and associated matrix. This reinforces the need to make technology relevant for students in modelling tasks (Greefrath et al., 2018) for constructing effective computer models that have an actual operating nature and represent the functioning of the real model. It also suggests the need for students to master the required algebra knowledge to take advantage of using technology in their modelling process (Galbraith & Stillman, 2006).

It should also be noted that the validation process was a step that students did not take, including the cases of those groups who reached the stage of getting real results. This can be possibly explained by the students' lack of experience in working on tasks with real contexts, as they did not have the chance to develop competencies to validate their mathematical results when working on tasks in pure mathematical contexts (Sokolowski, 2015), nor did they practice this activity in other tasks (Borromeo Ferri, 2018). Therefore, it is necessary to consider the development of this competency in the classroom, either in learning SLE or other topics of linear algebra, as the validation of results is crucial in several other tasks.

Looking at the students' learning and modelling competencies, the results are consistent with other studies on modelling tasks involving the SLE concepts (e.g., Possani et al. 2010), but they also raise new insights regarding their difficulties to relate the information given in tabular and graphical representations to formulate real models. The results also show that non-linear routes, when compared with the linear routes performed, are associated with a greater mobilization of students' algebraic knowledge on SLE concepts and with stronger modelling competencies. Moreover, this study may contribute to extend the previous research about the teaching and learning of linear algebra, namely the SLE concepts that are still little investigated. Particularly, the modelling routes performed by the students in the task may provide useful information for teachers to diagnose the students' knowledge on SLE and their modelling competencies, and also to detect their difficulties, which can be a starting point to guide the planning and implementation of more effective teaching practices. However, further research is needed, especially of an interventional nature, focusing on other topics of linear algebra and on the role of technology in the teaching and learning of linear algebra using a mathematical modelling approach, to illuminate how to make the use of technological tools relevant for the students' work on modelling tasks and for developing their knowledge on fundamental concepts.

CONFLICT OF INTEREST STATEMENT

On behalf of all authors, the corresponding author states that there is no conflict of interest.

REFERENCES

Alsina, C. (2007). Teaching applications and modelling at tertiary level. In W. Blum, P. Galbraith, H. W. Henn, & M. Niss (Eds.), *Modelling and Applications in Mathematics Education: The 14th ICMI study* (pp. 469–474). New York, NY: Springer.

Bernabeu, M., Moreno, M., & Llinares, S. (2019). Experimento de enseñanza como una aproximación metodológica a la investigación en Educación Matemática. *Uni-Pluriversidad, 19*(2), 103-123.

Bianchini, B. L., de Lima, G. L., & Gomes, E. (2019). Linear algebra in engineering: an analysis of Latin American studies**.** *ZDM Mathematics Education*, *51*(7), 1097-1110. https://doi.org/10.1007/s11858-019-01081-5

Blum, W. (2015). Quality teaching of mathematical modelling: What do we know, what can we do? In S. J. Cho (Ed.), *Proceedings of the 12th International Congress on Mathematical Education - Intellectual and Attitudinal Challenges* (pp. 73-96). New York, NY: Springer.

Blum, W., & Borromeo Ferri, R. (2009). Mathematical modelling: can it be taught and learnt? *Journal of Mathematical Modelling and Application, 1*(1), 45–58.

Borromeo Ferri, R. (2007). Personal experiences and extra-mathematical knowledge as an influence factor on modelling routes of pupils. In D. Pitta-Pantazi & G. Philippou (Eds.), *Proceedings of the 5th Congress of the European Society for Research in Mathematics Education* (CERME 5) (pp. 2080-2089). Larnaca, Cyprus: University of Cyprus and ERME.

Borromeo Ferri, R. (2018). *Learning how to teach mathematical modelling in school and teacher education*. Cham, Switzerland: Springer. https://doi.org/10.1007/978-3-319-68072-9

Cai, J., Cirillo, M., Pelesko, J. A., Borromeo Ferri, R., Borba, M., Geiger, V., …. Kwon, O. N. (2014). Mathematical modelling in school education: Mathematical, cognitive, curricular, instructional, and teacher education perspectives. In P. Liljedahl, C. Nicol, S. Oesterle, & D. Allen (Eds.), *Proceedings of the Joint Meeting of PME 38 and PME-NA 36* (Vol. 1, pp. 145–172). Vancouver, Canada: PME.

Cohen, L., Manion, L., & Mohinson, K. (2007). *Research methods in education* (*6th ed*.). New York, NY: Routledge.

Costa, V. A., & Rossignoli, R. (2017). Enseñanza del álgebra lineal en una facultad de ingeniería: Aspectos metodológicos y didácticos. *Revista Educación en Ingeniería*, *12*(23), 49-55.

Czocher, J. (2018). How does validating activity contribute to the modelling process? *Educational Studies in Mathematics*, *99*, 137–159. https://doi.org/10.1007/s10649-018-9833-4

Galbraith, P., & Stillman, G. (2006). A framework for identifying student blockages during transitions in the modelling process. *ZDM Mathematics Education*, *38*(2), 143–162.

Greefrath, G., Hertleif, C., & Siller, H. (2018). Mathematical modelling with digital tools - a quantitative study on mathematising with dynamic geometry software. *ZDM Mathematics Education*, *50*, 233–244.

Kaiser, G., & Sriraman, B. (2006). A global survey of international perspectives on modelling in mathematics education. *ZDM Mathematics Education*, *38*(3), 302–310.

Larson, C., & Zandieh, M. (2013). Three interpretations of the matrix equation Ax= b. *For the Learning of Mathematics*, *33*(2), 11-17.

Legorreta, C. O., & Andalón, J. L. (2016). Un sistema de audio asociado al concepto de combinación lineal. *Acta Latino americana de Matemática Educativa, 21*, 668–679.

Maaß, K. (2006). What are modelling competencies? *ZDM Mathematics Education, 38*(2), 113–142.

Mallet, D. G. (2007). Multiple representations for systems of linear equations via the computer algebra system Maple. *International Electronic Journal of Mathematics Education*, *2*(1), 16–32.

Niss, M., Blum, W., & Galbraith, P. (2007). Introduction. In W. Blum, P. Galbraith, H.-W. Henn, & M. Niss (Eds.), *Modelling and applications in mathematics education: The 14th ICMI study* (pp. 3–32). New York, NY: Springer.

Nomura, J. I., & Bianchini, B. L. (2009). Entendendo por que e como deve ser lecionada a disciplina Álgebra Linear em uma graduação de Engenharia Elétrica. In *Anais do IV SIPEM*. Taguatinga, Brazil: Sociedade Brasileira de Educação Matemática,

Parraguez, M. G. (2009). *Evolución Cognitiva del Concepto Espacio Vectorial*. (Ph.D. Thesis), Instituto Politécnico Nacional, Mexico.

Porras, P. C., Silverio, A. Q., & Vargas, E. H. (2010). Propuesta metodológica para la resolución de problemas de corrientes a través de sistemas de ecuaciones diferenciales lineales de primer orden utilizando valores y vectores propios. *Acta Latinoamericana de Matemática Educativa*, *23*, 437–444.

Possani, E., Trigueros, M., Preciado, J., & Lozano, M. D. (2010). Use of models in the teaching of linear algebra. *Linear Algebra and its Applications*, *432*, 2125–2140.

Roa-Fuentes, S., & Parraguez, M. (2017). Estructuras Mentales que Modelan el Aprendizaje de un Teorema del Álgebra Lineal: Un Estudio de Casos en el Contexto Universitario. *Formación Universitaria.*, *10*(4), 15–32.

Rosa, M., & Orey, D. (2012). A modelagem como um ambiente de aprendizagem para a conversão do conhecimento matemático. *Bolema*, *26*(42a), 261–290.

Salgado, H., & Trigueros, M. (2015). Teaching eigenvalues and eigenvectors using models and APOS theory. *The Journal of Mathematical Behavior*, *39*, 100–120.

Sánchez, J. (2019). *Carta al estudiante de curso MA1004: I semestre*. Universidad de Costa Rica.

Siller, H-S., & Greefrath, G. (2010). Mathematical modelling in class regarding to technology. In V. Durand-Guerrier, S. Soury-Lavergne, & F. Arzarello (Eds.), *Proceedings of the 6th Congress of the European Society for Research in Mathematics Education* (CERME 6) (pp. 2136–2145). Lyon, France: Institut National de Recherche Pédagogique and ERME.

Silva, M. E. S. C. (2016). *Concepção de transformação linear por estudantes de Licenciatura em Matemática.* (Ph.D. Thesis), Pontifícia Universidade Católica de São Paulo, São Paulo, Brasil.

Sokolowski, A. (2015). The effects of mathematical modelling on students' achievement – meta-analysis of research. *The IAFOR Journal of Education*, *3*(1), 93–104.

Stewart, S., Andrews-Larson, C., & Zandieh, M. (2019). Linear algebra teaching and learning: themes from recent research and evolving research priorities. *ZDM Mathematics Education 51*(7), 1017–1030. https://doi.org/10.1007/s11858-019-01104-1

Trigueros, M., & Bianchini, B. (2016). Learning linear transformations using models. In E. Nardi, C. Winsløw, & T. Hausberger (Eds.), *Proceedings of the First Conference of the International Network for Didactic Research in University Mathematics*. Montpellier, France: University of Montpellier and INDRUM. https://hal.archives-ouvertes.fr/hal-01337884

Trigueros, M., & Possani, E. (2013). Using an economics model for teaching linear algebra. *Linear Algebra and its Applications*, *438*(4), 1779–1792.

Wawro, M., Rasmussen, C., Zandieh, M., Sweeney, G. F., & Larson, C. (2012). An inquiry-oriented approach to span and linear independence: The case of the magic carpet ride sequence. *PRIMUS, 22*(8), 577–599.

Zandieh, M., & Andrews-Larson, C. (2019). Symbolizing while solving linear systems. *ZDM Mathematics Education, 51*(7), 1183-1197.