ISOMETRY GROUP OF BOREL RANDOMIZATIONS

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Abstract. We study global dynamical properties of the isometry group of the Borel randomization of a separable complete structure. In particular, we show that if properties such as the Rohklin property, topometric generics, extreme amenability hold for the isometry group of the structure, they also hold in the isometry group of the randomization.

1. Introduction

This paper deals with structural properties of isometry groups of randomizations of metric structures (see [BYK09]), in particular the existence of generic elements.

The setting is the following. Given G a Polish group, we say that an element $g \in G$ is generic if its orbit under conjugation $\{g^h : h \in G\}$ is comeager and we say that G has *generics* if it has a generic element. We say that an n-tuple $(g_1, \dots, g_n)$ is generic if its orbit under the action of G by pointwise conjugation $\{(g_1^h, \dots, g_n^h) : h \in G\}$ is comeager. Finally we say that G has ample generics if for every $n \geq 1$, G has generic n-tuples.

Ample generics where introduced by Hodges, Hodkinson, Lascar and Shelah in [HHLS93] and some of its consequences were explored by Kechris and Rosendal in [KR07]. Among other properties, they showed that if a Polish group has ample generics, then it has the automatic continuity property, namely, any homomorphism from G to a Polish group H is always continuous.

The examples studied in [KR07] are subgroups of S_∞ that arise as automorphism groups of Fraïssé limits whose partial automorphisms have nice amalgamation properties. They include the permutation group S_∞ of $\mathbb{N}$, the automorphism group of the random graph, the automorphism group of a countably dimensional vector space over a finite field and the automorphism group of the rational Urysohn space. A group can have generics and fail to have ample generics (for example $Aut(Q, <)$). All these examples are totally disconnected.

The work of Kaïchouh and Le Maître [KLM15] shows how to build connected examples. Let $L^0([0,1], G)$ denote the space of measurable functions from $[0,1]$ to G, which is a group with pointwise multiplication and Polish with the topology

Date: August 30, 2018.
2010 *Mathematics Subject Classification.* Primary: 03E15, 54H11 Secondary: 22F50, 22A05,
Key words and phrases. Topological groups, isometry groups, Borel randomizations, Rohklin property, topometric spaces, topometric groups.
We would like to thank Julien Melleray for suggesting the topological approach to the problem presented in section 5. We would also like to thank John K. Truss for the information he provided to us which allowed us to extend our result to Aut$(Q, <)$. The second author would like to thank the department of mathematics at Universidad de los Andes for its hospitality during his stay, which resulted in part of this work.

of convergence in measure. They prove that if G has generics (respectively ample generics), then $L^0([0,1],G)$ has generics (respectively ample generics).

On the other hand, several Polish groups that arise as the group of isometries of metric structures do not have generics or ample generics but weaker properties called *metric generics* and *metric ample generics* that were introduced and studied in [BYBM13]. This is the case for the group of isometries $Aut([0,1])$ of the measure algebra associated standard Lebesgue space, the group of isometries of the Urysohn space and the group of isometries of a separable Hilbert space [BYBM13]. The main idea behind this approach is to endow a group of isometries G of a metric space (or even a metric structure) (M,d) with two topologies, the one of pointwise convergence (which is Polish) and the one of uniform convergence (which is finer than the previous one and in general is not Polish). The interaction of the Polish topology and the uniform convergence topology gives a notion of relative continuity for group homomorphisms that replaces the usual notion of automatic continuity (see [BYBM13]). In this paper we apply the ideas of [BYBM13]) to the isometry group of a randomization of a countable first order structure M. Ibarlucía [Iba17] characterized this group in terms of the group of isometries of M. He showed that it can be writen as $\tilde{G} = L^0([0,1],G) \rtimes Aut([0,1])$ where $G = Isom(M)$ and for $\alpha \in L^0([0,1],M)$ and $(f,T) \in L^0([0,1],G) \rtimes Aut([0,1])$, the action is given by $((f,T)(\alpha))(\omega) = f(\omega)\alpha(T^{-1}(\omega))$.

The group $\tilde{G}$, being a group of isometries, can be endowed with the topology of pointwise convergence and the topology of uniform convergence. Ibarlucía [Iba17] proved that the pointwise convergence topology is the product topology of $L^0([0,1],G)$ and $Aut([0,1])$. We show a similar result and prove that the uniform convergence topology is the product of an induced uniform topology in $L^0([0,1],G)$ and the uniform topology in $Aut([0,1])$. With these tools we show that if G has metric generics (respectively metric ample generics), then $\tilde{G}$ has metric generics (respectively metric ample generics). We also prove that if G is extremely amenable, then so is $\tilde{G}$.

As a corollary of our work we can also show that if H and G are isometry groups and both have metric ample generics, then when we equip $H \rtimes G$ with the Polish product topology and the with the uniform convergence topology coming from the product of the corresponding uniform topologies, then $H \rtimes G$ also has metric ample generics. Similarly we prove that if the Rohklin property holds for H and G then it also holds for $H \rtimes G$.

We should point out that many model theoretic properties transfer from $T = Th(M)$ to the theory of its randomization T^R such as ω-categority, ω-stability and NIP [BYK09, BY09]. The work of Ibarlucía [Iba17] also shows that some properties of G are reflected in $\tilde{G}$ such as being reflexively representable. This, together with the results of Kaïchouh and Le Maître, was a strong indication that we also could expect some nice behavior at the level of generics in the automorphism group.

This paper is organized as follows. In section 2 we follow the work of [KLM15] and show that if G is a Polish group and has metric generics (respectively metric ample generics), so does $L^0([0,1],G)$. In section 3 we characterize the uniform convergence topology in $\tilde{G}$ when we see it as the isometry group of a randomization. In section 4 we approach the problem of existence of dense orbits in $\tilde{S}_\infty$. We handle the problem from an algebraic perspective and we extend the result to a robust class of Polish groups. In section 5, we follow a more topological approach and show that dense orbits are transferred from G to $\tilde{G}$ as well as topometric ample generics. In section 6 we show that extreme amenability transfers from G to $\tilde{G}$.

Finally, we should point out that there are other approaches to study automatic continuity in isometry groups such as those in the work of Kwiatkowska, Malicki and Sabok [KM, Mal16, Sab17]. The notes by Kaïchouh [Kaï] are a good reference for the subject.

2. Transferring topometric generics to $L^0([0,1],G)$

In this section we generalize the ideas of Kaïchouh-Le Maitre and show how the map that sends G to $L^0([0,1],G)$ not only preserves generics and ample generics but also metric generics and ample metric generics. We start by recalling some definitions from [BYBM13].

Definition. We say that a triple (X,τ,d_u) is a **topometric space** if X is a set, τ is a topology on X and d_u is a distance function such that:

(1) The topology induced by d_u refines τ.
(2) The metric d_u is lower semi-continuous with respect to τ, i.e., the set $\{(x,y)\in X^2 : d_u(x,y)\leq r\}$ is τ-closed for all $r\geq 0$.

We say that a triple (G,τ,d_u) is a **topometric group** if (G,τ) is a topological group, (G,τ,d_u) is a topometric space and d_u is bi-invariant under group multiplication.

In the setting above, whenever (X,τ) is a Polish space, we say that (X,τ,d_u) is a **Polish topometric space** and if (G,τ) is a Polish group, we say that (G,τ,d_u) is a **Polish topometric group**.

Definition. Let (X,τ,d_u) be a Polish topometric space and let d be a complete metric inducing the topology τ such that there is a constant C with $d(x,y)\leq Cd_u(x,y)$. Then we say that d_u is a distance that C**-uniformly refines** the metric space (X,d).

Note that this definition is equivalent to the identity function $i:(X,d_u)\to(X,d)$ being Lipschitz. Abusing notation, when we have a fixed metric d for τ we will write (X,d,d_u) instead of (X,τ,d_u). Also note that if (X,d,d_u) is a topometric space, so is (X,d',d'_u), where $d'=\min\{d,1\}$ and $d'_u=\min\{d_u,1\}$ and the topologies do not change. Thus we may always assume that both metrics are bounded.

Example 1. *Let (X,τ) be Polish, let d_X be a complete metric on X that induces the topology. Then (X,τ,d_X) is a topometric Polish space.*

Example 2. *Let (X,τ) be Polish, let d be a complete metric on X that induces the topology and assume that $diam_d(X)\leq 1$. Let $\{b_m\}_m$ be a dense subset of X. Let $G=Isom(X)$ be the group of isometries of X and for $g,h\in G$ let $d_p(f,h)=\sum_m \frac{1}{2^{m+1}}d(f(b_m),h(b_m))$. Then d_p induces the pointwise convergence topology on G which is Polish. Let $d_u(f,h)=\sup_{x\in X}d(f(x),h(x))$ which is a metric for the uniform convergence topology. Then d_u 1-uniformly refines d_p, and (G,d_p,d_u) is a topometric space. Note that d_u is bi-invariant under multiplication by G, d_u is lower semicontinuous with respect to d_p and so (G,d_p,d_u) is a Polish topometric group.*

Lemma 3. *Assume that (X,τ) is a Polish space. Let $f,g\in L^0([0,1],X)$, and $d:X\times X\to\mathbb{R}$ a measurable function. Then the function sending ω to $d(f(\omega),h(\omega))$ is measurable. Morever, if d is bounded, it is integrable in $[0,1]$.*

Proof. Since d is measurable, for any $r > 0$, $F_r = \{(x,y) \in X \times X : d(x,y) < r\}$ is measurable. But since f, h are measurable, $\{\omega \in [0,1] : d(f(\omega), h(\omega)) \leq r\} = \{\omega \in [0,1] : (f(\omega), h(\omega)) \in F_r\} = (f,h)^{-1}(F_r)$ is measurable and thus $d(f(\omega), h(\omega))$ is a measurable function on ω. Since d is bounded and measurable, then it is integrable and the integral is finite (see [WZ15]). □

Note that lower semicontinuous metrics are measurable.

Throughout this paper, we will work out with several induced distances. We will now standarize the notation.

Notation.

(1) Let (X,d) be a metric space. If $G = \mathrm{Isom}(X,d)$ is the group of isometries, we will usually denote by τ the pointwise-convergence topology on G. However, if there are several topologies that need to be considered, we will write τ_p for the topology of pointwise convergence.

(2) We will denote by d_u the uniform convergence distance on $\mathrm{Isom}(X,d)$ as introduced in Example 2.

(3) Let d be a bounded metric on X and for $f, h \in L^0([0,1], X)$ let $\hat{d}(f,h) = \int_0^1 d(f(\omega), h(\omega)) d\omega$. Note that by the argument above $d(f(\cdot), g(\cdot))$ is integrable in $[0,1]$, thus $\hat{d}$ is well defined.

(4) Let (X,d) be a metric space, then whenever $a, b \in X$ and $r > 0$, $B_r^d(a) = \{x \in X : d(a,x) < r\}$ and $B_r^d(a,b) = \{(x,y) \in X \times X : d(a,x) < r, d(b,y) < r\}$.

Proposition 4. *Assume that (X, d, d_u) is a topometric space, such that d_u C-uniformly refines d. Assume that d and d_u are bounded by 1. Then the triple $(L^0([0,1], X), \hat{d}, \hat{d}_u)$ is a topometric space and $\hat{d}_u$ C-uniformly refines $\hat{d}$.*

Proof. First we show that $\hat{d}_u$ refines the topology induced by $\hat{d}$. Indeed $\hat{d}(f,h) = \int_0^1 d(f(\omega), h(\omega)) d\omega \leq \int_0^1 C d_u(f(\omega), h(\omega)) d\omega \leq C \hat{d}_u(f,h)$. This shows that $\hat{d}_u$ C-uniformly refines the metric $\hat{d}$.

In order to prove that $\hat{d}_u$ is lower-semicontinuous, it suffices to prove that whenever $r > 0$, the set $V = \{(f,h) \in L^0 \times L^0 : \hat{d}_u(f,h) > r\}$ is open with respect to $\hat{d}$. Let $(f,h) \in V$. Then $\hat{d}_u(f,h) > r + \epsilon$ for some $\epsilon > 0$.

For each $m \geq 1$ and $0 \leq i < m$, let $A_{i/m,(i+1)/m} = \{\omega \in [0,1] : i/m < d_u(f(\omega), h(\omega)) \leq (i+1)/m\}$. Since d_u is lower semicontinuous, the sets $A_{i/m,(i+1)/m}$ are measurable. By the monotone convergence Theorem, so by choosing m large enough, we have

$$\sum_{i>0} \frac{i}{m} \mu(A_{i/m,(i+1)/m}) > r + \epsilon. \tag{1}$$

Since d_u is lower semicontinuous, for each $\omega \in A_{i/m,(i+1)/m}$ there is $\delta_\omega > 0$ such that for all $1/k \leq \delta_\omega$, if $(x_1, x_2) \in B^d_{1/k}(f(\omega), h(\omega))$, then $d_u(x_1, x_2) > i/m$.

For each $k > 0$, let

$$A_{i/m,(i+1)/m,k} = \{\omega \in A_{i/m,(i+1)/m} : \text{if } (x_1,x_2) \in B^d_{1/k}(f(\omega), h(\omega)), \text{ then } d_u(x_1,x_2) > i/m\}.$$

Then there is $N > 0$ such that $\mu(A_{i/m,(i+1)/m,N}) \geq \mu(A_{i/m,(i+1)/m}) - \frac{\epsilon}{4m}$ for all $i < m$. Let $C = (\cup_{1 \leq i < m} A_{i/m,(i+1)/m}) \setminus \cup_{1 \leq i < m} A_{i/m,(i+1)/m,N}$, then $\mu(C) < \epsilon/4$.

Claim The $\hat{d}$-open ball of center (f,h) and radius $\epsilon/(4N)$ is contained in V.

To check this, let $(f', h') \in B^{\hat{d}}_{\epsilon/(4N)}(f,h)$ and let

$$B = \{\omega \in [0,1] : d(f(\omega), f'(\omega)) \geq 1/N \text{ or } d(h(\omega), h'(\omega)) \geq 1/N\}.$$

Then $\mu(B) < \epsilon/4$. Now let $A = [0,1] \setminus (B \cup C)$, then $\mu(A^c) \le \mu(B) + \mu(C) \le \epsilon/2$.

Let $\omega \in A$ and assume that $d(f(\omega), h(\omega)) > 0$. Then there is some i such that $\omega \in A_{i/m,(i+1)/m}$. Then

$$d(f'(\omega), f(\omega)) < 1/N,$$
$$d(h'(\omega), h(\omega)) < 1/N,$$

and

$$\omega \in A_{i/m,(i+1)/m,N},$$

so $d(f'(\omega), h'(\omega)) > i/m$.

From this and equation (1) we get

$$\hat{d}_u(f', h') = \int d_u(f'(\omega), h'(\omega))d\omega \ge \int_A d_u(f'(\omega), h'(\omega))d\omega \ge$$
$$\sum_{i>0} \frac{i}{m} \mu(A \cap A_{i/m,(i+1)/m,N}) \ge [\sum_{i>0} \frac{i}{m} \mu(A_{i/m,(i+1)/m})] - \mu(A^c) \ge r + \epsilon/2$$

So $\hat{d}_u(f', g') > r$ and V is $\hat{d}$-open as we wanted. □

Corollary 5. *Let (G, d, d_u) be a Polish topometric group and assume $\hat{d}_u$ C-uniformly refines $\hat{d}$. Also assume that d and d_u are bounded by one. Then $(L^0([0,1], G), \hat{d}, \hat{d}_u)$ is a Polish topometric group.*

Proof. We just proved that $(L^0([0,1], G), \hat{d}, \hat{d}_u)$ is a topometric space. Since d_u is bi-invariant it is easy to check that $\hat{d}_u$ is also bi-invariant. Finally since (G, d) is Polish, then $L^0([0,1], G)$ is also Polish, see [Kec10, Iba17]. □

We recall the following definition from [BYBM13].

Definition. Let (G, τ, d_u) be a Polish topometric group.

(1) We say that (G, τ, d_u) has **metric generics** if there is $x \in G$ such that $\overline{Orb(x)}^{d_u}$ is comeager, where the closure is taken with respect to the metric d_u and the orbit with respect to the conjugacy action. We call such an x a **metric generic element**.
(2) We say that (G, τ, d_u) has **ample metric generics** if for every n there is $x \in G^n$ such that $\overline{Orb(x)}^{d_u}$ is comeager, where the closure is taken with respect to the metric d_u and the orbit with respect to the diagonal conjugacy action. We call such an x a **metric generic n-tuple**.

Theorem 6. *Let (G, d, d_u) be a Polish topometric group with metric generics and assume that d, d_u are bounded by one. Then $(L^0([0,1], G), \hat{d}, \hat{d}_u)$ has metric generics. Furthermore, if $g \in G$ is a metric generic in (G, d, d_u), then C_g is a metric generic in $(L^0([0,1], G), \hat{d}, \hat{d}_u)$, where $C_g(\omega) = g$ for all $\omega \in [0,1]$.*

Proof. Since (G, d, d_u) has metric generics, there is $g \in G$ such that $S = \overline{Orb(g)}^{d_u}$ is comeager in G (here we take the closure with respect to d_u). Then by [KLM15] $Orb(C_g)$ is dense. Let $F = \{f \in L^0([0,1], G) : f \in S \text{ a.e.}\}$. Since S contains a dense G_δ set in G, by [KLM15] F contains a dense G_δ set in $L^0([0,1], G)$. It remains to show that $F \subseteq \overline{Orb(C_g)}^{\hat{d}_u}$ (here we take the closure with respect to $\hat{d}_u$).

Let $f \in F$ and let $\epsilon > 0$. Since for a.e. $\omega \in [0,1]$, $f(\omega) \in S$, there is $h_\omega \in G$ such that $d_u(f(\omega), h_\omega^{-1} g h_\omega) \le \epsilon$. Since d_u is lower-semicontinuous, this is a Borel condition. By Jankov-von Neumann we can find $h \in L^0([0,1], G)$ such that

$d_u(f(\omega), h(\omega)^{-1}gh(\omega)) \leq \epsilon$ for a.e. $\omega \in [0,1]$ and thus $\hat{d}_u(f, h^{-1}gh) \leq \epsilon$. This shows that $f \in \overline{Orb(C_g)}^{\hat{d}_u}$ as we wanted. □

Theorem 7. *Let (G, d, d_u) be a Polish topometric group with topometric ample generics, then $(L^0([0,1], G), \hat{d}, \hat{d}_u)$ has topometric ample generics.*

Proof. Let $n \geq 1$ and let $\vec{g} = (g_1, \ldots, g_n) \in G^n$ be such that $S = \overline{Orb(\vec{g})}^{d_u}$ is comeager. Consider $C_{\vec{g}} = (C_{g_1}, \ldots, C_{g_n})$ the constant function with value $(g_1, \ldots, g_n)$. As in the previous proof, it is easy to prove that $\overline{Orb(C_{\vec{g}})}^{\hat{d}_u}$ is comeager in $(L^0([0,1], G), \hat{d})$. □

3. Topologies of the group $\mathrm{Isom}(L^0([0,1], X))$

Let X be a Polish space. Fix d a metric bounded by 1 that generates the topology and such that X is complete with respect to d. Let $G = \mathrm{Isom}(X)$, the group of isometries of (X, d). The topology of pointwise convergence on G is Polish.

For $\alpha, \beta \in L^0([0,1], X)$ let $\hat{d}(\alpha, \beta) = \int d(\alpha(\omega), \beta(\omega))d\omega$. This generates a Polish topology on $L^0([0,1], X)$. In [Iba17] Ibarlucía studies the group of isometries of $(L^0([0,1], X), \hat{d})$. In particular he characterizes the group of isometries as $\tilde{G} = L^0([0,1], G) \rtimes \mathrm{Aut}[0,1]$, where the action is given as follows: for $(f, T) \in L^0([0,1], G) \rtimes \mathrm{Aut}[0,1]$ and $\alpha \in L^0([0,1], X)$, then $((f,T)(\alpha))(\omega) = f(\omega)(\alpha(T^{-1}(\omega)))$. Note that since G is Polish, so is $L^0([0,1], G)$ and that $\mathrm{Aut}[0,1]$ is Polish with the topology of weak convergence.

Since $\tilde{G}$ is a group of isometries of a Polish metric space, we can also endow it naturally with two topologies:

Definition. For $(f, T) \in \tilde{G}$; $\alpha_1, \ldots, \alpha_n \in L^0([0,1], X)$ and $\epsilon > 0$, let

$$V((f,T)) = \{(g, S) \in \tilde{G} : \hat{d}((f,T)(\alpha_i), (g,R)(\alpha_i)) < \epsilon, i \leq n\}.$$

We call the topology generated by these sets, where ϵ varies over the positive numbers, $\alpha_1, \ldots, \alpha_n$ belong to $L^0([0,1], X)$ and n ranges over the positive natural numbers the **pointwise convergence topology**. It is Polish, see example to find a complete metric that generates the topology.

Definition. For $(f, T), (g, S) \in \tilde{G}$, the uniform distance is given by:

$$L_u((f,T), (g,S)) = \sup_{\alpha \in L^0([0,1], X)} \hat{d}((f,T)(\alpha), (g,S)(\alpha)).$$

Since the collection of simple functions is dense in $L^0([0,1], \mathbb{N})$ we can also take the supremum above over simple functions. This is the topology of **uniform convergence** as explained in example 2.

The goal of this section is to study and characterize the two topologies in $\tilde{G}$ in terms of topologies of $L^0([0,1], G)$ and $\mathrm{Aut}[0,1]$.

3.1. **Pointwise convergence topology.** As we said earlier, the spaces $L^0([0,1], G)$, $\mathrm{Aut}[0,1]$ are both Polish, so the group $\tilde{G}$, being the semidirect product of these two groups, also carries the product topology which is Polish. In [Iba17] Ibarlucía shows that the product topology coincides with the topology of pointwise convergence. In this section we will prove again this fact, our proof is very soft, we will prove by double containment that the topologies agree. We assume that X is not trivial, that is, $|X| \geq 2$. We will write Id for the map from $[0,1]$ to $[0,1]$ defined by $Id(\omega) = \omega$.

First, we show that the product topology is finer than the pointwise convergence topology.

Lemma 8. *Let $(f,T) \in \tilde{G}$, let $\alpha \in L^0([0,1],X)$ be a simple function and $\epsilon > 0$. Consider the subbasic open set*

$$V = \{(g,R) \in \tilde{G} : \int d(f(\omega)\alpha(T^{-1}(\omega)), g(\omega)\alpha(R^{-1}(\omega)))d\omega < \epsilon\}$$

in the topology of pointwise convergence. Then there are open sets $U \subseteq L^0([0,1],G)$, $W \subseteq Aut[0,1]$ such that $(f,T) \in U \times W \subseteq V$.

Proof. We may write $\alpha = \sum_{i=1}^k a_i \chi_{A_i}$ where $A_1, A_2, \ldots, A_k$ is a partition of $[0,1]$ and $a_1, \ldots, a_k \in X$.

First let $W := \{R \in \text{Aut}[0,1] : \mu(T(A_i) \triangle R(A_i)) < \epsilon/(2k), i \leq k\}$ and let $U := \{g \in L^0([0,1],G) : \int d(f(\omega)(a_i), g(\omega)(a_i))d\omega < \epsilon/(2k) : i \leq k\}$.

Then whenever $g \in U$ and $R \in W$ we have that:

$$\begin{aligned}
&\int d(f(\omega)\alpha(T^{-1}(\omega)), g(\omega)\alpha(R^{-1}(\omega)))d\omega \\
&\leq \sum_{i,j\leq k} \int_{T(A_i)\cap R(A_j)} d(f(\omega)(a_i)), g(\omega)(a_j))d\omega \\
&\leq \sum_{i\leq k} \int_{T(A_i)\cap R(A_i)} d(f(\omega)(a_i)), g(\omega)(a_i))d\omega \\
&\qquad + \sum_{i,j\leq k, i\neq j} \int_{T(A_i)\cap R(A_j)} d(f(\omega)(a_i)), g(\omega)(a_j))d\omega \\
&\leq \sum_{i\leq k} \epsilon/(2k) + \sum_{i\leq k} \mu(T(A_i) \triangle R(A_i)) \leq \epsilon/2 + \sum_{i\leq k} \epsilon/(2k) \leq \epsilon.
\end{aligned}$$

Finally notice that $f \in U$ and $T \in W$. □

The next lemma is the other direction: the pointwise convergence topology is finer than the product topology.

Lemma 9. *Let $(f,T) \in \tilde{G}$, let $\alpha \in L^0([0,1],X)$ be simple, $B \subseteq [0,1]$ measurable and $\epsilon > 0$. Consider the open sets $U \subseteq L^0([0,1],G)$ defined by $U = \{g \in L^0([0,1],G) : \int d_X(f(\omega)\alpha(\omega), g(\omega)\alpha(\omega))d\omega < \epsilon\}$ and $W \subseteq Aut[0,1]$ given by $\{R \in Aut[0,1] : \mu(T(B) \triangle R(B)) < \epsilon\}$. Then there are open sets V_1, V_2 in $\tilde{G}$ in the topology of pointwise convergence with $(f,T) \in V_1 \subseteq L^0([0,1],X) \times W$ and $(f,T) \in V_2 \subseteq U \times Aut[0,1]$.*

Proof. Since X has more than one point, we may find $c_1, c_2 \in X$ with $d_X(c_1,c_2) = s > 0$. Let $\beta_1 = c_1\chi_B + c_1\chi_{B^c}$ and let $\beta_2 = c_1\chi_B + c_2\chi_{B^c}$ and consider

$$V((f,T),\beta_1,\beta_2,\epsilon s/4) = \{(g,R) : \int d_X((f,T)(\beta_i),(g,R)(\beta_i)) < \epsilon s/4, i \leq 2\}.$$

Then whenever $(g,R) \in V((f,T),\beta_1,\beta_2,\epsilon s/4)$, we have

$$\int_{T(B)\cap R(B)^c} d_X(f(\omega)(c_1), g(\omega)(c_1))d\omega < \epsilon s/4, \tag{2}$$

and

$$\int_{T(B)\cap R(B)^c} d_X(f(\omega)(c_1), g(\omega)(c_2))d\omega < \epsilon s/4, \tag{3}$$

so adding inequalities (1) and (2)

$$\int_{T(B)\cap R(B)^c} d_X(f(\omega)(c_1), g(\omega)(c_1)) + d_X(f(\omega)(c_1), g(\omega)(c_2))d\omega < \epsilon s/2.$$

This shows, using the triangle inequality, that

$$\int_{T(B)\cap R(B)^c} sd\omega < \epsilon s/2, \text{ so } \mu(T(B)\cap R(B)^c) < \epsilon/2.$$

Similarly, $\mu(T(B)^c \cap R(B)) < \epsilon/2$, so $\mu(T(B) \vartriangle R(B)) < \epsilon$ as desired.

For the second part, write $\alpha = \sum_{i=1}^k a_i \chi_{A_i}$ where $A_1, A_2, \ldots, A_k$ is a partition of $[0,1]$ and $a_1, \ldots, a_k \in X$. By applying the previous argument, we can find a basic open set V_4 with $(f,T) \in V_4$ such that whenever $(g,R) \in V_4$ we have $\mu(T(A_i) \vartriangle R(A_i)) < \epsilon/2k$ for $i \le k$. Now consider $V_3 = \{(g,R) : \int d_X((f,T)(\gamma), (g,R)(\gamma))d\omega < \epsilon/2\}$, where $\gamma(\omega) = \alpha(T(\omega))$. Notice that $(h, Id)(\alpha) = (h,T)(\gamma)$ for all $h \in L^0([0,1], G)$.

Choose $(g,R) \in V_4 \cap V_3$. Then

$$\begin{aligned}
&\int d_X(f(\omega)\alpha(\omega), g(\omega)\alpha(\omega))d\omega \\
=&\int d_X((f,Id)\alpha(\omega), (g,Id)\alpha(\omega))d\omega \\
=&\int d_X((f,T)\gamma(\omega), (g,T)\gamma(\omega))d\omega \\
\le&\sum_{i\le k}\int_{A_i\cap R(T^{-1}(A_i))} d_X(f(\omega)\gamma(T^{-1}\omega), g(\omega)\gamma(R^{-1}(\omega))d\omega + \sum_{i\le k}\mu(T(A_i)\vartriangle R(A_i)) \\
\le&\epsilon/2 + \epsilon/2 \le \epsilon.
\end{aligned}$$

□

Thus, we proved that the product topology is indeed the pointwise convergence topology.

3.2. **Uniform convergence topology.** In this section we characterize the metric for uniform convergence in terms of the metrics for uniform convergence for Aut$[0,1]$ and for $L^0([0,1], G)$. By the metric of uniform convergence in Aut$[0,1]$ we mean $\Delta_u(T,R) = \mu\{\omega \in [0,1] : T(\omega) \ne R(\omega)\}$. Note that we could also use $\Delta'_u(T,R) = \sup\{\mu(T(A) \vartriangle R(A)) : A \subseteq [0,1] \text{ measurable }\}$ (see [Hal60]). Similarly, since $G = Isom(X,d)$ has a metric of uniform convergence d_u, we have by section 2 that $(L^0([0,1], G), \tau, \hat{d_u})$ is topometric where τ is the topology of convergence in measure, so we can consider $\hat{d_u}$ as a natural uniform metric for $L^0([0,1], G)$.

For clarity, we first do the argument for $X = \mathbb{N}$ (where $d(n,m) = 1$ if $n \ne m$) and $G = S_\infty$ and then we consider the general case. Note that for $\sigma, \rho \in S_\infty$ distinct $d_u(\sigma,\rho) = 1$. In what follows, we write e for the identity in S_∞, C_e for the function from $[0,1] \to S_\infty$ with constant value e and Id for the map from $[0,1]$ to $[0,1]$ defined by $Id(\omega) = \omega$.

Proposition 10. *For $(f,T) \in L^0([0,1], S_\infty) \rtimes Aut[0,1]$, we have:*

$$L_u((f,T), (C_e, Id)) = \mu(\{\omega \in [0,1] : f(\omega) \ne e\} \cup \{\omega \in [0,1] : T^{-1}(\omega) \ne \omega\}).$$

Proof. Let $C = \{\omega \in [0,1] : f(\omega) = e \text{ and } T^{-1}(\omega) = \omega\}$. Then for any function $\alpha \in L^0([0,1], \mathbb{N})$ and for any $\omega \in C$ we have

$$(f,T)(\alpha)(\omega) = f(\omega)(\alpha(T^{-1}(\omega))) = f(\omega)(\alpha(\omega)) = (\alpha)(\omega)$$

and thus $d_u((f,T),(C_e,Id)) \leq \mu(C^c)$.

For the other inequality, let us define $A = \{\omega \in [0,1] : f(\omega) \neq e \wedge T^{-1}(\omega) = \omega\}$.

For each $\omega \in A$ let n_ω be the minimum natural number such that $f(\omega)(n_\omega) \neq n_\omega$ and let $\alpha(\omega) = n_\omega$. Note that α has been defined on A and in this set it is measurable. Also note that for $\omega \in A$, $(f,T)(\alpha)(\omega) = f(\omega)(\alpha(T^{-1}(\omega))) = f(\omega)(n_\omega) \neq n_\omega = \alpha(\omega)$ so the two automorphisms (f,T), (C_e,Id) disagree in every $\omega \in A$ when they act in α.

Now let $B = \{\omega \in [0,1] : T^{-1}(\omega) \neq \omega\}$. We may write $B = B_0 \cup \cup_{i\geq 2} B_i$ where B_0 are the points where T is an aperiodic map and B_i are the points where T is a cycle of period i. All the sets B_i are measurable.

First we will deal with $B_2 = \{\omega \in B : T^2(\omega) = \omega\}$. Let C_2 be a measurable subset of B_2 such that C_2, $T(C_2)$ are disjoint and $B_2 = C_2 \cup T(C_2)$.

For $n \geq 1$, define α_n as follows. For $\omega \in C_2$, let $\alpha_n(\omega) = 0$ and $\alpha_n(T(\omega)) = n$. It is easy to observe that α_n satisfies $(f,T)\alpha_n(\omega) \neq (C_e,Id)\alpha_n(\omega)$ for $\omega \in B_2$ outside the set $\{\omega \in C_2 : f(\omega)^{-1}(0) = n\} \cup \{\omega \in T(C_2) : f(\omega)(0) = n\}$. Since f is fixed, $\mu(\{\omega \in C_2 : f(\omega)^{-1}(0) = n\}) \to 0$ as $n \to \infty$. Similarly, $\mu(\{\omega \in T(C_2) : f(\omega)(0) = n\}) \to 0$ as $n \to \infty$.

Thus $\lim_{n\to\infty} \int_{B_2} \hat{d}((f,T)(\alpha_n),(C_e,Id)(\alpha_n))d\omega = \mu(B_2)$, so restricting to B_2 we get $d_u((f,T),(C_e,Id)) \geq \mu(B_2)$.

We deal with $B_3 = \{\omega \in B : T^3(\omega) = \omega\}$ in a similar way. Let C_3 be a measurable subset of B_3 such that C_3, $T(C_3)$, $T^2(C_3)$ are pairwise disjoint and $B_3 = C_3 \cup T(C_3) \cup T^2(C_3)$. For $\omega \in C_3$ and $n \geq 1$, let $\beta_n(\omega) = 0$ and $\beta_n(T(\omega)) = n$, $\beta_n(T^2(\omega)) = 2n$.

As above, β_n separates (f,T) from (C_e,Id) in B_3 outside the set of exceptional points $\{\omega \in C_3 : f(\omega)^{-1}(0) = n\} \cup \{\omega \in T(C_3) : f^{-1}(\omega)(n) = 2n\} \cup \{\omega \in T^2(C_3) : f(\omega)(0) = 2n\}$. Notice that the measure of the set of exceptional points goes to zero as n goes to infinty, so $\lim_{n\to\infty} \int_{B_3} \hat{d}((f,T)(\beta_n),(C_e,Id)(\beta_n))d\omega = \mu(B_3)$.

Using a similar approach for all the periodic pieces and approximating the aperiodic piece by periodic pieces using Rohklin's Lemma we get the desired result.

□

Corollary 11. *For $(f,T) \in L^0([0,1],S_\infty) \rtimes Aut[0,1]$, we have:*
$\max\{\int d_u(f(\omega),e)d\mu, \Delta_u(T,Id)\} \leq L_u((f,T),(C_e,Id)) \leq \int d_u(f(\omega),e)d\mu + \Delta_u(T,Id)$
this shows that the metric of uniform continuity is uniformly equivalent to the product distance of the metrics $\hat{d}_u$ (in $L^0([0,1],S_\infty)$) and Δ_u (in $Aut[0,1]$).

Now we consider the general case, so (X,d) is a Polish metric space with $diam(X) \leq 1$ and $G = \text{Isom}(X,d)$. We also assume the set X has more than one point, so we may find $a,b \in X$ with $d(a,b) = r > 0$.

Proposition 12. *For $(f,T),(h,S) \in L^0([0,1],G) \rtimes Aut[0,1]$, the functions sending $\omega \in [0,1]$ to $d_u(f(T^{-1}(\omega)),h(S^{-1}(\omega)))$ and $d_G(f(T^{-1}(\omega)),h(S^{-1}(\omega)))$ are both measurable and integrable.*

Proof. First observe that since f is measurable and T is an invertible measurable preserving transformation, the map that sends ω to $f(T^{-1}(\omega))$ is measurable. So is the map that sends ω to $h(S^{-1}(\omega))$. Finally, since the map $d_u : G \times G \to [0,1]$ is Borel measurable, then the map from $[0,1]$ to $[0,1]$ that sends ω to $d_u(f(T^{-1}(\omega)),h(S^{-1}(\omega)))$ is measurable.

Similarly, since the function $d : G \times G \to [0,1]$ is Borel measurable, the map from $[0,1]$ to $[0,1]$ that sends ω to $d_G(f(T^{-1}(\omega)),h(S^{-1}(\omega)))$ is measurable.

Since both functions are bounded by 1, they are also integrable. □

Theorem 13. *Let*

$$A = \{\omega \in [0,1] : f(\omega)) \neq \omega \wedge T^{-1}(\omega) = \omega\},$$
$$B = \{\omega \in [0,1] : T(\omega) \neq \omega\}.$$

Then

$$\frac{r}{8}\mu(B) + \int_A d_u(f(\omega), e) d\mu \leq L_u((f,T),(C_e, Id)) \leq \mu(B) + \int_A d_u(f(\omega), e) d\mu.$$

Proof. Let $\alpha \in L^0([0,1], X)$ and let $B = \{\omega \in [0,1] : T(\omega) \neq \omega\}$. Then for each $\omega \in B$, $d(f(\omega)(\alpha(T^{-1}(\omega)), \alpha(\omega))) \leq 1$. On the other hand for any $\omega \in A$, $d_u(f(\omega), e) \geq d(f(\omega)(\alpha(\omega)), \alpha(\omega))$, so $\int_A d_u(f(\omega), e) d\mu \geq \int_A d(f(\omega)(\alpha(\omega)), \alpha(\omega)) d\mu$. This shows that $L_u((f,T),(C_e, Id)) \leq \mu(B) + \int_A d_u(f(\omega), e) d\mu$.

We now prove the other inequality. Let $\epsilon > 0$.

For each $\omega \in A$ let $\alpha_\omega \in X$ be such that $d(f(\omega)(\alpha_\omega), \alpha_\omega)) + \epsilon \geq d_u(f(\omega), e)$. Since d, d_u are measurable, we may define a measurable function α on A so that $d(f(\omega)(\alpha(\omega)), \alpha(\omega)) + \epsilon \geq d_u(f(\omega), e)$. Note that for $\omega \in A$, $\int_A d(f(\omega)(\alpha(\omega)), \alpha(\omega)) d\omega + \epsilon \geq \int_A d_u(f(\omega), e) d\omega$.

Recall that $B = \{\omega \in [0,1] : T^{-1}(\omega) \neq \omega)\}$ and as before, write $B = B_0 \cup \cup_{i \geq 2} B_i$ where B_0 are the points where T is an aperiodic map and B_i are the points where T is a cycle of period i. All the sets B_i are measurable and $T(B_n) = B_n$ for all n.

For each $n \geq 2$ we define a function in B_n. Consider the sets $D_n = \{\omega \in B_n : d(a, f(\omega)(a)) \geq r/2\}$ and $E_n = \{\omega \in B_n : d(a, f(\omega)(b)) \geq r/2\}$. By the triangle inequality, one of the two sets must have measure $\geq \mu(B_n)/2$.

Case 1 $\mu(D_n) \geq \mu(B_n)/2$.

Define $\alpha_1 = a\chi_{B_n}$, then whenever $\omega \in D_n$, we have $d((f,T)(\alpha_1)(\omega), (C_e, Id)(\alpha_1)(\omega)) = d(f(\omega)(a), a) \geq r/2$, so $\int_{B_n} d((f,T)(\alpha_1)(\omega), (C_e, Id)(\alpha_1)(\omega)) d\omega \geq \mu(D_n) r/2 \geq \mu(B_n) r/4$.

Case 2 $\mu(E_n) \geq \mu(B_n)/2$.

Assume for the sake of simplicity $n = 2k$ and that there is a measurable set C_n such that $C_n, \ldots, T^{2k-1}(C_n)$ are pairwise disjoint, form a partition of B_n and all of them are independent from E_n (this last part can be assured up to δ for any $\delta > 0$). Define $\alpha_2 = a\chi_{C_n \cup T^2(C_n) \cup \cdots \cup T^{2k-2}(C_n)} + b\chi_{T(C_n) \cup \cdots \cup T^{2k-1}(C_n)}$. Then whenever $\omega \in E_n \cap (C_n \cup T^2(C_n) \cup \cdots \cup T^{2k-2}(C_n))$, we have that $d((f,T)(\alpha_2)(\omega), (C_e, Id)(\alpha_2)(\omega)) = d(f(\omega)(b), a) \geq r/2$ so $\int_{B_n} d((f,T)(\alpha_2)(\omega), (C_e, Id)(\alpha_2)(\omega)) d\omega \geq (\mu(E_n)/2) \cdot (r/2) \geq \mu(B_n) r/8$.

□

Note that the above formula implies that

$$\frac{r}{8} \max\{\mu(B), \hat{d}_u(f, C_e)\} \leq L_u((f,T),(C_e, Id)) \leq \mu(B) + \hat{d}_u(f, C_e)$$

so the uniform metric corresponds to the product topology of the uniform metrics $\hat{d}_u$ in $L^0([0,1], G)$ and Δ_u in $\mathrm{Aut}[0,1]$.

4. The Rohklin property in $\tilde{S}_\infty$

In this section we study the Rohklin property on $\tilde{S}_\infty$. We notice first that the strong Rohklin property is not satisfied by any $\tilde{G}$, so $\tilde{G}$ will not have ample generics. We then show that $\tilde{S}_\infty$ has the Rohklin property. From this proof, we can extract a sufficient condition for $\tilde{G}$ to have the Rohklin property, namely the Rohklin property under powers defined below. We also show that this is a rather robust definition, by showing several groups that have this property. In the next section, we will show via a topological argument the general case, however, we keep this argument as it shows a different approach (more algebraic in nature) but interesting on itself.

Proposition 14. *Let G be a topological group. Then $\tilde{G}$ does not have the strong Rokhlin property.*

Proof. Suppose that $(f,R) \in \tilde{G}$ has a comeager orbit. By the Kuratowski-Ulam Theorem, there is a comeager set X_G such that for every $g \in X_G$, $\{T : (g,T) \in [(f,R)]_{\tilde{G}}\}$ is comeager. So fix such g and notice that if $(g,T) \in [(f,R)]_{\tilde{G}}$, then T and R are conjugates. Therefore the orbit of R is comeager, which we know it is not the case. ☐

Theorem 15. *$\tilde{S}_\infty$ has the Rohklin property.*

Proof. Choose $\sigma \in S_\infty$ such that its conjugacy orbit is comeager. Remember that σ must have infinite copies of any k-cycle for every $k \geq 1$. Denote by C_σ the constant function in $L^0([0,1],G)$ with value σ. Let $S \in \mathrm{Aut}[0,1]$ with a dense conjugacy orbit, so S is aperiodic. We claim that (C_σ, S) has a dense orbit.

Let $U \times V$ be an open subset of $\tilde{S}_\infty$. Since the orbit of S is dense, there is $T \in \mathrm{Aut}[0,1]$ such that $T^{-1}ST \in V$.

Let us also fix a simple function $h(x) \coloneqq \Sigma_{i<M} \tau_i \chi_{A_i}(x)$ in TU. Furthermore, we can suppose that the A_i are Borel sets. Notice there is $\varepsilon > 0$ and there is $K \in \mathbb{N}$ such that

$$\{f \in L^0([0,1],G) | \mu(\{x \in [0,1] \mid \ \forall n < K \ f(x)(n) = h(x)(n)\}) \geq 1 - \varepsilon\} \subseteq TU.$$

So we need to find $g \in L^0([0,1],G)$ such that

$$g(x)h(x)(n) = \sigma g(S^{-1}(x))(n), \tag{4}$$

for all $n \leq K$ and all x except for a set of measure smaller than ε.

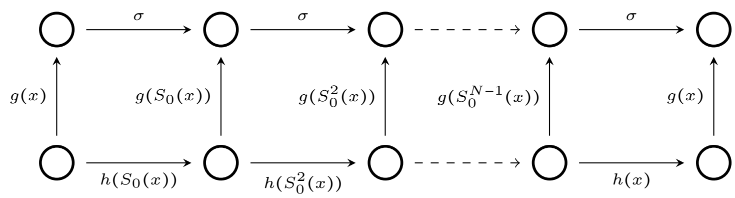
Since S is aperiodic, by Rohklin's Lemma, given ε as above and any $N \geq 1$ there is a measurable subset $E \subseteq [0,1]$ such that $E, S(E), \ldots, S^{N-1}(E)$ are pairwise disjoint and $\mu(\cup_{0 \leq i \leq N-1} S^i(E)) \geq 1 - \epsilon/2$. Choose N such that $1/N < \epsilon/2$. We now define S_0 so that it coincides with S on $\cup_{0 \leq i \leq N-2} S^i(E)$, as $S^{-N+1}(x)$ for $x \in S^{N-1}(E)$ and as a periodic map of period N in $[0,1] \smallsetminus \cup_{0 \leq i \leq N-1} S^i(E)$. Thus the map S_0 is a measure preserving transformation with period N such that $d_u(S,S_0) = \mu(\{x \in [0,1] | S(x) \neq S_0(x)\}) \leq \epsilon/2 + 1/N < \epsilon$ (this is called by Halmos the Uniform Approximation Theorem). By enlarging E if necessary and using the fact that S_0 is periodic with period N, we may find a new set $E_0 \supseteq E$ such that $E_0, S_0(E_0), \ldots, S_0^{N-1}(E_0)$ are pairwise disjoint and $\mu(\cup_{0 \leq i \leq N-1} S_0^i(E_0)) = 1$.

As said earlier, we need to find $g \in L^0([0,1],G)$ that satisfies equation 4 for all x but a set of measure smaller than ε and all $n \leq K$; thus it suffices to find $g \in L^0([0,1],G)$ such that

$$g(x)h(x)(n) = \sigma g(S_0^{-1}(x))(n) \tag{5}$$

for all x and all $n < K$.

Let us see what we need. Assume we have defined a function g that satisfies (5) for all $n < K$. We fix $x \in [0,1]$. If we consider all the conditions that we can derive for each $n \leq K$ and its orbit under S_0, we obtain the following diagram, which must "commute" (over $n \leq K$):



In other words, the diagram represents, for each $n \leq K$ the different equations that all of its corresponding images must satisfy. Notice that for each n, the corresponding images are going to be finite, so that we can also think of the circles from the previous diagram representing the union of all the images where we need the equations to be satisfied.

Then we get $g(S_0(x))h(S_0(x))(n) = \sigma(g(x)(n))$ and $g(S_0^2(x))(h(S_0^2(x)(n)) = \sigma g((S_0(x))(n)$ and thus (putting together the first two pieces of the diagram) $g(S_0^2(x))h(S_0^2(x))h(S_0(x))(n) = \sigma^2(g(x)(n))$. Applying the same argument $N-1$ times we get

$$g(x)h(S_0^{N-1}(x))\cdots h(S_0(x))h(x)(n) = \sigma^N g(x)(n). \tag{6}$$

We build the desired function $g(x)$ backwards, starting from equation (6). So fix $x \in E_0$ and define

$$f(x) := h(S_0^{N-1}(x))h(S_0^{N-2}(x))\dots h(S_0(x))h(x). \tag{7}$$

Since σ^N is a generic element in S_∞, for each $x \in E_0$ we can find a permutation $\rho_x \in S_\infty$ such that $\rho_x^{-1}\sigma^N\rho_x(n) = f(x)(n)$ for all $n \leq K$. Thus, whenever $x \in E_0$, define $g(x)$ to be one of such permutations ρ_x.

Now suppose that g is defined in E_0 for all $n < K$ and satisfies the above property. We extend g to $\cup_{1\leq i\leq N-1}S_0^i(E_0)$ so that on $S_0^i(x) \in S_0^i(E_0)$ it satisfies the following equation:

$$g(S_0^i(x))(h(S_0^{i-1}(x))\dots h(S_0(x))h(x)(n)) = \sigma^i g(x)(n): \quad n \leq K$$

This might not define $g(S_0^i(x))$ for all $n < K$. But for those values not defined by this equation, we use the same argument as in E_0. This is well defined, since S_0 is periodic, so that no value will be repeated if it was chosen before. An easy verification shows us that $g(x)(h(x)(n)) = \sigma(g(S_0^{-1}(x)))(n)$ for all $x \in E_0 \cup S_0(E_0) \cup \dots S_0^{N-1}(E_0)$ as desired.

This process does not necesarilly defines $g(x)(n)$ for all n and as constructed the function g need not be measurable. However, this can be solved using Jankov-von Neumann uniformization, as in [KLM15]. Indeed, let us now show that we can find such a $g \in L^0([0,1],G)$. Let $P \subseteq E_0 \times S_\infty$ be defined by

$$(x,\rho) \in P \iff \rho^{-1}\sigma^N\rho(n) = f(x)(n): \quad x \in E_0, n < K.$$

By Jankov-von Neuman uniformization, we can choose for each $x \in E_0$ a $g_0(x) \in S_\infty$ in a measurable way.

Now let $P_1 \subseteq S_0(E_0) \times S_\infty$ be defined by $(x,\rho) \in P_1$ if and only if:

$$\rho h(S_0(x))h(x)(n) = \sigma g_0(S_0^{-i}(x))(n) : n < K, x \in S^i(E_0).$$

Recall that $h(x)$ is a simple measurable function and since S_0 is Borel measurable, so is $h(S_0(x))$. This proves that $h(S_0(x))h(x)$ is also a simple measurable function. Since we have a finite intersection of Borel conditions, P_1 is a Borel set. Again by Jankov-von Neuman uniformization, we can choose for each $x \in S_0(E_0)$ a $g_0(x) \in S_\infty$ in a measurable way.

Repeating this process for all i, we can choose $g(x) \in S_\infty$ in a measurable way.

□

Finally we see how to generalize the above result to a wide class of Polish groups.

Definition. Let G be a Polish group. We say that G has the *Rohklin property under powers* if there is $g \in G$ such that for all $N \geq 1$, the orbit of g^N under conjugation is dense.

Theorem 16. *If G is a Polish group that has the* Rohklin property under powers*, then $\tilde{G}$ has the Rohklin property.*

Instead of doing the proof again in the general setting, we point out how to modify the previous proof.

Proof. We choose a right-invariant complete metric on G, and call it d_G.

Choose $\sigma \in G$ that witnesses the Rohklin property under powers, and $S \in Aut([0,1])$ aperiodic. As in the previous proof, we will show that the tuple (C_σ, S) has a dense orbit. Let $U \times V$ be open in $\tilde{G}$. Once T has been chosen, we can choose $h \in TU$, and choose $\varepsilon > 0$ such that

$$\{f \in L^0([0,1], G) | \mu(\{x \in [0,1] | d_G(f(x), h(x)) < \varepsilon\}) \geq 1 - \varepsilon\} \subseteq TU.$$

Notice that the choices of S_0 and E_0 in the proof for S_∞ do not depend at all on $L^0([0,1], G)$. Thus we can choose them in the same way. As before, the idea is to define $g(x)$ for x in E_0, and then just expand it to all of the interval.

However, we no longer have an initial segment of the positive integers to do so. Instead, we want the following inequality to hold for all x in $\cup_{i<n} S_0^n(E_0)$.

$$d_G(g(x)^{-1}\sigma g(S_0^{-1}(x)), h(x)) < \varepsilon.$$

With the same idea in mind as in the other proof, we define the Borel relation $P \subseteq E_0 \times G$ by:

$$(x, \rho) \in P \Leftrightarrow d(\rho^{-1}\sigma^n \rho, h(x)h(S_0^{N-1}(x)) \dots h(S_0(x))) \leq \varepsilon.$$

Notice this set is not empty, since G has the Rohklin property under powers. Thus, it has a measurable uniformization. We define $g : E_0 \to G$ to be this uniformization. We expand it to all of $\cup_{i<n} S_0^n(E_0)$ by the following equation.

$$g(S_0^i(x)) = \sigma^i g(x) h(S_0(x))^{-1} h(S_0^2(x))^{-1} \dots h(S_0^i(x))^{-1}.$$

We claim that g so defined satisfies that for all $x \in \cup_{i<n} S_0^n(E_0)$

$$d_G(g(x)^{-1}\sigma g(S_0^{-1}(x)), h(x)) \leq \epsilon.$$

The calculation is straightforward, but as an illustration of what is happening, let us show it for $x \in E_0$.

$$\begin{aligned} d(g(x)^{-1}\sigma g(S_0^{-1}(x)), h(x)) &= d(g(x)^{-1}\sigma g(S_0^{N-1}(x),)h(x)) \\ &= d(g(x)^{-1}\sigma g(x)\sigma^{N-1}h(S_0(x))^{-1}\dots h(S_0^{N-1}(x))^{-1}, h(x)) \\ &= d(g(x)^{-1}\sigma^N g(x), h(x)h(S_0^{N-1}(x))\dots h(S_0(x))) \\ &\leq \varepsilon \end{aligned}$$

Since the last claim is valid for all $x \in \cup_{i<n} S_0^n(E_0)$, and this set has measure at least $1-\varepsilon$, then (C_σ, T) has a dense orbit. □

Proposition 17. *The group $G = Aut[0,1]$ has the Rohklin property under powers. In particular, $\tilde{G}$ has the Rohklin property.*

Proof. Notice that if T is aperiodic, then so is T^n for any $n \neq 0$. □

Proposition 18. *Let $\mathcal{U}$ be the group of unitary transformations on a separable Hilbert space. Then $\mathcal{U}$ has then Rohklin property under powers and $\tilde{\mathcal{U}}$ has the Rohklin property.*

Proof. Notice that $T \in \mathcal{U}$ is generic when its spectrum $\sigma(T) = S^1$. Then for any $n \geq 1$, $\sigma(T^n) = S^1$, so T^n is also generic. □

Proposition 19. *The group $G = Aut(\mathbb{Q}, \leq)$ has the Rohklin property under powers. In particular, $\tilde{G}$ has the Rohklin property.*

Proof. We use the characterization of the generic element found in [Tru92]. Given $g \in G$ and $x \in \mathbb{Q}$, define the orbital of x by g as the following set:

$$\text{obt}(x,g) := \{y \in \mathbb{Q} | \exists m, n \in \mathbb{Z}\, g^n(x) \leq y \leq g^m(x)\}.$$

and the sign of this orbital as:

$$\text{sgn}(x,g) := \begin{cases} 1 & \text{if } x < g(x), \\ -1 & \text{if } x > g(x), \\ 0 & \text{if } x = g(x). \end{cases}$$

Notice that both $\text{obt}(x,g) = \text{obt}(x,g^n)$ and $\text{sgn}(x,g) = \text{sgn}(x,g^n)$ for any n.

In [Tru92], Truss showed that g is generic if the set of orbitals with sign ϵ is a dense linear order without endpoints and is dense in the union of the other two. Thus, if g is generic, g^n is generic.

□

5. Transferring the Rohklin property to $\tilde{G}$

In this section we prove that the Rohklin property transfers from G to $\tilde{G}$. Instead of the algebraic approach from the previous section we follow a topological approach that works for all Polish groups.

Theorem 20. *If G has the Rohklin property, then so does $\tilde{G}$.*

Proof. Recall that the Polish topology in $\tilde{G}$ is the product topology, so it suffices to show that for any non-empty open sets $U_1, U_2 \subseteq L^0([0,1], G)$ and $V_1, V_2 \subseteq \text{Aut}[0,1]$ there is $(f,T) \in \tilde{G}$ such that $(U_1 \times V_1)^{(f,T)} \cap (U_2 \times V_2) \neq \varnothing$.

So assume we have such sets. Since $\text{Aut}[0,1]$ has a dense orbit, we can find $T \in \text{Aut}[0,1]$ such that $V_1^T \cap V_2 \neq \varnothing$. Since conjugation is a homeomorphism of the space $\tilde{G}$, $W_1 = (U_1 \times V_1)^{(C_e, T)}$ is open in $\tilde{G}$.

Notice that the projection on the second component of W_1 coincides with V_1^T and intersects V_2. Thus we can find open sets $U_3 \subseteq L^0([0,1],G)$ and $V_3 \subseteq \mathrm{Aut}[0,1]$ such that $U_3 \times V_3 \subseteq W_1$ and $V_3 \subseteq V_2$.

Since G has a dense orbit, $L^0([0,1],G)$ also has a dense orbit (see [KLM15]). Thus we can find $f \in L^0([0,1],G)$ such that $U_3^f \cap U_2 \neq \varnothing$. This proves that $(U_3 \times V_3)^{(f,Id)} \cap (U_2 \times V_3) \neq \varnothing$ and so we obtain that $(U_3 \times V_3)^{(f,Id)} \cap (U_2 \times V_2) \neq \varnothing$. Likewise, $(W_1)^{(f,Id)} \cap (U_2 \times V_2) \neq \varnothing$ and thus $((U_1 \times V_1)^{(C_e,T)})^{(f,Id)} \cap (U_2 \times V_2) \neq \varnothing$. □

From the proof, we obtain some corollaries.

Corollary 21. *Let H, G are topological groups such that H acts continuously on G. If H and G have the Rohklin property, so does $G \rtimes H$.*

Corollary 22. *Consider the action of G on G^n by diagonal conjugation, that is, each $g \in G$ sends $(g_1, \dots, g_n)$ to $(g_1^g, \dots, g_n^g)$. Assume G^n has a dense orbit under diagonal conjugation, then so does $\tilde{G}^n$.*

Proof. The proof is the same as before but now we consider open subsets $U_1, U_2 \subseteq L^0([0,1],G)^n$ and $V_1, V_2 \subseteq (\mathrm{Aut}[0,1])^n$ and use the fact that $\mathrm{Aut}[0,1]$ has ample metric generics and that if G^n has a dense orbit under diagonal conjugation so does $L^0([0,1],G)^n$ (see [KLM15]). □

Below we will use the following notation. We write d_u for the metric of uniform convergence for G, $\hat{d}_u$ for the induced metric of uniform convergence in $L^0([0,1],G)$, L_u for the metric of uniform convergence (see Definition 3) in $\tilde{G}$. For $B \subseteq G$, the set $\overline{B}^{d_u}$ stands for its closure with respect to the metric d_u and for $A \subseteq \tilde{G}$, $\overline{A}^{L_u}$ stands for its closure with respect to the metric L_u.

Proposition 23. *Let $h \in G$, let $C_h : [0,1] \to G$ be the function with constant value h and let $T \in Aut[0,1]$ be aperiodic. Then $\overline{(C_h,T)^{\tilde{G}}}^{L_u} \supseteq \{(C_h,S) : S \text{ is aperiodic } \}$.*

Proof. Let S be aperiodic and $\epsilon > 0$. By Rokhlin's Lemma there is $R \in \mathrm{Aut}[0,1]$ such that $\mu\{\omega \in [0,1] : R^{-1}TR(\omega) \neq S(\omega)\} < \epsilon$. Note that since C_h is a constant function the action by R on C_h is trivial. Thus by Theorem 13 we get that $L_u((C_h,T)^{(C_e,R)}, (C_h,S)) = L_u((C_h, R^{-1}TR), (C_h,S)) < \epsilon$. □

Observation 24. *Let $f, h \in L^0([0,1],G)$, let $R, T \in Aut[0,1]$ and assume that $(f,R) \in \overline{(h,T)^{\tilde{G}}}^{L_u}$. Then whenever $(f_1,R_1) \in (f,R)^{\tilde{G}}$ we also have $(f_1,R_1) \in \overline{(h,T)^{\tilde{G}}}^{L_u}$.*

Proof. Let $\epsilon > 0$ and let $(g,S) \in \tilde{G}$ be such that $L_u((f,R),(h,T)^{(g,S)}) < \epsilon$. Also let $(f_2,R_2) \in \tilde{G}$ be such that $(f_1,R_1) = (f,R)^{(f_2,R_2)}$ Since L_u is biinvariant, $L_u((f_1,R_1),((h,T)^{(g,S)})^{(f_2,R_2)}) = L_u((f,R)^{(f_2,R_2)},((h,T)^{(g,S)})^{(f_2,R_2)}) < \epsilon$ as we wanted. □

Theorem 25. *If (G,τ,d_u) has metric generics, then so does $\tilde{G}$.*

Proof. We will use again that the Polish topology in $\tilde{G}$ is the product topology and that the uniform convergence topology is also a product topology (see Theorem 13). Let $g \in G$ be such that $\overline{g^G}^{d_u}$ is comeager, so g is a metric generic. Let $T \in \mathrm{Aut}[0,1]$ be aperiodic, so it is a metric generic in the space of invertible measure preserving transformation. We will prove that (C_g,T) is a metric generic in $\tilde{G}$.

Notice that $(C_g, T)^{\tilde{G}}$ is Borel as well as $O = \overline{(C_g, T)^{\tilde{G}}}^{L_u}$. By Proposition 23, for S in the comeager subset $\{S \in \mathrm{Aut}[0,1] : S \text{ is aperiodic}\}$ of $\mathrm{Aut}[0,1]$, $(C_g, S) \in O$. Thus by Kuratowski-Ulam and Observation 24, it suffices to check that the fibers $\pi_1\{\overline{(C_g, S)^{(f,Id)}}^{Lu} : f \in L^0([0,1], G)\}$ are comeager for all such S.

First notice that by Theorem 13 and Section 2, the set $F = \overline{C_g^{L^0([0,1],G)}}^{\hat{d}_u}$ is comeager in $L^0([0,1], G)$. Now consider $\{(C_g, S)^{(f,Id)} : f \in L^0([0,1], G)\}$. Then by Theorem 13 we get that

$$\overline{\{(C_g, S)^{(f,Id)} : f \in L^0([0,1], G)\}}^{L_u} = (\overline{\{C_g^f : f \in L^0([0,1], G)\}}^{\hat{d}_u}, S) = (F, S)$$

The result follows from the fact that $\{S \in \mathrm{Aut}[0,1] : S \text{ is aperiodic}\}$ and F are comeager. □

Theorem 26. *Assume (G, τ, d_u) has metric ample generics, then so does $\tilde{G}$.*

Proof. The proof is very similar to the one of the previous theorem. For each $n \geq 1$ let $\vec{g} = (g_1, \dots, g_n) \in G^n$ be such that $\overline{\vec{g}^G}^{d_u}$ is comeager, where the action by G is given by diagonal conjugation. Let $\vec{T} = (T_1, \dots, T_n) \in \mathrm{Aut}[0,1]^n$ be a tuple such that $\overline{\vec{T}^{Aut([0,1])}}^{\Delta_u}$ is comeager, where again the action is given by diagonal conjugation. Then $(C_{g_1}, \dots, C_{g_n}, T_1, \dots, T_n)$ is a metric generic under the action by diagonal conjugation by $\tilde{G}$. □

6. Extreme amenability

A topological group G is **extremely amenable** if every action of G on a compact space has a fixed point. Notice however that discrete groups G have a free action on βG. Likewise, it can be shown that locally compact groups have a free action on some compact space. So this notion, although inspired in amenability on locally compact groups is only interesting on non-locally compact groups.

Theorem 27.
(1) *(Pestov, Schneider,* [PS17]*) If G is an amenable group, then $L^0([0,1], G)$ is extremely amenable.*
(2) *(Giordano, Pestov,* [GP02]*) $Aut[0,1]$ is extremely amenable.*
(3) *(Pestov, see* [Pes06]*) Let H' be a closed subgroup of a topological group H. If the topological groups H' and H/H' are extremely amenable, then so is G.*

As a corollary of this, we obtain the following.

Theorem 28. *If G is an amenable group, then $\tilde{G}$ is extremely amenable.*

Proof. If e is the identity on G, let C_e denote the constant function with value e. Take $H = \{(C_e, S) | S \in \mathrm{Aut}[0,1]\}$. This is a closed subgroup of $\tilde{G}$ which is extremely amenable as a topological group. Note that G/H isomorphic as a topological group to $L^0([0,1], G)$. In fact, take the function $F : L^0([0,1], G) \to G/H$ defined by $F(f) = (f, \mathrm{id})H$. Since G is amenable by Theorem 27 we get that $L^0([0,1], G)$ is extremely amenable and again by Theorem 27 $\tilde{G}$ is extremely amenable. □

7. Questions

Here are a few open questions that follow from the results in these papers.

(1) Does $\tilde{S}_\infty$ have automatic continuity?
(2) If G has Rohklin property does it have Rohklin property under powers?
(3) If H and G have the strong Rohklin property, does $G \rtimes H$ have the strong Rohklin property?

References

[BY09] Itaï Ben Yaacov. Continuous and random Vapnik-Chervonenkis classes. *Israel J. Math.*, 173:309–333, 2009.

[BYBM13] Itaï Ben Yaacov, Alexander Berenstein, and Julien Melleray. Polish topometric groups. *Trans. Amer. Math. Soc.*, 365(7):3877–3897, 2013.

[BYK09] Itaï Ben Yaacov and H. Jerome Keisler. Randomizations of models as metric structures. *Confluentes Math.*, 1(2):197–223, 2009.

[GP02] Thierry Giordano and Vladimir Pestov. Some extremely amenable groups. *C. R. Math. Acad. Sci. Paris*, 334(4):273–278, 2002.

[Hal60] Paul R. Halmos. *Lectures on ergodic theory*. Chelsea Publishing Co., New York, 1960.

[HHLS93] Wilfrid Hodges, Ian Hodkinson, Daniel Lascar, and Saharon Shelah. The small index property for ω-stable ω-categorical structures and for the random graph. *J. London Math. Soc. (2)*, 48(2):204–218, 1993.

[Iba17] Tomas Ibarlucia. Automorphism groups of randomized structures. *Journal of Symbolic Logic*, 82(3):1150–1179, 2017.

[Kaï] Adriane Kaïchouh. Variations on automatic continuity. http://math.univ-lyon1.fr/homes-www/kaichouh/research.html.

[Kec10] Alexander S. Kechris. *Global aspects of ergodic group actions*, volume 160 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2010.

[KLM15] Adriane Kaïchouh and François Le Maître. Connected Polish groups with ample generics. *Bull. Lond. Math. Soc.*, 47(6):996–1009, 2015.

[KM] Aleksandra Kwiatkowska and Maciej Malicki. Groups of measurable functions. arXiv:1612.03106 [math.LO].

[KR07] Alexander S. Kechris and Christian Rosendal. Turbulence, amalgamation, and generic automorphisms of homogeneous structures. *Proc. Lond. Math. Soc. (3)*, 94(2):302–350, 2007.

[Mal16] Maciej Malicki. Consequences of the existence of ample generics and automorphism groups of homogeneous metric structures. *J. Symb. Log.*, 81(3):876–886, 2016.

[Pes06] Vladimir Pestov. *Dynamics of infinite-dimensional groups*, volume 40 of *University Lecture Series*. American Mathematical Society, Providence, RI, 2006. The Ramsey-Dvoretzky-Milman phenomenon, Revised edition of ı Dynamics of infinite-dimensional groups and Ramsey-type phenomena [Inst. Mat. Pura. Apl. (IMPA), Rio de Janeiro, 2005; MR2164572].

[PS17] Vladimir G. Pestov and Friedrich Martin Schneider. On amenability and groups of measurable maps. *J. Funct. Anal.*, 273(12):3859–3874, 2017.

[Sab17] Marcin Sabok. Automatic continuity for isometry groups. *Journal of the Institute of Mathematics of Jussieu*, 2017.

[Tru92] J. K. Truss. Generic automorphisms of homogeneous structures. *Proc. London Math. Soc. (3)*, 65(1):121–141, 1992.

[WZ15] Richard L. Wheeden and Antoni Zygmund. *Measure and integral*. Pure and Applied Mathematics (Boca Raton). CRC Press, Boca Raton, FL, second edition, 2015. An introduction to real analysis.

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