

# $S$ -matrix from the metaplectic representation

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## Abstract

We show how the  $S$ -matrix for bosons in an external field can be derived directly from the infinite dimensional metaplectic representation, in terms of the classical scattering operator.

## 1 Introduction

For bosons or fermions interacting with an external field, the full  $S$ -matrix can be written in closed form. Explicit expressions have been exploited, in the fermion case, to study gauge group representations [1], “charged vacua” [2], and for standard QED calculations [3]. To the best of our knowledge, a similar formula in the boson case has been given only in [4] (implicit equivalent expressions are found, for instance, in [5, 6]).

In this letter, we point out that a clean and short derivation for the  $S$ -matrix of a scalar boson field in the presence of an external classical field is available in terms of the (infinite dimensional) metaplectic representation. Moreover, our derivation is completely rigorous and works both for the neutral and charged fields.

We begin with a brief account of the metaplectic representation in Bargmann space, extending work by Vergne [7] and Robinson and Rawnsley [8]; the latter concern themselves only with the finite dimensional case, but much of their treatment carries over to the present context. Then, we spell out the dictionary to translate from Bargmann’s to the standard version of Fock space. The main general result follows therewith. We finally give a second version of it, valid and more suitable for charged fields.

## 2 The metaplectic representation in Bargmann space

The classical manifold underlying the boson field is just a symplectic vector space, i.e., a real vector space  $V$  with a symplectic form  $s$ . The primary examples are spaces of solutions of the

Klein–Gordon type. Pick a complex structure  $J$ , i.e., a real-linear operator on  $V$  satisfying  $J^2 = -1$ , which leaves  $s$  invariant and such that  $s(v, Jv) > 0$ , for  $v \neq 0$ . Then  $d_J(u, v) := s(u, Jv)$  is a positive definite symmetric form on  $V$ . Thus  $V$  can be regarded as a complex vector space with the obvious rule  $(\alpha + i\beta)v := \alpha v + \beta Jv$  for  $\alpha, \beta$  real, and then the hermitian form

$$\langle u | v \rangle := s(u, Jv) + is(u, v) = d_J(u, v) + id_J(Ju, v) \quad (1)$$

is a positive definite inner product on  $V$ . We will henceforth assume that  $V$  is complete for the inner product (1), and is thus the underlying real space to a complex Hilbert space; and we will suppose that  $V$  is separable. However, the real part  $d_J$  of the inner product (whose imaginary part is  $s$ ) is not unique, since it depends on  $J$ . We will denote that Hilbert space by  $V$  also. Note that the topology of  $V$  determined by  $d_J$  is independent of  $J$ .

An (antiholomorphic, homogeneous) polynomial of degree  $m$  on  $V$  is a complex function on  $V$  of the form  $v \mapsto E(v, v, \dots, v)$ , where  $E$  is continuous and  $m$ -antilinear. An antiholomorphic function on  $V$  is the sum of a series of such polynomials, absolutely and uniformly convergent on bounded sets of finite dimension. (That is, we use Gâteaux holomorphicity; the usual Fréchet holomorphicity, involving convergence on norm bounded sets would yield much too small a space for our purposes.)

If  $V'$  is any finite dimensional subspace of  $V$ , we denote by  $e^{-\frac{1}{2}\langle u|u \rangle} du$  for  $u \in V'$  the gaussian measure on  $V'$ , which we suppose normalized so that the integral of 1 equals 1. Denote by  $T(V)$  the space of *tame* functions, i.e., functions of the form  $f(u) = F(P_{V'}u)$ , where  $F$  is a polynomial. For such a function we define:

$$\|f\|_B^2 := \int_{V'} |F(u)|^2 e^{-\frac{1}{2}\langle u|u \rangle} du.$$

Let  $\mathcal{B}(V)$  be the completion of  $T(V)$  for this Hilbert norm, and  $\langle - | - \rangle_B$  the corresponding inner product. It can be shown [9] that  $\mathcal{B}(V)$  coincides with the space of antiholomorphic functions on  $V$  such that  $\|f\|_B^2 := \sup_{V'} \int_{V'} |f(u)|^2 e^{-\frac{1}{2}\langle u|u \rangle} du < \infty$ , where  $V'$  ranges over the finite dimensional complex subspaces of  $V$ .

For  $f \in \mathcal{B}(V)$  and  $v \in V$ , we consider

$$\beta(v)F(u) := \exp(\frac{1}{4}\langle 2u - v | v \rangle)F(u - v). \quad (2)$$

This defines an irreducible Weyl system:

$$\beta(v)\beta(v') = \beta(v + v') \exp[-\frac{i}{2}s(v, v')].$$

Denote  $E_v(u) := \exp(\frac{1}{2}\langle u | v \rangle)$ . Then  $E_v \in \mathcal{B}(V)$  and for any  $\Psi \in \mathcal{B}(V)$ , we have  $\langle E_v | \Psi \rangle_B = \Psi(v)$ . These are well-known properties of the system of coherent states in Bargmann space, that generalize to the infinite dimensional case. We write  $|0\rangle$  to denote the vacuum (the constant function  $E_0 = 1$ ).

Consider the group  $\text{Sp}$  of invertible continuous real-linear transformations leaving  $s$  invariant. One has  $g \in \text{Sp}$  iff  $gJ = Jg^{-t}$ , where the superscript ' $t$ ' means transposition with respect to  $d$ , and  $g^{-t} := (g^t)^{-1}$ . We may decompose any real-linear operator  $g$  on  $V$  into linear and antilinear parts by

$$p_g := \frac{1}{2}(g - JgJ), \quad q_g := \frac{1}{2}(g + JgJ).$$

Then  $g \in \text{Sp}$  iff  $p_g = \frac{1}{2}(g + g^{-t})$ ,  $q_g = \frac{1}{2}(g - g^{-t})$ . If  $g \in \text{Sp}$ , then  $p_g$  is invertible; we define  $T_g := q_g p_g^{-1}$  and abbreviate  $\widehat{T}_g := T_g^{-1}$ .

We can parametrize  $g \in \text{Sp}$  by the pair  $(p_g, T_g)$ ; for  $g$  can be written in a unique way as  $g = (1 + T_g)p_g$ , where  $T_g$  is antilinear and symmetric and  $1 - T_g^2$  is positive definite;  $p_g$  is linear and satisfies  $p_g^t(1 - T_g^2)p_g = 1$ . Moreover, for any pair  $(p, T)$  of real-linear operators on  $V$  satisfying these conditions, the operator  $g := (1 + T)p$  belongs to  $\text{Sp}$ .

Consider now the Weyl system  $\beta_g(v) := \beta(gv)$ . This is unitarily equivalent to  $\beta$  iff  $T_g$  is a real Hilbert–Schmidt operator. This is Shale’s theorem [10], which we readily prove using Bargmann space techniques. By definition, the metaplectic representation intertwines  $\beta$  and  $\beta_g$ :

$$\nu(g)\beta(v)\nu(g)^{-1} = \beta(gv). \quad (3)$$

This defines  $\nu(g)$  up to multiplication by a complex number of absolute value 1.

If  $d\beta$  is the derived representation:

$$d\beta(v)f(u) := \left. \frac{d}{dt} \right|_{t=0} \beta(tv)f(u), \quad (4)$$

then  $\nu(g)d\beta(v) = d\beta(gv)\nu(g)$  from (3). Since only the vacuum  $|0\rangle$  is annihilated by any  $d\beta(v) + i d\beta(Jv)$ ,  $\nu(g)|0\rangle$  is annihilated by any  $d\beta(gv) + i d\beta(Jgv)$ ; using (2), this yields a differential equation whose solution is  $\nu(g)|0\rangle = c_g \exp(\frac{1}{4}\langle u | T_g u \rangle) =: c_g f_{T_g}(u)$ , and from unitarity of  $\nu(g)$ , the constant  $c_g$  satisfies  $|c_g| = \det^{1/4}(1 - T_g^2)$ . Given that  $1 - T_g^2 > 0$ , this converges if and only if  $T_g$  is Hilbert-Schmidt.

All operators on  $\mathcal{B}(V)$  whose domains contain the principal vectors  $E_v$ , have integral kernels:

$$Af(u) = \langle \overline{K}_A(u, -) | f \rangle_B.$$

In particular:

$$K_{\nu(g)}(u, v) = c_g \exp \frac{1}{4} \{ \langle u | T_g u \rangle + 2 \langle p_g^{-1} u | v \rangle + \langle \widehat{T}_g v | v \rangle \}. \quad (5)$$

Indeed,  $K_{\nu(g)}(u, v) = \nu(g)E_v(u) = e^{\frac{1}{4}\langle v | v \rangle} \beta(gv)\nu(g)|0\rangle(u)$ , from which (5) follows.

Adopt  $c_g := \det^{1/4}(1 - T_g^2)$  for definiteness. Then it is found by a direct computation, using the theory of Gaussian integration in infinite dimensional spaces [11], that  $\nu(g)\nu(h) = c(g, h) \nu(gh)$ , where the group cocycle is given by

$$c(g, h) = \exp(i \arg \det^{-1/2}(1 - T_h \widehat{T}_g)). \quad (6)$$

### 3 The S-matrix

The space  $\mathcal{B}(V)$  is just the boson Fock space under a thin disguise. To make the necessary translation, consider the correspondence:

$$(u \mapsto 2^{-n/2} \langle u | v_1 \rangle \cdots \langle u | v_n \rangle) \longleftrightarrow v_1 \vee \cdots \vee v_n \in V^{\vee n} \quad (7)$$

between antiholomorphic polynomials and elements of the symmetric algebra  $S(V)$  of  $V$ . With the obvious inner product on  $S(V)$ , this extends to an isometry between  $\mathcal{B}(V)$  and boson Fock space  $\overline{S(V)}$ .

Introduce now the boson field  $\phi := -i d\beta$ . The annihilation and creation operators for the boson field  $\phi$  are the real-linear operators on  $\mathcal{B}(V)$ :

$$a(v) := \frac{1}{\sqrt{2}}[\phi(v) + i\phi(Jv)], \quad a^\dagger(v) := \frac{1}{\sqrt{2}}[\phi(v) - i\phi(Jv)]. \quad (8)$$

It is clear that  $a(v)$  is antilinear and  $a^\dagger(v)$  is linear in  $v$ , and the canonical commutation relations:  $[a(v), a(v')] = 0$  and  $[a(v), a^\dagger(v')] = \langle v | v' \rangle$  hold.

It follows from (2) and (4) that  $a(v) = i\sqrt{2}D_v$  and  $a^\dagger(v) = -(i/\sqrt{2})v$  as differentiation and multiplication operators on  $\mathcal{B}(V)$ . In particular, each  $a(v)$  annihilates the vacuum  $|0\rangle$ . Note that

$$a^\dagger(v_1)a^\dagger(v_2)\cdots a^\dagger(v_n)|0\rangle = (-i)^n v_1 \vee v_2 \vee \cdots \vee v_n, \quad (9)$$

in  $\mathcal{B}(V)$ , by the convention (7).

The principal vectors are generated from the vacuum by

$$E_v = \exp\left(\frac{i}{\sqrt{2}}a^\dagger(v)\right)|0\rangle.$$

These are smooth vectors for all creation and annihilation operators, and we have

$$\exp\left(\frac{i}{\sqrt{2}}a^\dagger(v)\right)E_{v'} = E_{v+v'}, \quad \exp\left(-\frac{i}{\sqrt{2}}a(v)\right)E_{v'} = e^{\frac{1}{2}\langle v|v'\rangle}E_{v'}.$$

The effect of the metaplectic representation on the creation and annihilation operators is readily determined. Let us write

$$a_g(v) := \frac{1}{\sqrt{2}}[\phi(v) + i\phi(gJg^{-1}v)], \quad a_g^\dagger(v) := \frac{1}{\sqrt{2}}[\phi(v) - i\phi(gJg^{-1}v)],$$

in accordance with (8), for any  $g \in \mathrm{Sp}'(V)$ . Since  $gJv = (p_g + q_g)Jv = J(p_g - q_g)v$ , we obtain the Bogoliubov transformation:

$$a_g(gv) = a(p_g v) + a^\dagger(q_g v), \quad a_g^\dagger(gv) = a(q_g v) + a^\dagger(p_g v).$$

We have immediately:

$$\nu(g)a(v) = a_g(gv)\nu(g), \quad \nu(g)a^\dagger(v) = a_g^\dagger(gv)\nu(g),$$

so that each  $a_g(gv)$  annihilates the “out-vacuum”  $|0_{\mathrm{out}}\rangle := \nu(g)|0\rangle = c_g f_{T_g}$ .

We are ready to translate (5) to give a closed expression for the  $S$ -matrix in Fock space. Indeed, this is an abstract result about Bogoliubov transformations. However, to fix ideas, one can think of  $V$  as the space of solutions for a Klein–Gordon equation (say, covariant in a static, globally hyperbolic spacetime) with external interaction,  $s$  as the usual symplectic form on  $V$ , and  $J$  is chosen as the unique complex structure commuting with the time evolution generator for the Klein–Gordon equation when the external interaction is switched off.

Then the classical scattering operator  $S_{\mathrm{cl}}$  belongs to  $\mathrm{Sp}$ . If, moreover,  $T_{S_{\mathrm{cl}}}$  is of the Hilbert–Schmidt class, then its quantum counterpart  $\nu(S_{\mathrm{cl}})$  is the  $S$ -matrix, except for the phase factor.

We may factorize (5) as follows:

$$\nu(g) E_v = c_g \exp\left(\frac{1}{4}\langle \widehat{T}_g v \mid v \rangle\right) f_{T_g} E_{p_g^{-t} v}.$$

For  $v = 0$ , we see that  $c_g = |0_{\text{out}}\rangle|$ , so that  $c_g$  is just the absolute value of the vacuum persistence amplitude. We shall maintain the notation  $c_g$  for brevity's sake. We want now to factorize  $\nu(g)$  as

$$\nu(g) = c_g \mathbf{S}_1 \mathbf{S}_2 \mathbf{S}_3,$$

where the  $\mathbf{S}_i$ ,  $1 \leq i \leq 3$ , are operators on  $\mathcal{B}(V)$ , to be chosen so that

$$\mathbf{S}_3 E_v = \exp\left(\frac{1}{4}\langle \widehat{T}_g v \mid v \rangle\right) E_v, \quad \mathbf{S}_2 E_v = E_{p_g^{-t} v}, \quad \mathbf{S}_1 E_{v'} = f_{T_g} E_{v'}.$$

Since the  $a^\dagger(v)$  act as multiplication operators,  $\mathbf{S}_1$  need only satisfy  $\mathbf{S}_1 |0\rangle = f_{T_g}$  if made up of creation operators only. From (7), (9) with  $n = 2$ , we have  $a^\dagger(v_1) a^\dagger(v_2) |0\rangle(u) = -\frac{1}{2}\langle u \mid v_1 \rangle \langle u \mid v_2 \rangle$ , so we take

$$\mathbf{S}_1 := \exp\left(-\frac{1}{2} a^\dagger T_g a^\dagger\right),$$

with

$$a^\dagger T_g a^\dagger := \sum_{j,k} a^\dagger(f_k) \langle f_k \mid T_g e_j \rangle a^\dagger(e_j), \quad (10)$$

where  $\{e_j\}, \{f_k\}$  are orthonormal bases for  $V$ ; this series converges (to an operator whose domain includes all  $E_v$ ) iff  $T_g$  is Hilbert–Schmidt; and the sum is independent of the orthonormal bases used. We may take  $f_k = e_k$  to be eigenvectors for the positive traceclass operator  $T_g^2$ ; since  $T$  is symmetric, we may choose it so that  $T e_k = \lambda_k J e_k$  and  $T J e_k = \lambda_k e_k$ , with each  $\lambda_k \geq 0$ . Then  $a^\dagger T_g a^\dagger = \sum_k i \lambda_k a^\dagger(e_k) a^\dagger(e_k)$ .

Similarly we define a formal adjoint to (10):

$$a T_g a := \sum_{j,k} a(e_j) \langle T_g e_j \mid f_k \rangle a(f_k),$$

and check that  $(a T_g a) E_v = -\frac{1}{2} \langle T_g v \mid v \rangle E_v$ . Substituting  $g^{-1}$  for  $g$ , we get at once:

$$\mathbf{S}_3 = \exp\left(-\frac{1}{2} a \widehat{T}_g a\right).$$

To obtain an expression for  $\mathbf{S}_2$ , we mix creation and annihilation operators with the usual normal ordering rule. Then  $\mathbf{S}_2$  is a Wick-ordered exponential:

$$\begin{aligned} \mathbf{S}_2 &= :\exp(a^\dagger C a): \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\substack{k_1 \dots k_n \\ l_1 \dots l_n}} a^\dagger(f_{k_1}) \cdots a^\dagger(f_{k_n}) \langle f_{k_1} \mid C e_{l_1} \rangle \cdots \langle f_{k_n} \mid C e_{l_n} \rangle a(e_{l_1}) \cdots a(e_{l_n}), \end{aligned} \quad (11)$$

where  $C$  is some bounded linear operator on  $V$ . Now,  $\mathbf{S}_2$  makes sense in principle as a quadratic form whose domain includes all  $E_v$ . Indeed:

$$\langle E_{v'} \mid \mathbf{S}_2 E_v \rangle = \sum_{p=0}^{\infty} \frac{1}{2^p (p!)^2} \langle 0 \mid a(v')^p \mathbf{S}_2 a^\dagger(v)^p \mid 0 \rangle,$$

with

$$\begin{aligned}
& \langle 0 | a(v')^p \mathbf{S}_2 a^\dagger(v)^p | 0 \rangle \\
&= \sum_{n=0}^p \frac{1}{n!} \sum_{k_1 \dots k_n l_1 \dots l_n} \left\langle 0 \left| a(v')^p a^\dagger(f_{k_1}) \cdots a^\dagger(f_{k_n}) \prod_{j=1}^n \langle f_{k_j} | C e_{l_j} \rangle a(e_{l_1}) \cdots a(e_{l_n}) a^\dagger(v)^p \right| 0 \right\rangle \\
&= \sum_{n=0}^p \binom{p}{n} \langle v' | v \rangle^{p-n} \sum_{k_1 \dots k_n l_1 \dots l_n} \prod_{j=1}^n \langle v' | f_{k_j} \rangle \langle f_{k_j} | C e_{l_j} \rangle \langle e_{l_j} | v \rangle \\
&= \sum_{n=0}^p \binom{p}{n} \langle v' | v \rangle^{p-n} \langle v' | C v \rangle^n = \langle v' | v + C v \rangle^p,
\end{aligned}$$

so that  $\langle E_{v'} | \mathbf{S}_2 E_v \rangle$  converges and equals  $\langle E_{v'} | E_{(1+C)v} \rangle$ . Thus we have  $:\exp(a^\dagger C a): E_v = E_{(1+C)v}$  and so  $\mathbf{S}_2 := :\exp(a^\dagger(p_g^{-t} - 1)a):$  is what we need. We finally arrive at the explicit  $S$ -matrix:

$$\mathbf{S} = e^{i\theta} \nu(g) = e^{i\theta} c_g \exp(-\tfrac{1}{2} a^\dagger T_g a^\dagger) :\exp(a^\dagger(p_g^{-t} - 1)a): \exp(-\tfrac{1}{2} a \widehat{T}_g a)$$

in terms of the  $(p_g, T_g)$  parametrization of the element  $g = S_{\text{cl}}$  of the symplectic group on  $V$ ; it is now clear that the  $S$ -matrix exists if and only if  $T_g$  is Hilbert–Schmidt.

The phase factor  $e^{i\theta}$  remains to be computed. For that, we use the fact that generally  $S_{\text{cl}} = g(+\infty, -\infty)$ , where  $g(t, s)$ , for  $t \geq s$ , gives the evolution in the interaction picture [5] and satisfies  $\frac{\partial}{\partial t} g(t, s) = e^{-At} V(t) e^{At} g(t, s)$ , where  $A$  is the generator of the free evolution (which commutes with  $J$ ) and  $V(t)$  is the external interaction “potential”. We make the Ansatz that the quantum evolution operator is given by

$$U(t, s) := e^{i\theta(t, s)} \nu(g(t, s)), \quad \mathbf{S} = U(+\infty, -\infty).$$

From  $U(t, r) = U(t, s)U(s, r)$  for  $t \geq s \geq r$ , we obtain

$$e^{i\theta(t, r)} = e^{i\theta(t, s)} e^{i\theta(s, r)} c(g(t, s), g(s, r)). \quad (12)$$

We may as well suppose that  $\frac{\partial}{\partial t} \Big|_{t=s} \theta(t, s) = 0$  (which is a kind of “normal ordering” rule), whereupon (12) yields

$$\theta(t, r) = -i \int_r^t \frac{\partial}{\partial \tau} \Big|_{\tau=s} c(g(\tau, s), g(s, r)) ds.$$

From (6) we obtain

$$\frac{\partial}{\partial \tau} \Big|_{\tau=s} c(g(\tau, s), g(s, r)) = -\tfrac{1}{4} \text{Tr}_{\mathbb{C}} \left( \left[ \frac{\partial}{\partial \tau} \Big|_{\tau=s} \widehat{T}_{g(\tau, s)}, T_{g(s, r)} \right] \right),$$

(which, as the complex trace of a commutant of antilinear operators, is imaginary) and we thus find for the phase of the scattering matrix:

$$e^{i\theta} = e^{i\theta(+\infty, -\infty)} = \exp \left\{ \frac{1}{8} \int_{-\infty}^{\infty} \text{Tr}_{\mathbb{C}} ([e^{-At} (V(t) + JV(t)J) e^{At}, T_{g(t, -\infty)}]) dt \right\}.$$

## 4 The charged field

It is standard practice to identify the classical phase space for a particle-antiparticle field with the complex space  $V_{\mathbb{C}} = V \oplus iV$ , instead of  $V \oplus V$ . The operators  $g, T_g$ , etc., in the present context are complex-linear operators defined on  $V_{\mathbb{C}}$ . Thus we have  $J = i(P_+ - P_-)$ , where  $P_+, P_-$  are the projectors on the respective subspaces  $W, W^*$  of positive- and negative-energy one-particle solutions. Since  $V_{\mathbb{C}} = W \oplus W^*$  as vector spaces, we can write the operators in matrix form:

$$g = \begin{pmatrix} S_{++} & S_{+-} \\ S_{-+} & S_{--} \end{pmatrix} \in \text{Sp}.$$

Then  $p_g$  is the diagonal part and  $q_g$  is the off-diagonal part of  $g$ . Hence

$$T_g = \begin{pmatrix} 0 & S_{+-}S_{--}^{-1} \\ S_{-+}S_{++}^{-1} & 0 \end{pmatrix}, \quad \widehat{T}_g = \begin{pmatrix} 0 & -S_{++}^{-1}S_{+-} \\ -S_{--}^{-1}S_{-+} & 0 \end{pmatrix},$$

which are Hilbert–Schmidt operators iff  $S_{+-}$  or  $S_{-+}$  are Hilbert–Schmidt.

Denote by  $\langle\langle - | - \rangle\rangle$  the Hilbert space product on  $V_{\mathbb{C}}$ , given by  $\langle\langle w_1 | w_2 \rangle\rangle := 2s(w_1^*, Jw_2)$ , so that  $\langle\langle P_+u | P_+v \rangle\rangle = \langle u | v \rangle$  and  $\langle\langle P_-u | P_-v \rangle\rangle = \langle v | u \rangle$ . Let  $\{\varphi_1, \varphi_2, \dots\}$ , respectively  $\{\psi_1, \psi_2, \dots\}$  be orthonormal bases in  $W, W^*$  for this inner product, and let  $f_k, e_k$  be the basis vectors in separate copies of  $V$  for which  $P_+(f_k) = \varphi_k, P_-(e_k) = \psi_k$ . We distinguish the positive and negative charge sectors identifying  $b^\dagger(\varphi_k) := a^\dagger(f_k), d^\dagger(\psi_k) := a^\dagger(e_k)$ , etc. Then we find that  $-\frac{1}{2}a^\dagger T_g a^\dagger$  goes over to

$$-\sum_{j,k} b^\dagger(\varphi_k) \langle\langle \varphi_k | T_g \psi_j \rangle\rangle d^\dagger(\psi_j) =: -b^\dagger S_{+-} S_{--}^{-1} d^\dagger.$$

Note the absence of charge-nonpreserving terms  $b^\dagger b^\dagger$  and  $d^\dagger d^\dagger$ . With similar notational conventions, we obtain that  $-\frac{1}{2}a \widehat{T}_g a$  goes over to  $d S_{--}^{-1} S_{-+} b$  and the Wick-ordered exponential to  $:\exp(b^\dagger((S_{++}^\dagger)^{-1} - 1)b + d(S_{--}^{-1} - 1)d^\dagger):$ . In summary, the  $S$ -matrix for the charged boson field has the explicit form:

$$\mathbf{S} = e^{i\theta} c_g \exp(-b^\dagger S_{+-} S_{--}^{-1} d^\dagger) : \exp(b^\dagger((S_{++}^\dagger)^{-1} - 1)b + (d(S_{--}^{-1} - 1)d^\dagger) : \exp(d S_{--}^{-1} S_{-+} b). \quad (13)$$

## 5 Remarks about the fermionic S-matrix

It is interesting to compare formula (11) with the one for Fermi fields given – minus the phase factor – in [3, Eq. (2.4.61)], with the analogous notation. Indeed, it is possible to derive the fermion scattering matrix in a completely parallel way with the approach outlined here for boson fields. For fermions, the one-particle space  $V$  carries a symmetric bilinear form  $d$ , and its symmetries form an orthogonal group. Before quantizing, one must again choose a complex structure  $J$ , this time an orthogonal one; and the appropriate symmetry group is the restricted orthogonal group of those orthogonal transformations  $g$  of  $V$  for which  $[J, g]$  is Hilbert–Schmidt.

The fermion Fock space is an exterior algebra over  $V$ , regarded as a complex Hilbert space under the inner product  $\langle u | v \rangle := d(u, v) + id(Ju, v)$ . Although no Bargmann representation is available for the fermion Fock space, it does have a holomorphic presentation [12].

The (infinite dimensional) pin representation  $\nu_F$  takes the group of restricted orthogonal operators into unitary operators on the fermion Fock space, and the  $S$ -matrix is  $\nu_F(S_{\text{cl}})$ . Writing  $g = S_{\text{cl}}$ ,

one may express the out-vacuum vectors in the form  $|0_{\text{out}}\rangle := e^{i\theta} \nu_F(g) |0\rangle = e^{i\theta} c_g f_{T_g}$ , where  $T_g = q_g p_g^{-1}$  as before, but only locally, as  $p_g$  is generally not invertible for the orthogonal group; nevertheless, since  $p_g$  is a Fredholm operator of index zero for restricted orthogonal  $g$ , this obstacle may be overcome and an explicit expression for  $\nu_F(g)$  may be derived in all cases. On rewriting  $\nu_F(g)$  in terms of creation and annihilation operators satisfying the canonical anticommutation relations, the path *ut supra* leads us to the analogue of (13) for the  $S$ -matrix.

The pin representation is again projective and its cocycle is again directly related to the phase of the vacuum persistence amplitude. In fact, this phase and thus the renormalized polarization tensor in QED, can be computed from the cocycle without ever encountering the divergence that results from the naïve use of Feynman diagrams. We shall deal with these questions elsewhere.

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