A GEOMETRIC SPLITTING THEOREM

JOSÉ ROSALES-ORTEGA

ABSTRACT. Let $G = G_1 \cdots G_l$ be a connected noncompact semisimple Lie group with Lie algebra $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_l$ acting topologically transitive on a manifold M. We obtain a geometric splitting of the metric on M that consider metrics on each G_i. Also we obtained a result about the isometry group of the manifold $G \times \tilde{N}$, where $\tilde{N}$ is the universal covering of a leaf N of the normal foliation to the G-orbits.

1. Introduction

It is well known that interesting examples of pseudo-Riemannian manifolds appear when we consider Lie groups with bi-invariant pseudo-Riemannian metrics. In addition, this class of Lie groups that support bi-invariant pseudo Riemannian metrics is quite large.

The isometry group of a Lie group G, denoted by $I(G)$, has been the subject of a systematic study. It is well known that $I(G)$, among other properties, is a Lie group, see [6].

Under certain conditions it can be proven that a manifold M is isometric to a product of two manifolds, and this is known as a splitting theorem, see [8].

Our main goal in this paper is to prove an isometric splitting theorem for the metric of the pseudo Riemannian manifold M on which a semisimple Lie group G acts.We also calculate the group of isometries of a product of manifolds that appear in the main result.

The organization of this article is as follows. In section 2 we collect some basic results about bi-invariant pseudo-Riemannian metrics on a Lie group G. Also we give the classification of the $\mathrm{Ad}(G)$-invariant bilinear forms on a semisimple Lie algebra. This is mentioned in [3], but the generalization to semisimple Lie groups is new, also see [11].

Date: April 10, 2010.

2000 *Mathematics Subject Classification.* Primary: 53C05; Secondary: 53C10.

Key words and phrases. semisimple Lie groups, bi-invariant metric, local freeness.

As a consequence we give the classification of the bi-invariant pseudo-Riemannian metrics on G. In section 3 we give some results about the foliation given by the G-orbits. In section 4 we show the main

I would like to thank David Lovelady for useful comments that allowed to simplify the exposition of this work.

2. Bi-invariant metrics on a semi-simple Lie group

We are going to study the geometry of the orbits and the normal bundle in the case where a semi-simple Lie group G acts on a pseudo Riemannian manifold M, with the aim of obtaining a description of the pseudo Riemannian manifold on which the Lie group acts.

The following lemma, although trivial, will be of great importance in the subsequent results, and therefore we present your proof by the sake of completeness of the work.

Lemma 1. *Let* $F\colon \mathfrak{g}\times\mathfrak{g}\to\mathbb{R}$ *be a symmetric bilinear form that is* $\mathrm{Ad}(G)$*-invariant, then* $F([X,W],Y)=F(X,[W,Y])$ *for* $W,X,Y\in\mathfrak{g}$*. The converse holds if* G *is connected.*

Proof. It is sufficent to show that $F([W,X],X)=0$ for all $W,X\in\mathfrak{g}$. If we assume the above observation, then

$$\begin{aligned}
F([W,&X+Y],X+Y)\\
&=F([W,X+Y],X)+F([W,X+Y],Y)\\
&=F([W,Y],X)+F([W,X],Y)\\
&=F(X,[W,Y])-F([W,X],Y)=0.
\end{aligned}$$

We consider the following function $f(s)=F(\mathrm{Ad}(\alpha(s))X,\mathrm{Ad}(\alpha(s))X)$, where $\alpha(s)=\exp(sY)$ and $Y\in\mathfrak{g}$.

We affirm that f is constant on each one-parameter subgroup α of G.

Using the $\mathrm{Ad}(G)$-invariance of F it follows that

$$\begin{aligned}
f'(s)&=\lim_{t\to 0}\frac{1}{t}\big\{f(s+t)-f(s)\big\}\\
&=\lim_{t\to 0}\frac{1}{t}\big\{F(\mathrm{Ad}(\alpha(s+t))X,\mathrm{Ad}(\alpha(s+t))X)\\
&\qquad\qquad -F(\mathrm{Ad}(\alpha(s))X,\mathrm{Ad}(\alpha(s))X)\big\}\\
&=\lim_{t\to 0}\frac{1}{t}\big\{F(\mathrm{Ad}(\alpha(s))X,\mathrm{Ad}(\alpha(s))X)-F(\mathrm{Ad}(\alpha(s))X,\mathrm{Ad}(\alpha(s))X)\big\}\\
&=0.
\end{aligned}$$

From this the direct assertion follows. ∎

The classification of the $\mathrm{Ad}(G)$-invariant bilinear forms on a simple Lie algebra can be found in [4]. The next result gives the classification of the $\mathrm{Ad}(G)$-invariant bilinear forms on a semisimple Lie algebra.

Theorem 1. *Let G be a connected semisimple Lie group such that $\mathrm{Lie}(G) = \mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_l$, where each $\mathfrak{g}_i$ is a simple ideal of the Lie algebra $\mathfrak{g}$. We shall suppose the following:*

- *The complexification of each $\mathfrak{g}_i$, for $i = 1, \ldots, k$ is simple; and*
- *The complexification of each $\mathfrak{g}_i$, for $i = k+1, \ldots, l$ is not simple and so there exists a complex structure J_i for each $\mathfrak{g}_i$.*

Then every $\mathrm{Ad}(G)$-invariant bilinear form B on $\mathfrak{g}$ is given by

$$B = \lambda_1 B_{\mathfrak{g}_1} \oplus \cdots \oplus \lambda_k B_{\mathfrak{g}_k} \oplus (\mu_1^{k+1} B_{\mathfrak{g}_{k+1}} + \mu_2^{k+1} B_{\mathfrak{g}_{k+1}}^{J_{k+1}}) \oplus \cdots \oplus (\mu_1^l B_{\mathfrak{g}_l} + \mu_2^l B_{\mathfrak{g}_l}^{J_l}),$$

where each $B_{\mathfrak{g}_i}$ is the Killing–Cartan form on $\mathfrak{g}_i$, for $i = 1, \ldots, l$, all λ_i and μ_i^j are real numbers, and $B_{\mathfrak{g}_j}^{J_i}(X, Y) = B_{\mathfrak{g}_j}(X, J_i Y)$.

Proof. By the previous lemma it follows that $B(X, [Y, Z]) = B([X, Y], Z)$, for all $X, Y, Z \in \mathfrak{g}$.

On the other hand, it is easy to show the following properties: $[\mathfrak{g}_i, \mathfrak{g}_j] = \{0\}$ for all $i \neq j$, and $[\mathfrak{g}_i, \mathfrak{g}_i] = \mathfrak{g}_i$, for all i.

We have for $Y \in \mathfrak{g}_j$ there exists $Z, W \in \mathfrak{g}_j$ such that $Y = [Z, W]$, and for $X \in \mathfrak{g}_i$ we conclude $B(X, Y) = B([X, Z], W) = 0$.

Therefore $\mathfrak{g}_i \perp \mathfrak{g}_j$ for all $i \neq j$ with respect to B. From this it follows that $B = B_{\mathfrak{g}_1} \oplus \cdots \oplus B_{\mathfrak{g}_l}$.

Now we use the classification of $\mathrm{Ad}(G)$-invariants bilinear forms on a simple Lie algebra given in [4]. □

Using the previous results we can give a classification of the Bi-invariant metrics on semisimple Lie groups.

Theorem 2. *Let G be a connected semisimple Lie group such that $\mathrm{Lie}(G) = \mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_l$, where each $\mathfrak{g}_i$ is a simple ideal of the Lie algebra $\mathfrak{g}$. We shall suppose the following:*

- *The complexification of each $\mathfrak{g}_i$, for $i = 1, \ldots, k$ is simple; and*
- *The complexification of each $\mathfrak{g}_i$, for $i = k+1, \ldots, l$ is not simple and so there exists a complex structure J_i for each $\mathfrak{g}_i$.*

Then every Bi-invariant pseudo Riemannian metric ϕ on G is given by

$$\phi = \lambda_1 B_{\mathfrak{g}_1} \oplus \cdots \oplus \lambda_k B_{\mathfrak{g}_k} \oplus (\mu_1^{k+1} B_{\mathfrak{g}_{k+1}} + \mu_2^{k+1} B_{\mathfrak{g}_{k+1}}^{J_{k+1}}) \oplus \cdots \oplus (\mu_1^l B_{\mathfrak{g}_l} + \mu_2^l B_{\mathfrak{g}_l}^{J_l}),$$

where each $B_{\mathfrak{g}_i}$ is the Killing–Cartan form on $\mathfrak{g}_i$, for $i = 1, \ldots, l$, all λ_i and μ_i^j are real numbers, and $B_{\mathfrak{g}_j}^{J_i}(X, Y) = B_{\mathfrak{g}_j}(X, J_i Y)$.

3. Properties of the foliation given by a G-action

From now on $G = G_1 \cdots G_l$ will be a connected noncompact semisimple Lie group with Lie algebra $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_l$. We shall suppose the following:

- The complexification of each $\mathfrak{g}_i$, for $i = 1, \ldots, k$ is simple; and
- The complexification of each $\mathfrak{g}_i$, for $i = k+1, \ldots, l$ is not simple and so there exists a complex structure J_i for each $\mathfrak{g}_i$.

We know that G admits bi-invariant pseudoRiemannian metrics and all of them can be described in terms of the Killing-Cartan form on each $\mathfrak{g}_i$.

Let M be a connected finite volume pseudo Riemannian manifold. We always assume that the action of G on M is analytic , faithful, and preserves a finite measure on M and the metric.

Definition 1. The dimension of maximal lightlike tangent subspaces for M will be denoted with $m_0 = \min\{m_1, m_2\}$, where (m_1, m_2) represent the signature of M, i.e., that m_1 correspond to the dimension of maximal timelike tangent subspaces and m_2 correspond to the dimension of the maximal spacelike tangent subspaces.

Definition 2. The dimension of maximal lightlike tangent subspaces for G_i, $i = 1, \ldots, l$, will be denoted with $n_0^i = \min\{n_1^i, n_2^i\}$, where (n_1^i, n_2^i) represent the signature of G_i.

Gromov remarked in [3] that if (n_1, n_2) is the signature of the metric given by the Killing–Cartan form on $\mathfrak{g}$, then any other bi-invariant pseudoRiemannian metric on G has signature given by either (n_1, n_2) or (n_2, n_1).

We will denote with $T\mathcal{O}$ the tangent bundle to the orbits of the G-action on M. If $X \in \mathfrak{g}$, we define the infinitesimal generator X^* as the vector field on M induced by X. This new vector field is given by

$$X^*_p \, = \, \frac{d}{dt}\,|_{t=0} \exp{(tX)} \cdot p \; .$$

It is clear that X^* is a Killing vector field, and $X^*_p \in T_p(G \cdot p)$, for $p \in M$. We will use the following map: $\varphi_x : \mathfrak{g} \to T_x(Gx)$, given by $\varphi_x(X) = X^*_p$, where $x \in M$.

We will consider the so-called smooth normal bundle $T\mathcal{O}^{\perp}$. In a previous paper, [12], we proved that this normal bundle is integrable under the same conditions on M and G that we will stablish in the next section.

4. A Geometric Splitting

Theorem 3. *For G and M as before suppose G acts topologically transitively on M, i.e., there is a dense G-orbit, preserving its pseudoRiemannian metric and satisfying $n_0^1 + \cdots + n_0^l = m_0$. Then the metric h, of M, restricted to the orbits defines a leafwise pseudo Riemannian metric which is given by*

$$\sum_{i=1}^{k} f_i B_{\mathfrak{g}_i} + \sum_{j=k+1}^{l} \left(f_{1,j} B_{\mathfrak{g}_j} + f_{2,j} B^{\mathfrak{g}_j, J_j} \right),$$

where $f_i, f_{1,j}, f_{2,j} : \mathbb{R} \to \mathbb{R}$ are G-invariant smooth functions, for all $i = 1, \ldots, k$, and $j = k+1, \ldots, l$.

Proof. Let $x \in M$ and consider $\Psi : M \to \mathfrak{g}^* \otimes \mathfrak{g}^*$ given by the bilinear symmetric map $\Psi_x = h_x(\varphi_x \cdot, \varphi_x \cdot) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$. It is know that Ψ_x is $\mathrm{Ad}(G)$-invariant, and we only show that Ψ_x is independient of x.

If $x \in M$, $X, Y \in \mathfrak{g}$, and $g \in G$, then because G preserves the metric, we have $\Psi_{gx}(X, Y) = h_x(dg_{gx}^{-1} X_{gx}^*, dg_{gx}^{-1} Y_{gx}^*)$. On the other hand the fact that $dg(X^*) = \mathrm{Ad}(g) X^*$ implies $h_x(dg_{gx}^{-1} X_{gx}^*, dg_{gx}^{-1} Y_{gx}^*) = h_x(\mathrm{Ad}(g^{-1}) X_x^*, Ad(g^{-1}) Y_x^*)$. Based on the previous result, we conclude that the map Ψ satifies $\Psi_{gx}(X, Y) = \Psi_x(\mathrm{Ad}(g^{-1})X, Ad(g^{-1})Y)$, which means that Ψ is G-equivariant. Therefore Ψ is G-invariant, and the result follows because G acts topologically transitively on M.

By a result in [11] the action is locally free everywhere and each orbit is nondegenerate. Then the metric h on M restricted to each orbit is nondegenerate, therefore Ψ_x, the metric induced on G, is Bi-invariant, then we can use the classification of Bi-invariant pseudo Riemannian metrics for semisimple groups and the claim follows.

□

The next corollary is an easy consequence of the above and it will be useful in the main theorem of this work.

Corollary 1. *With the same hypothesis of theorem 3, there is a G-equivariant isometric immersion from G to M when G is considereted with a bi-invariant pseudo Riemannian metric.*

We will need the following result in order to prove some of our theorems. Given a pseudo Riemannian manifold M we will denote with $\tilde{M}$ a covering space of M, and with $\mathrm{Kill}(\tilde{M}, x)$ the Lie algebra of germs at x of local Killing vector fields defined in a neighborhood of x.

In [1] it is introduced the concept of $\mathcal{H}$-curve in a connected Riemannian manifold as a piecewise smooth curve all of whose tangent vectors are perpendiculars to the leaves of a smooth totally geodesic foliation of the manifold. We apply this to $T\mathcal{O}$ and its normal bundle. $T\mathcal{O}^{\perp}$.

Theorem 4. *For G and M as before suppose G acts topologically transitively on M and preserves a finite volume pseudo Riemannian metric. There exists a dense conull subset $S \subset \tilde{M}$ such that for every $x \in S$ we have a Lie algebra $Kill(\tilde{M}, x)$ and a homomorphism $\rho_x : \mathfrak{g} \to Kill\ (\tilde{M}, x)$ satisfying*

- *all elements of $Kill(\tilde{M}, x)$ vanish at x.*
- *ρ_x is an isomorphism onto its image $\rho_x(\mathfrak{g}) = \mathfrak{g}(x)$.*
- *$[\rho_x(\mathfrak{g}), \mathcal{Y}] \subset \mathcal{Y}$, where $\mathcal{Y} = \{Y^* : Y \in \mathfrak{g}\}$.*

Theorem 5. *For G and M as before suppose G acts topologically transitively on M, i.e., there is a dense G-orbit, preserving its pseudoRiemannian metric, h, and satisfying $n_0^1 + \cdots + n_0^l = m_0$. If S is an open conull set in M, then with the metric induced by M, the leaves of the foliation $T\mathcal{O}$ lying in a fixed component of S have isometric universal coverings.*

Proof. As the orbits are nondegenerate then $dL_g\left(T_x(Gx)^{\perp}\right) \subset T_x(Gx)^{\perp}$, for every $x \in S$, and $g \in G$. In fact, for every $x \in S$, we have $T_x(S) = T_x(Gx) \oplus T_x(Gx)^{\perp}$. If $v^{\perp} \in T_x(Gx)^{\perp}$, then $dL_g(v^{\perp}) = w^{\perp} + w$, where $w \in T_x(Gx)$. Therefore, $h(dL_g(v^{\perp}), v) = h(w, v)$. On the other hand, it follows that $h(v^{\perp}, dL_{g^{-1}}v) = 0$, and then $dL_g(v^{\perp}) \in T_x(Gx)^{\perp}$. This proves that the action preserves the normal bundle $T\mathcal{O}^{\perp}$.

We now prove that GL is a connected component of S if L is a leaf of the foliation $T\mathcal{O}^{\perp}$. Using the same argument that appears in corollary 2.8 in [1], we define an equivalence relation on the leaves of $T\mathcal{O}^{\perp}$ of S by saying $L_1 \sim L_2$ if $L_2 = gL_1$ for some $g \in G$. It is easy to prove that $GL = [L]$, the equivalent class of L. Therefore GL is open in S, since $GL = \phi(G \times L)$, where the local diffeomorphism $\phi : G \times L \to M$ is the restricion of the G-action map to the leaf L. Also, GL is open because is the complement of the union of open sets. Therefore GL is a connected component of S.

The theorem is now a direct consequence of the above. □

The next result is the main theorem of this work, and will give us a kind of splitting geometric theorem on the manifold M.

We observe this theorem is in spirit, of the same class than theorem A in [1] and theorem A in [10].

Theorem 6. *For G and M as before suppose G acts topologically transitively on M, i.e., there is a dense G-orbit, preserving its pseudoRiemannian metric, h, and satisfying $n_0^1 + \cdots + n_0^l = m_0$. If L is a leaf* *of the normal foliation $\mathcal{F}^{\perp}$ in S, then there is a $\tilde{G}$-equivariant map $\tilde{G}\times\tilde{L}\to S$ that isometrically covers the connected component S_0 of S that contains the leaf L, where $\tilde{G}\times\tilde{L}$ is furnished with a metrix k given as follows.*

$$k=\sum_{i=1}^{k} f_i B_{\mathfrak{g}_i}+\sum_{j=k+1}^{l}\left(f_{1,j}B_{\mathfrak{g}_j}+f_{2,j}B^{\mathfrak{g}_j,J_j}\right)+h^{\mathcal{F}^{\perp}},$$

where $f_i, f_{1,j}, f_{2,j}:\tilde{L}\to\mathbb{R}$ are smooth functions, for all $i=1,\dots,k$, and $j=k+1,\dots,l$ and $h^{\mathcal{F}}$ is the metric on the common universal cover of the leaves of $\mathcal{F}$, induced by the restriction of the metric h on S to the leaves of the foliation $\mathcal{F}$.

Proof. Let $\psi=\tilde{G}\times\tilde{S}_0\to\tilde{S}_0$ be the lifted action. We consider $\hat{L}$ a leaf of the normal bundle to the orbits in $\tilde{S}_0$ that is mapped onto L by the covering map $\pi|_{\tilde{S}_0}:\tilde{S}_0\to S_0$.

Let $\phi=\psi|_{\tilde{G}\times\hat{L}}$ denote the restriction of ψ to $\tilde{G}\times\hat{L}$. It is an easy consequence of corollary 1 and theorem 5 that ϕ is an isometric immersion of $\tilde{G}\times\hat{L}$ to $\tilde{S}_0$, where $\tilde{G}\times\hat{L}$ is considered a pseudo Riemannian manifold with metric, k, given by

$$k=\sum_{i=1}^{k} f_i B_{\mathfrak{g}_i}+\sum_{j=k+1}^{l}\left(f_{1,j}B_{\mathfrak{g}_j}+f_{2,j}B^{\mathfrak{g}_j,J_j}\right)+h^{\mathcal{F}^{\perp}}.$$

We will show that ϕ is injective and that the universal cover ol L is equal to $\hat{L}$.

Using theorem 4.1 from [12] we conclude that the leaves of the foliation $\mathcal{F}^{\perp}$ on S_0 and $\tilde{S}_0$ are totally geodesic.

For the rest of the proof we are going to use some ideas established in theorem A of [1].

In the proof of theorem 5 we obtained that $GL=S_0$. From this it follows there are open sets $V_1\subset G, V_2\subset L, V_x\subset S_0$, for every $x\in S_0$, such that the map $H_x:V_1\times V_2\to V_x$ is a diffeomorphism.

If we denote with $\pi_2:V_1\times V_2\to V_2$, the proyection on the second factor, then we obtain a submersion $\pi_2\circ H_x^{-1}:V_x\to V_2$. This submersion locally defines the foliation on S_0 given by the G-orbits. Note that $\pi_2\circ H_x^{-1}$ is a pseudo Riemannian submersion.

We can obtain, $\{U_\alpha\}_\alpha$, an open covering of S_0 for which we have pseudo Riemannian submersions $H_\alpha:U_\alpha\to L$. For each α, the open set U_α is connected, evenly covered by the universal covering $\pi:\tilde{S}_0\to S_0$, and $H_\alpha(U_\alpha)$ contained in an open set evenly covered by $\pi_1:\hat{L}\to L$.

There exists a pseudo Riemannian submersion $H : \tilde{S}_0 \to \tilde{L}$ such that $\pi_1 \circ H|_{\tilde{U}_{\alpha,k}} = H_\alpha \circ \pi$, for every α, and $\pi^{-1}(U_\alpha) = \cup_k \tilde{U}_{\alpha,k}$. The proof of this is based on known arguments of algebraic topology.

It is concluded that the foliation defined by submersion H is the foliation defined by the orbits of the action of $\tilde{G}$ on $\tilde{S}_0$. In particular, H is a local isometry when is restricted to every leaf $\hat{L}_x$ of the foliation in $\tilde{L}$ given by $\mathcal{F}^\perp$. Moreover, $H|_{\hat{L}_x} : \hat{L}_x \to \tilde{L}$ is a bijection. This is proven by seeing that for each $x \in \hat{L}_x$, every geodesic $\hat{\sigma} : [0,1] \to \hat{L}$ with $\hat{\sigma}(0) = H(w)$ can be lifted a geodesic $\tilde{\sigma} : [0,1] \to \hat{L}_x$ with $\tilde{\sigma}(0) = w$. Therefore, $H|_{\hat{L}_x} : \hat{L}_x \to \tilde{L}$ is a covering map, and $\hat{L}_x$ is the universal cover of L_x for every $x \in S$.

If $\phi(g_1, x_1) = \phi(g_2, x_2)$, then $g_1 x_1 = g_2 x_2$. By using the fact that H is G-invariat, it follows that $H(x_1) = H(g_1 x_1) = H(g_2 x_2) = H(x_2)$, therefore $x_1 = x_2$.

On the other hand, it is easy to see that $g = g_1^{-1} g_2 \in \mathrm{Stab}(x_1)$ and we can consider $V_{x_1} \subset \hat{L}$ a normal neighborhood of x_1. Let σ be a geodesic from x_1 to x, where $x \in V_{x_1}$. Then $H(\sigma(1)) = H(g\sigma(1))$, so we obtain $\sigma(1) = g\sigma(1) \in \tilde{L}$. Therefore $g \in \mathrm{Stab}(\sigma(1))$ and g fixes V_{x_1}. By using that the action is analytic it follows that g fixes $\hat{L}$.

Let $\mathrm{Stab}(\hat{L})$ be denoted the subgroup of G that fixes the points in $\hat{L}$. The map $\Phi : \tilde{G}/\mathrm{Stab}(\hat{L}) \times \tilde{L} \to \tilde{S}_0$ given by $\Pi\left(g + \mathrm{Stab}(\hat{L}), x\right) = \phi(g, x)$ is a diffeomorphism. It follows that $\mathrm{Stab}(\hat{L}) = \{e\}\}$ because $\tilde{S}_0$ is simply connected, so we obtain $g_1 = g_2$. Therefore, ϕ is injective.

We obtain the main part of the theorem by performing the following composition: $\pi \circ \phi : \tilde{G} \times \tilde{S}_0 \to S_0$.

□

5. Isometric Splitting

To obtain an isometric splitting, in the case where M is compact, we will need the following result.

Lemma 2. *If G acts topologically transitive on M preserving its pseudo Riemannian metric and satisfying $n_0^1 + \cdots n_0^l = m_0$, then the leaves of the foliation defined by $T\mathcal{O}$ are complete for the metric induced by M.*

Proof. We know that $T\mathcal{O}^\perp$ is either Riemannian or antiRiemannian, see [11]. Hence, the foliation by G-orbits on M carries a Riemannian or antiRiemannian structure obtained from $T\mathcal{O}^\perp$.

On the other hand, using the compactness of M it follows that geodesic completeness is satisfied for geodesics orthogonal to the G-orbits, then we get the completeness for leaves of the foliation given by $T\mathcal{O}^{\perp}$, see [5]. □

The next theorem describes properties of the manifold M.

Theorem 7. *Suposse G acts topologically transitive and by isometries on M, and satisfying $n_0 = m_0$. Let N be a leaf of the foliation defined by $T\mathcal{O}^{\perp}$, and consider it as a pseudo Riemannian manifold with the metric inherited by M. Then the map $G \times N \to M$, obtained by restricting the G-action to N, is a G-equivariant pseudo Riemannian covering map. Also, we obtained a G-equivariant pseudo Riemannian covering map $G \times \widetilde{N} \to M$, where $\widetilde{N}$ is the universal covering space of N.*

Proof. By lemma 2 we have that N is a complete manifold. It is know that G is complete, see chapter II of [2]. Hence $G \times N$ is a complete pseudo Riemannian manifold. As $G \times N \to M$ is an local isometry it follows(see [7]) that $G \times N \to M$ is a pseudo Riemannian covering map. The G-invariancy follows of this $h \cdot (g, n) = (hg, n)$.

If we called Φ to the map $G \times N \to M$, then we obtained a local isometry $\psi : G \times \widetilde{N} \to M$ if we defined $\psi(g, \widetilde{n}) = \Psi(g, \pi(\widetilde{n}))$.

The map ψ is G-equivariant because the following:

$$\begin{aligned} \psi(g_1 \cdot (g, \widetilde{n})) &= \psi(g_1 g, \widetilde{n}) \\ &= \Phi(g_1 g, \pi(\widetilde{n})) \\ &= g_1 \cdot \Phi(g, \pi(\widetilde{n})) \\ &= g_1 \cdot \psi(g, \widetilde{n}). \end{aligned}$$

□

Definition 3. Let (M, g) be a pseudoRiemannian space. We say that (M, g) is a pseudoRiemannian symmetric space if each $x \in M$ is an isolated fixed point of an involutive isometry s_x of M. Any isometry of the form $s_x \circ s_y$ is called a transvection of M. the group T generated by transvections is called the transvection group of M.

Definition 4. A pseudo-Riemannian symmetric triple is a triple $(\mathfrak{g}, \sigma, B)$ consisting of the following objects:

(1) a finite dimensional real Lie algebra $\mathfrak{g}$,
(2) an involutive automorphism σ of $\mathfrak{g}$, and
(3) a symmetric bilinear form B on $\mathfrak{g}$, which is non-degenerate and invariant by T and by σ.

We will be interested in the following symmetric triple $(\mathrm{Lie}(T), \sigma, B)$, where B is a suitable bilinear form on $\mathrm{Lie}(T)$ and σ is the differential at $e \in T$ of the conjugation by some involution s_0.

Definition 5. A connected pseudoRiemannian manifold is called weakly irreducible if the tangent space in every point has no non-singular proper subspaces invariant by the holonomy group at that point.

Since G is semisimple, we know that $G = G_1 \times \cdots \times G_l$. The universal covering space for G is $\widetilde{G_1} \times \cdots \times \widetilde{G_l}$. It is known that for any bi-invariant pseudoRiemannian metric on G_i, the universal covering space is weakly irreducible, therefore we can conclude that $\widetilde{G_1} \times \cdots \times \widetilde{G_l}$ is weakly irreducible.

Lemma 3. *The universal covering $\widetilde{G}$ is weakly irreducible for $G = G_1 \cdots G_l$ a connected noncompact semisimple Lie group, and for any pseudoRiemannian bi-invariant metric on G.*

Theorem 8. *Let N be a connected complete Riemannian, or antiRiemannian, manifold. Then*

$$Iso(G_1 \times \cdots \times G_l \times N) = Iso(G_1 \times \cdots \times G_l) \times Iso(N)$$

.

Proof. The universal covering of $G_1 \times \cdots \times G_l \times N$ is $\widetilde{G_1} \times \cdots \times \widetilde{G_l} \times \widetilde{N}$

Let $\widetilde{N} = N_0 \times N_1 \times \cdots \times N_k$ be the de Rham decomposition of $\widetilde{N}$ as Riemannian manifold. By the de Rham-Wu decomposition for pseudoRiemannian manifolds, see [**?**], and by lemma 3 , it follows that $\widetilde{G_1} \times \cdots \times \widetilde{G_l} \times \widetilde{N}$ has a de Rham decomposition and its is given by

$$\widetilde{G_1} \times \cdots \times \widetilde{G_l} \times N_0 \times N_1 \times \cdots \times N_k.$$

It is known that such decomposition is unique up to order. In particular, every isometry of $\widetilde{G_1} \times \cdots \times \widetilde{G_l} \times N_0 \times N_1 \times \cdots \times N_k$ permutes the factors, but since each N_i is Riemannian and $\widetilde{G}_i$ is not, then every isometry of $\widetilde{G_1} \times \cdots \times \widetilde{G_l} \times N_0 \times N_1 \times \cdots \times N_k$ preserves the factors $\widetilde{G_1} \times \cdots \times \widetilde{G_l}$ and $N_1 \times \cdots \times N_k$.

Now let $f \in \mathrm{Iso}(G_1 \times \cdots \times G_l \times N)$ and lift it to an isometry $\widetilde{f}$ of $\widetilde{G_1} \times \cdots \times \widetilde{G_l} \times \widetilde{N}$. By the previous arguments, $\widetilde{f}$ preserves the product, i.e, if we write $\widetilde{f} = (\widetilde{f_1}, \widetilde{f_2})$, then $\widetilde{f_1}$ does not depend on $\widetilde{N_1} \times \cdots \times \widetilde{N_k}$ and $\widetilde{f_2}$ does not depend on $\widetilde{G_1} \times \cdots \times \widetilde{G_l}$. From this we obtain isometries $f_1 \in \mathrm{Iso}(G_1 \times \cdots \times G_k)$ and $f_2 \in \mathrm{Iso}(N)$ such that $f = (f_1, f_2)$.

□

References

[1] Blumenthal, R. A. and Hebda, J.J. *De Rham Decomposition Theorems for Foliated Manifolds*, Ann. Inst. Fourier **33**(1983), 183-198.

[2] S. Helgason, *Differential Geometry, Lie Groups and Symmetric Spaces*, Academic Press, New York, 1978.

[3] M. Gromov, *Rigid transformations groups*, Géométrie différentielle (Paris 1986), Travaux en Cours **33**, Hermann, Paris, 1988, pp. 65–139.

[4] F. Hernández-Zamora, *Formas bilineales en álgebras de Lie y acciones de grupos. Morfismos, Vol 3, No. 2, 199, pp. 58–67.*

[5] P. Molino, Riemannian Fotiations*. Progress in Math. Vol 73. Birkhausser, Boston, 1988.*

[6] D. Muller, Isometries of Bi-invariant pseudo-Riemannian metrics on Lie Groups. Geometria Dedicata **29** (1989), 65-96.

[7] B. O'neill, *SEMI-RIEMANNIAN GEOMETRY*, Academic Press, New York, 1983.

[8] P. Petersen, *Riemannian Geometry*, Springer, New York, 2016.

[9] M. Cahen and M. Parker, *Pseudo-riemannian Symetric Spaces, Memoirs of the AMS, 229, 1980*

[10] R. Quiroga-Barranco, Isometric actios of simple Lie groups on pseudo Riemannian manifolds*. Ann. of Math. (2) 164 (2006), no. 3, 941-969.*

[11] J. Rosales-Ortega, The signature in actions of semisimple Lie groups on pseudo-Riemannian manifolds*. Proyecciones (Antofagasta), Mar 2012, vol.31, no.1, p.51-63.*

[12] J. Rosales-Ortega, Foliations by G-actions*. Bol. Soc. Paran. Mat. (3s.)* ***v. 33*** *2 (2015): pp. 79-87.*

[13] Rosales-Ortega, J. Bol. Soc. Mat. Mex. (2017) 23: 825. https://doi.org/10.1007/s40590-015-0084-4

[14] J. Szaro, Isotropy of semisimple group actions on manifolds with geometric structures*, Amer. J. Math.**120** (1998), 129–158.*

[15] R. J. Zimmer, Ergodic Theory and Semisimple Lie Groups*, Birkhäuser, Boston, 1984.*

Department of Mathematics, ITCR, Cartago, Costa Rica

Current address: Department of Mathematics, University Of Costa Rica, San José, Costa Rica

E-mail address: `jrosales@itcr.ac.cr// jose.rosales@ucr.ac.cr`