

ZILBER'S DICHOTOMY FOR DIFFERENTIALLY CLOSED FIELDS WITH AN AUTOMORPHISM

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ABSTRACT. The theory of difference-differential fields of characteristic zero has a model-companion denoted by $DCFA$. In [2] we prove its main properties. In [3] we proved a weak version of Zilber's dichotomy for $DCFA$. In this paper we use arc spaces techniques as developed by Moosa, Pillay and Scanlon in [10] to suppress the extra hypothesis needed in [3] and prove the full Zilber's dichotomy for $DCFA$, we also state how these techniques generalise to partial differential fields with an automorphism.

1. INTRODUCTION AND PRELIMINARIES

A difference-differential field is a differential field with an automorphism which commutes with the derivation. For the case of characteristic zero, the theory of difference-differential has a model-companion that we denote $DCFA$.

If (K, σ, D) is a model of $DCFA$, the fixed field of K is $\text{Fix } \sigma = \{x \in K : \sigma(x) = x\}$ and the field of constants of K is $\mathcal{C} = \{x \in K : Dx = 0\}$.

Our goal is to prove the following theorem, known as Zilber dichotomy.

Theorem 1. *If p is a type of SU -rank 1, then it is either 1-based or non orthogonal to $\text{Fix } \sigma \cap \mathcal{C}$.*

This dichotomy states, intuitively speaking, that a "minimal" set is either "algebraic" or "geometrically simple". Analogues for this dichotomy are satisfied in both differentially closed fields (models of DCF , the model companion or the theory of differential fields) and existentially closed difference fields (models of $ACFA$, the model companion of the theory of difference fields). Both proofs ([7], and [5]) depend heavily on Zariski geometries. These results lead to Hrushovsky's proofs of the Mordell-Lang conjecture and the Manin-Mumford conjecture, respectively. Pillay and Ziegler proved both dichotomies for the case of characteristic zero [14] using algebraic jet spaces. A key point in their proofs is the fact that in models of DCF $tp(a/F)$ has finite rank if and only if the transcendence degree of the differential field generated by the tuple a and F over F is finite, where F is a differential subfield of a the model. This transcendence degree is what we call the dimension of a over F , denoted $\dim(a/F)$. An analogous statement holds for models of $ACFA$. These equivalences allow the authors to replace an element of finite rank with something interdefinable which, by finite-dimensionality, turns out to be

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a finite sequence. Unfortunately this equivalence does not hold for $DCFA$. In [4] we give an example of a type with SU -rank 1, but with infinite dimension. So following the jet spaces techniques we were able to prove the following weaker dichotomy.

Theorem 2. [3]. *Let (\mathcal{U}, σ, D) be a saturated model of $DCFA$ and let $K = acl(K) \subset \mathcal{U}$. Let $a \in \mathcal{U}$ such that $SU(tp(a/K)) = 1$ and $tp(a/K)$ is finite-dimensional. Then $tp(a/K)$ is either 1-based or non orthogonal to $\text{Fix } \sigma \cap \mathcal{C}$.*

In this paper we want to suppress the finite-dimensional hypothesis. For this we follow [10] where the authors use algebraic arc spaces to prove a dichotomy for differentially closed fields with finitely many commuting derivations (DCF_n). The main theorem states that if p is a regular non-locally modular type then there is a definable subgroup of the additive group whose generic type is regular and non orthogonal to p .

We want to prove an analogue theorem for our case, namely:

Theorem 3. *Let p be a regular non-locally modular type in \mathcal{U} . Then there is a definable subgroup of the additive group whose generic type is regular and non-orthogonal to p .*

How can we link Theorem 2 and Theorem 3 to prove Theorem 1? The next proposition on definable subgroups of the additive group will do the job:

Proposition 1.1. [4]. *Let G be a definable subgroup of \mathbb{G}_a^n .*

- (1) *G has no proper subgroup of finite index.*
- (2) *G is quantifier-free definable.*
- (3) *If H a definable subgroup of G . Then G/H is definably isomorphic to a subgroup of \mathbb{G}_a^l for some l .*
- (4) *If G has infinite dimension then $SU(G) \geq \omega$.*

Now, provided that Theorem 3 holds, we can easily prove Theorem 1.

Proof of Theorem 1:

Suppose that p is not 1-based, then it is non-locally modular. As $SU(p) = 1$ it is regular. By Theorem 3 there is a definable subgroup G of \mathbb{G}_a whose generic type q is regular and non-orthogonal to p .

$p \not\perp q$ implies that $SU(q) = \alpha + 1$ for some α . Then, by 5.4.3 of [15], G contains a definable subgroup N such that $SU(G/N) < \omega$, and by 1.1 G must be finite-dimensional.

By Theorem 2, $q \not\perp \text{Fix } \sigma \cap \mathcal{C}$ and by transitivity $p \not\perp \text{Fix } \sigma \cap \mathcal{C}$.

□

DCF is complete, ω -stable and eliminates quantifiers, these facts are key in [10], on the other hand $DCFA$ is not complete, no completion is stable nor eliminates quantifiers. However its completions are easily described, they are supersimple, quantifier-free stable and eliminates imaginaries. So we need to state some facts that allow us to build a set-up similar to the one exposed in [10]. Section 2 is devoted to this. In section 3 we give a brief description and list the main theorems on $DCFA$. In section 4 we use arc spaces to give a proof of 3. Finally, in section 5 we give details on how the same reasonings apply to partial differential fields with an automorphism.

2. TYPES IN SUPERSIMPLE THEORIES

Throughout this section T shall denote a supersimple theory which eliminates imaginaries and M a saturated model of T . This implies that there is a good notion of independence, and thus of forking. Moreover all types are ranked by the SU -rank. With independence we define orthogonality: if $A \subset M$ is a set of parameters and p, q are complete types over A . We say that p and q are orthogonal ($p \perp q$) if for every $B \supseteq A$ and realisations a of p and b of q , we have that a is independent of b over B ($a \perp_B b$).

Proposition 2.1. ([15], 5.1.12) *Let $a \in M$ and $A \subset M$. Let us suppose that $SU(a/A) = \beta + \omega^\alpha \cdot n$, with $n > 0$ and $\omega^{\alpha+1} \leq \beta < \infty$ or $\beta = 0$. Then $tp(a/A)$ is non-orthogonal to a type of SU -rank ω^α . Moreover there is $b \in \text{acl}(Aa)$ with $SU(b/A) = \omega^\alpha n$.*

Definition 2.2. *Let $p, q \in S(A)$. We say that q is p -internal if for every realization a of q there is a set B such that $B \perp_A a$ and a tuple c of realizations of p such that $a \in \text{dcl}(Bc)$. A set X definable over A is p -internal if for every tuple a of X , $tp(a/A)$ is p -internal. If we replace dcl by acl above we say that q (or X) is almost p -internal.*

Definition 2.3. *Let p be a (possibly partial) type over A and $q = tp(a/B)$ a type. The p -weight of q , denoted by $w_p(q)$, is the largest integer n such that there are $C \supset A \cup B$, a tuple a_1, \dots, a_n of realizations of p which are independent over C , and a realization b of q such that $(a_1, \dots, a_n) \perp_A C$, $b \perp_B C$ and $a_i \not\perp_C b$ for every $i = 1, \dots, n$. If p is the partial type $x = x$ we say weight instead of p -weight and it is denoted by $w(q)$.*

Definition 2.4. *Let A, B and C be sets. We say that A dominates B over C if for every set D , $D \perp_C A$ implies $D \perp_C B$. Let p, q be two types. We say that p dominates q if there is a set C containing the domains of p and q and realizations a and b of non-forking extensions of p and q to C respectively, such that a dominates b over C . We say that p and q are equidominant if p dominates q and q dominates p .*

A type is said to be regular if it is orthogonal to all its forking extensions. For regular types, equidominance is an equivalence relation and non-orthogonality is transitive ([15], section 5.2).

Let p and q be two complete types. We say that q is hereditarily orthogonal to p if every extension of q is orthogonal to p .

Definition 2.5. (1) *Let p be a type and A a set. The p -closure of A , $cl_p(A)$ is the set of all a such that $tp(a/A)$ is hereditarily orthogonal to p .*
 (2) *A type p is called locally modular if for any A containing the domain of p , and any tuples a and b of realizations of p , we have $a \perp_C b$ where $C = cl_p(Aa) \cap cl_p(Ab)$.*

For the dichotomy we talk about 1-basedness, which is related with local modularity.

Definition 2.6. (1) *Let $A \subset M$ and let S be an (∞) -definable set over A . We say that S is 1-based if for every $m, n \in \mathbb{N}$, and $a \in S^m, b \in S^n$, a and b are independent over $\text{acl}(Aa) \cap \text{acl}(Ab)$.*

(2) A type is 1-based if the set of its realizations is 1-based.

For an important characterization of locally modular types we need to introduce the following definitions.

Definition 2.7. (1) Let $p \in S(A)$ be regular and let q be a type over a set $B \supset A$. We say that q is p -simple if there is $C \supset B$ and a set X of realizations of p and a realization a of q with $a \perp_B C$ such that $tp(a/CX)$ is hereditarily orthogonal to p .

(2) Let p be a regular type over A and let q be a p -simple type. We say that q is p -semi-regular if it is domination equivalent to a non-zero power of p .

Remark 2.8. If a, b are tuples of M , the canonical base of a over b denoted by $Cb(a/b)$ the smallest algebraically closed subset of M over which $tp(a/b)$ does not fork. As T is simple, canonical bases exist (see [1] for the details). 3.3 of [15] implies that for $B \subset M$, a tuple a of M , and a sequence a_i of realizations of $tp(a/B)$ independent over B , there is some m for which $Cb(a/B) \subseteq dcl(a_1 \cdots a_m)$.

Proposition 2.9. ([15], 3.5.17) A type p is locally modular if and only if for any two models M and N with $N \prec M$, and any tuple of realizations a of p over M such that $tp(a/N)$ is p -semi-regular, $Cb(a/M) \subset cl_p(Na)$.

Lemma 2.10. Let A be a subset of M , a a tuple and $b \in acl(Aa)$. Let $q = tp(a, b/A)$ be a regular type and let $p = tp(b/A)$. Then p is locally modular if and only if q is locally modular.

Proof:

As being hereditarily orthogonal to p is the same as being hereditarily orthogonal to q , by definition, for any set B , $cl_p(B) = cl_q(B)$.

Let (a_1, b_1) and (a_2, b_2) be tuples (of tuples) of realizations of q .

Claim: $a_i \in cl_p(b_i)$ for $i = 1, 2$:

Proof of the claim: Suppose that $a_1 \notin cl_p(b_1)$. Then there exists a non-forking extension r of $tp(a_1)$ over B such that r is non-orthogonal to $tp(b_1)$. Let c and d be realisations of the partial type $r \cup q$. Then $tp(c, d/B)$ is a forking extension of q . The element c can be chosen dependent over B from a realisation (c', d') of a forking extension of q over B , and this contradicts the regularity of q . The same argument applies to a_2 and $cl_p(b_2)$ and the claim is proved.

The following equation holds:

$$cl_q(a_1, b_1) \cap cl_q(a_2, b_2) = cl_p(b_1) \cap cl_p(b_2) =_{\text{def}} C.$$

It follows immediately that the local modularity of q implies the local modularity of p . Conversely, assume that p is locally modular. Then $b_1 \perp_C b_2$. Let $D = Cb(a_1 b_1 / acl(Cb_2))$. By 2.8 D is contained in the algebraic closure of independent realizations of $tp(a_1 b_1 / acl(Cb_2))$ this implies that $tp(D/C)$ is almost-internal to the set of conjugates of $tp(a_1 / Cb_1)$, and is therefore hereditarily orthogonal to p . Hence $D \subset cl_p(C) = C$, and $a_1 b_1 \perp_C b_2$. A similar reasoning gives that $Cb(a_2 b_2 / acl(Ca_1 b_1)) \subset C$.

□

Lemma 2.11. *Let $A = \text{acl}(A)$ a subset of M of T and let a be a tuple in M . Assume that $\text{tp}(a/A)$ has SU -rank $\beta + \omega^\alpha = \beta \oplus \omega^\alpha$, with $\beta > \omega^\alpha$, and has weight 1. Then there is $b \in \text{acl}(Aa)$ such that $SU(b/A) = \omega^\alpha$.*

Proof:

By 2.1, there is some $C = \text{acl}(C) \supset A$ independent from a over A and a tuple c such that $SU(c/C) = \omega^\alpha$, and c and a are not independent over C . Let B be the algebraic closure of $Cb(Cc/\text{acl}(Aa))$. Then by 2.8 B is contained in the algebraic closure of finitely many (independent over Aa) realizations of $\text{tp}(Cc/\text{acl}(Aa))$, say C_1c_1, \dots, C_nc_n . Let $D = \text{acl}(C_1, \dots, C_n)$. Then D is independent from a over A , and each c_i is not independent from a over D . Since $\text{tp}(a/A)$ has weight 1, so does $\text{tp}(a/D)$, and therefore for each $1 < i \leq n$, c_1 and c_i are not independent over D . Thus $SU(c_i/Dc_1) < \omega^\alpha$, and therefore $SU(c_1, \dots, c_n/D) < \omega^{\alpha 2}$. As D is independent from a over A , and $B \subset \text{acl}(D, c_1, \dots, c_n) \cap \text{acl}(Aa)$, we get $SU(B/A) < \omega^{\alpha 2}$. Since $SU(c/C) = \omega^\alpha$ and $SU(c/CB) < \omega^\alpha$, then $SU(B/C) \geq \omega^\alpha$, and as $B \downarrow_A C$ we have $SU(B/A) \geq \omega^\alpha$. By Lascar's inequalities we have $SU(a/AB) + SU(B/A) \leq \beta + \omega^\alpha$. As $SU(B/A) \geq \omega^\alpha$ we have that $SU(B/A) = \delta + \omega^\alpha$ with $\delta \geq \omega^\alpha$ or $\delta = 0$, and $SU(B/A) < \omega^{\alpha 2}$ implies that $\delta = 0$.

□

We end this section with two useful results proved in [15], section 5.2.

Proposition 2.12. *A type in a supersimple theory is equidominant with a finite product of regular types.*

Proposition 2.13. *If $p \not\perp q$ and $q \not\perp r$ then there is a conjugate r' of r such that $p \not\perp r'$.*

3. DIFFERENCE-DIFFERENTIAL FIELDS

A difference-differential field is a differential field (K, D) with an automorphism σ of K which commutes with D .

The theory of difference-differential fields of characteristic zero has a model-companion which we denote $DCFA$ ([2]). Before we give an axiomatization of this theory we need to introduce some definitions regarding varieties defined in differential fields.

Definition 3.1. *Let (K, D) be a differential field, and let $V \subset \mathbb{A}^n$ be a variety, let $F(X)$ be a finite tuple of polynomials over K generating $I(V)$ where $X = (X_1, \dots, X_n)$.*

(1) *We define the first prolongation of V , $\tau_1(V)$ by the equations:*

$$F(X) = 0, J_F(X)Y_1^t + F^D(X) = 0$$

where Y_1 is an n -tuple, F^D denotes the tuple of polynomials obtained by applying D to the coefficients of each polynomial of F , and $J_F(X)$ is the Jacobian matrix of F (i.e. if $F = (F_1, \dots, F_k)$ then $J_F(X) = (\partial F_i / \partial X_j)_{1 \leq i \leq k, 1 \leq j \leq n}$).

(2) *For $m > 1$, we define the m -th prolongation of V by induction on m :*

Assume that $\tau_{m-1}(V)$ is defined by $F(X) = 0, J_F(X)Y_1^t + F^D(X) = 0, \dots,$

$J_F(X)Y_{m-1}^t + f_{m-1}(X, Y_1, \dots, Y_{m-2}) = 0$. Then $\tau_m(V)$ is defined by:

$$(X, Y_1, \dots, Y_{m-1}) \in \tau_{m-1}(V)$$

and

$$J_F(X)Y_m^t + J_F^D(X)Y_{m-1}^t + J_{f_{m-1}}(X, Y_1, \dots, Y_{m-2})(Y_1, \dots, Y_{m-1})^t \\ + f_{m-1}^D(X, Y_1, \dots, Y_{m-2}) = 0.$$

- (3) Let $W \subset \tau_m(V)$ be a variety. We say that W is in normal form if, for every $i \in \{0, \dots, m-1\}$, whenever $G(X, Y_1, \dots, Y_i) \in I(W) \cap K[X, Y_1, \dots, Y_i]$ then

$$J_G(X, Y_1, \dots, Y_i)(Y_1, \dots, Y_{i+1})^t + G^D(X, Y_1, \dots, Y_i) \in I(W).$$

- (4) Let $W \subset \tau_m(V)$ be a variety in normal form.

A point a (in some extension of K) is an (m, D) -generic of W over K if $(a, Da, \dots, D^m a)$ is a generic of W over K and for $i > m$, we have $\text{tr.dg}(D^i a/K(a, \dots, D^{i-1} a)) = \text{tr.dg}(D^m a/K(a, \dots, D^{m-1} a))$.

Now we can give an axiomatization for $DCFA$.

Fact 3.2. [2]. (K, D, σ) is a model of $DCFA$ if

- (1) (K, D) is a differentially closed field.
 - (2) σ is an automorphism of (K, D) .
 - (3) If U, V, W are varieties such that:
 - (a) $U \subset V \times V^\sigma$ projects generically onto V and V^σ .
 - (b) $W \subset \tau_1(U)$ projects generically onto U .
 - (c) $\pi_1(W)^\sigma = \pi_2(W)$ (we identify $\tau_1(V \times V^\sigma)$ with $\tau_1(V) \times \tau_1(V)^\sigma$ and let $\pi_1 : \tau_1(V \times V^\sigma) \rightarrow \tau_1(V)$ and $\pi_2 : \tau_1(V \times V^\sigma) \rightarrow \tau_1(V)^\sigma$ be the natural projections).
 - (d) A $(1, D)$ -generic point of W projects onto a $(1, D)$ -generic point of $\pi_1(W)$ and onto a $(1, D)$ -generic point of $\pi_2(W)$.
- Then there is a tuple $a \in V(K)$, such that $(a, \sigma(a)) \in U$ and $(a, Da, \sigma(a), \sigma(Da)) \in W$.

This theory is not complete, but its completions are easily described. It's properties in fact are in general very similar to the ones of $ACFA$: independence is defined by linear disjointness, the models of $DCFA$ eliminates imaginaries (moreover, they satisfy the Independence Theorem over algebraically closed sets), the completions of $DCFA$ are supersimple and thus types are ranked by the SU -rank; forking is determined by quantifier-free formulas, this implies that $DCFA$ is quantifier-free ω -stable; in a model of $DCFA$ the difference-differential Zariski topology (defined in analogy with Zariski topology in algebraically closed fields) is Noetherian. All these properties are proved in [2].

Since $DCFA$ is quantifier-free ω -stable, we can define canonical bases for quantifier-free types as in stable theories as in [6]. We denote the canonical basis of the quantifier-free type of a over K as $Cb(qftp(a/K))$. It does not coincide with the canonical base of $tp(a)$ as defined for simple theories. However, as $DCFA$ satisfies the independence theorem over algebraically closed sets, $Cb(p)$ will be contained in $\text{acl}(Cb(qftp(a/K)))$.

4. ARC SPACES IN DIFFERENCE DIFFERENTIAL FIELDS

First we define algebraic arc spaces, for the formal definition we refer to [10].

Let K be a field, and $K^{(m)}$ the K -algebra $K[\epsilon]/(\epsilon^{m+1})$. Then, identifying $K^{(m)}$ with $K \cdot 1 \oplus K \cdot \epsilon \dots \oplus K \cdot \epsilon^m$, we see that the K -algebra $K^{(m)}$ is quantifier-free interpretable in K , if we encode elements of $K^{(m)}$ by $(m+1)$ -tuples of K .

Let $V \subset \mathbb{A}^\ell$ be a variety defined over K . For $m \in \mathbb{N}$, we consider the set $V(K^{(m)})$ of $K^{(m)}$ -rational points of V .

Using the quantifier-free interpretation of $K^{(m)}$ in K , we may identify $V(K^{(m)})$ with a subvariety $\mathcal{A}_m V(K)$ of $\mathbb{A}^{(m+1)\ell}(K)$. The variety $\mathcal{A}_m V$ is called the m -th arc bundle of V . More precisely, if $f_1, \dots, f_k \in K[X_1, \dots, X_\ell]$ generate the ideal $I(V)$, then the ideal of $\mathcal{A}_m V$ is generated by the polynomials $f_{j,t} \in K[X_{i,t} : 1 \leq i \leq \ell, 0 \leq t \leq m]$, $1 \leq j \leq k, 0 \leq t \leq m$, which are defined by the identity

$$f_j\left(\sum_{t=0}^m x_{i,t}\epsilon^t\right)_{1 \leq i \leq \ell} = \sum_{t=0}^m f_{j,t}\left((x_{i,t})_{1 \leq i \leq \ell}\right)\epsilon^t$$

modulo (ϵ^{m+1})

If $m = 1$ the polynomials defining $\mathcal{A}_1 V$ are determined by $f_j((x_{i,0} + x_{i,1}\epsilon)_{1 \leq i \leq \ell}) = f_{j,t}((x_{i,0})_{1 \leq i \leq \ell}) + f_{j,t}((x_{i,1})_{1 \leq i \leq \ell})\epsilon$ modulo $(\epsilon)^2$ and we identify $\mathcal{A}_1 V$ with the tangent bundle $T(V)$.

If $r > m$, the natural map $K^{(r)} \rightarrow K^{(m)}$ induces a map $V(K^{(r)}) \rightarrow V(K^{(m)})$, which in turn induces a morphism $\rho_{r,m} : \mathcal{A}_r V \rightarrow \mathcal{A}_m V$.

In general, given a morphism of varieties $f : U \rightarrow V$ defined over K , the natural morphism $U(K^{(m)}) \rightarrow V(K^{(m)})$ induced by f gives rise to a morphism $\mathcal{A}_m f : \mathcal{A}_m U \rightarrow \mathcal{A}_m V$.

Let us write ρ_m for $\rho_{m,0}$. For $a \in V(K)$ the m -th arc space of V at a , $\mathcal{A}_m V_a$ is the fiber of ρ_m over a .

We follow [10] to prove our theorem. Our propositions 4.6, 4.7, 4.10, 4.12, 4.14, 4.17, 4.19, 4.20, 4.21, 4.22, 4.24, 4.25 and 4.26 are difference-differential versions of 2.9, 2.10, 2.11, 2.12, 3.1, 3.6, 3.9, 3.10, 3.11, 3.12, 3.15 and 3.16 of [10]. Some of the proofs are very similar, we give the details where the proofs are different or in case they give some useful insights.

The following three lemmas are crucial, and allow us to characterise varieties by their arc spaces.

Lemma 4.1. [10]. *Let U, V be two algebraic varieties, and let $f : U \rightarrow V$ be a morphism, all defined over K . Let $m \in \mathbb{N}$ and $a_m \in \mathcal{A}_m U(K)$ be such that $\bar{a} = \rho_m(a)$ and $f(\bar{a}) = \rho_m(f(a))$ are non-singular. Let U' be the fiber of $\rho_{m+1,m} : \mathcal{A}_{m+1} U \rightarrow \mathcal{A}_m U$ over a and V' the fiber of $\rho_{m+1,m} : \mathcal{A}_{m+1} V \rightarrow \mathcal{A}_m V$ over $\mathcal{A}_m f(a)$. Let $\bar{a} = \rho_m(a)$. Then there are biregular maps $\varphi_U : U' \rightarrow T(U)_{\bar{a}}$ and $\varphi_V : V' \rightarrow T(V)_{f(\bar{a})}$ such that the following diagram is commutative:*

$$\begin{array}{ccc} U' & \xrightarrow{\mathcal{A}_m(f)} & V' \\ \downarrow \varphi_U & & \downarrow \varphi_V \\ T(U)_{\bar{a}} & \xrightarrow{df_{\bar{a}}} & T(V)_{f(\bar{a})} \end{array}$$

Lemma 4.2. [10]. *Let U, V be algebraic varieties defined over K , and let $f : U \rightarrow V$ be a dominant map defined over K . Let $a \in U(K)$ be non-singular such that $f(a)$ is non-singular and the rank of df_a equals $\dim V$. Then for every $m \in \mathbb{N}$ the map $\mathcal{A}_m(f) : \mathcal{A}_m U_a(K) \rightarrow \mathcal{A}_m V_{f(a)}(K)$ is surjective.*

Lemma 4.3. [10]. *Let U, V, W be algebraic varieties defined over K such that $U, V \subset W$. Let $a \in U(K) \cap V(K)$ be non-singular. Then $U = V$ if and only if $\mathcal{A}_m U_a(K) = \mathcal{A}_m V_a(K)$ for all $m \in \mathbb{N}$.*

Let (\mathcal{U}, σ, D) be a saturated model of *DCFA*, let K be a difference-differential subfield of \mathcal{U} .

Let V be an algebraic variety of the affine space of dimension k over \mathcal{U} .

Let $\nabla_m : V \rightarrow \tau_m(V)$ be defined by $x \mapsto (x, Dx, \dots, D^m x)$ and let $\pi_{l,m} : \tau_l(V) \rightarrow \tau_m(V)$ be the natural projection for $l \geq m$. $S_m(V)$ will denote the Zariski closure of $\{(x, \dots, \sigma^m(x)) : x \in V\}$. Let $q_m : V \rightarrow S_m(V)$ be defined by $x \mapsto (x, \dots, \sigma^m(x))$ and let $p_{l,m} : S_l(V) \rightarrow S_m(V)$ be the natural projections for $l \geq m$.

We now define a notion of difference-differential prolongation. Let $\Phi_m(V) = \tau_m(S_m(V))$, let $\psi_m : V \rightarrow \Phi_m(V)$ be such that $x \mapsto \nabla_m(q_m(x))$ and for $l \geq m$ let $t_{l,m} : \Phi_l(V) \rightarrow \Phi_m(V)$ be defined by $t_{l,m} = \pi_{l,m} \circ p_{l,m}$. Let us denote $\pi_l = \pi_{l,0}$, $p_l = p_{l,0}$, $t_l = t_{l,0}$, $\Phi(V) = \Phi^1(V) = \Phi_1(V)$ and $\Phi^{m+1}(V) = \Phi(\Phi^m(V))$.

We define $\psi = \psi^1 = \psi_1 : V \rightarrow \Phi(V)$ and $\psi^{m+1}(V) = \psi(\psi^m) : V \rightarrow \Phi^{m+1}(V)$.

We extend σ and D to $K^{(m)}$ by defining $\sigma(\epsilon) = \epsilon$ and $D\epsilon = 0$. Since σ and D commute, $\nabla_m q_m(x)$ is a permutation of $q_m \nabla_m x$. For $x = (x_0, \dots, x_m) \in K^{(m)}$ we identify it with $\sum_{i=0}^m x_i \epsilon^i$, we can identify $\Phi_m(x)$ with $\sum_{i=0}^m \Phi_m(x_i) \epsilon^i$. Then we can identify $\mathcal{A}_r(S_m(V))(K)$ with $S_m(\mathcal{A}_r(V))(K)$. We can thus assume that $\mathcal{A}_r(\Phi_m(V))(K) = \Phi_m(\mathcal{A}_r(V))(K)$. All this is done with greater generality in [12] and [13]

Let X be a (σ, D) -variety given as a (σ, D) -closed subset of an algebraic variety \bar{X} . We define $\Phi_m(X)$ as the Zariski closure of $\psi_m(X)$ in $\Phi_m(\bar{X})$. Thus X is determined by the prolongation sequence $\{t_{l,m} : \Phi_l(X) \rightarrow \Phi_m(X) : l \geq m\}$, since $X(\mathcal{U}) = \{a \in \bar{X}(\mathcal{U}) : \psi_l(a) \in \Phi_l(X) \forall l\}$. We call this sequence the prolongation sequence of X .

Proposition 4.4. *Let $\{V_l \subset \Phi_l(\bar{V}) : l \geq 0\}$ be a sequence of algebraic varieties and $\{t_{m,l} : V_m \rightarrow V_l, m \geq l\}$ a sequence of morphisms such that:*

- (1) $t_{l+1,l} \upharpoonright V_{l+1} \rightarrow V_l$ is dominant.
- (2) After embedding $\Phi_l(\bar{V})$ in $\Phi^l(\bar{V})$ and $\Phi_{l+1}(\bar{V})$ in $\Phi^{l+1}(\bar{V})$,
 - (a) V_{l+1} is a subvariety of $\Phi(V_l)$.
 - (b) Let $\pi'_1 : \Phi(V_l) \rightarrow \tau(V_l)$ and $\pi'_2 : \Phi(V_l) \rightarrow \tau(V_l^\sigma)$ be the projections induced by $\Phi(V_l) \subset \tau(V_l) \times \tau(V_l^\sigma)$; then $\pi'_1(V_{l+1})^\sigma$ and $\pi'_2(V_{l+1})$ have the same Zariski closure.

Then there is a (unique) (σ, D) -variety V with prolongation sequence $\{t_{m,l} : V_m \rightarrow V_l, m \geq l\}$.

Proof:

For each l , as the maps $\pi_{m,j}$ are dominant, the system $\{p_{m,l}(V_m), \pi_{m,j} : m > j \geq l\}$ defines a differential subvariety W_l of $\bar{V} \times \dots \times \bar{V}^{\sigma^l}$.

Condition (1). implies that for m sufficiently large, an (m, D) -generic of $p_{m,l+1}(V)$ is sent by $p_{l+1,l}$ to an (m, D) -generic of $p_{m,l}(V)$. Hence, a D -generic of W_{l+1} is sent by $p_{l+1,l}$ to a D -generic of W_l (that is, a generic in the sense of *DCF*).

By conditions (2) (b) and (1), the map $t'_{l+1,l} : V_{l+1} \rightarrow V_l^\sigma$ induced by $\Phi(V_l) \rightarrow V_l^\sigma$ is dominant. Hence, considering the natural projection $p'_{l+1,l} : S_{l+1}(\bar{V}) \rightarrow S_l(\bar{V})^\sigma$,

and reasoning as above, we obtain that $p'_{l+1,l}$ sends a D -generic of W_{l+1} to a D -generic of W_l^σ . Hence by 3.2, for every l there is a such that $\psi_l(a)$ is a generic of V_l over K . By saturation, there is a such that for all l , $\psi_l(a)$ is a generic of V_l . Then $\{t_{m,l} : V_m \rightarrow V_l, m \geq l\}$ is the prolongation sequence of the (σ, D) -locus of a over K .

□

We define non-singular points in analogy with the corresponding notion in [10].

Definition 4.5. *Let X be a (σ, D) -subvariety of the algebraic variety \bar{X} . We say that a point $a \in X$ is non-singular if, for all l , $\psi_l(a)$ is a non-singular point of $\Phi_l(X)$, the maps $dt_{l+1,l}$ and $dt'_{l+1,l}$ at $\psi_{l+1}(a)$ have rank equal to $\dim X_l$ and the maps $d\pi'_1$ and $d\pi'_2$ (as defined above) at $\psi_{l+1}(a)$ have rank equal to the dimension of the Zariski closure of $\pi'_1(\Phi_{l+1}(X))$.*

The next proposition is essentially 4.7 of [13].

Proposition 4.6. *Let (K, σ, D) be a model of DCF. Let V be a (σ, D) -variety given as a closed subvariety of an algebraic variety \bar{V} . Let $m \in \mathbb{N}$ and $a \in V(K)$ a non-singular point. Then $\{\mathcal{A}_m(t_{r,s}) : \mathcal{A}_m\Phi_r(V)_{\psi_r(a)} \rightarrow \mathcal{A}_m\Phi_s(V)_{\psi_s(a)}, r \geq s\}$ form the (σ, D) -prolongation sequence of a (σ, D) -subvariety of $\mathcal{A}_m\bar{V}_a$. We define the m -th arc space of V at a , \mathcal{A}_mV_a , to be this subvariety. We have also that $\Phi_r(\mathcal{A}_mV_a) = \mathcal{A}_m\Phi_r(V)_{\psi_r(a)}$ for all r .*

Proof:

As $\mathcal{A}_m\Phi_r(\bar{V}) = \Phi_r(\mathcal{A}_m\bar{V})$, we look at $\mathcal{A}_m\Phi_r(V)_{\psi_r(a)}$ as an algebraic subvariety of $\Phi_r(\mathcal{A}_m\bar{V})_{\psi_r(a)}$. We have that $\Phi_{r+1}(V) \subset \Phi(\Phi_r(V))$ for all r . Since \mathcal{A} preserves inclusion we have $\mathcal{A}_m\Phi_{r+1}(V)_{\psi_{r+1}(a)} \subset \mathcal{A}_m\Phi(\Phi_r(V)_{\psi_r(a)}) = \Phi(\mathcal{A}_m\Phi_r(V)_{\psi_r(a)})$. This proves conditions 1. and 2.(a) of 4.4.

Moreover, the maps $t_{r,s} : \Phi_r(V) \rightarrow \Phi_s(V)$ are dominant, and as a is non-singular, by 4.2, the maps $\mathcal{A}(t_{r,s}) : \mathcal{A}_m\Phi_r(V)_{\psi_r(a)} \rightarrow \mathcal{A}_m\Phi_s(V)_{\psi_s(a)}$, are dominant. Applying \mathcal{A}_m to the dominant maps $\pi'_1 : \Phi_{r+1}(V) \rightarrow \tau(\Phi_r(V))$ and $\pi'_2 : \Phi_{r+1}(V) \rightarrow \tau(\Phi_r(V))^\sigma$, using the hypothesis on a and 4.2, we get

$$\mathcal{A}_m\pi'_1(\mathcal{A}_m(\Phi_{r+1}(V)_{\psi_{r+1}(a)})) = \mathcal{A}_m(\pi'_1(\Phi_{r+1}(V)))_{\pi'_1(\psi_{r+1}(a))}$$

and

$$\mathcal{A}_m\pi'_2(\mathcal{A}_m(\Phi_{r+1}(V)_{\psi_{r+1}(a)})) = \mathcal{A}_m(\pi'_2(\Phi_{r+1}(V)))_{\pi'_2(\psi_{r+1}(a))}$$

and since $\pi'_1(\Phi_{r+1}(V))^\sigma$ and $\pi'_2(\Phi_{r+1}(V))$ have the same Zariski closure, and $\sigma(\pi'_1\psi_{r+1}(a)) = \pi'_2\psi_{r+1}(a)$ we get condition 2(b). Hence $\{\mathcal{A}_m(t_{r,s}) : \mathcal{A}_m\Phi_r(V)_{\psi_r(a)} \rightarrow \mathcal{A}_m\Phi_s(V)_{\psi_s(a)}, r \geq s\}$ is the (σ, D) -prolongation sequence of a (σ, D) -subvariety W of $\mathcal{A}_m\bar{V}_a$, where $W(K) = \{x \in \mathcal{A}_m\bar{V}_a(K) : \psi_r(x) \in \mathcal{A}_m\Phi_r(V)_{\psi_r(a)}(K), r \geq 0\}$ and $\mathcal{A}_m\Phi_r(V)_{\psi_r(a)} = \Phi_r(W)$ for all r . We define then $\mathcal{A}_mV_a = W$.

□

Lemma 4.7. *Let U, V be two (σ, D) -subvarieties of an algebraic variety \bar{V} . Let $a \in U(K) \cap V(K)$ be a non-singular point of U . Then $U = V$ if and only if $\mathcal{A}_m\Phi_l(U)_{\psi_l(a)} = \mathcal{A}_m\Phi_l(V)_{\psi_l(a)}$ for all m, l .*

Proof:

If $\mathcal{A}_mU_a(K) = \mathcal{A}_mV_a(K)$ for all m , then $\Phi_r(\mathcal{A}_mU_a)(K) = \Phi_r(\mathcal{A}_mV_a)(K)$. Thus, by 4.6, $\mathcal{A}_m\Phi_r(U)_{\psi_r(a)}(K) = \mathcal{A}_m\Phi_r(V)_{\psi_r(a)}(K)$. Hence, for all r and m , we have

$\mathcal{A}_m \Phi_r(U)_{\psi_r(a)} = \mathcal{A}_m \Phi_r(V)_{\psi_r(a)}$. Lemma 4.3 implies that U and V have the same (σ, D) -prolongation sequence. Hence $U = V$.

□

Definition 4.8. Let V be a (σ, D) -variety and a a non-singular point of V . We define the (σ, D) -tangent space $T_{\sigma, D}(V)_a$ of V at a as follows:

Let P_r be a finite tuple of polynomials generating $I(\Phi_r(V)_{\psi_r(a)})$. Then $T_{\sigma, D}(V)_a$ is defined by the equations $J_{P_r}(\psi_r(a)) \cdot (\psi_r(Y)) = 0$. In other words, the prolongation sequence of $T_{\sigma, D}(V)_a$ is $dt_{l,r} : T(\Phi_l(V))_{\psi_l(a)} \rightarrow T(\Phi_r(V))_{\psi_r(a)}, l \geq r$, where T denotes the usual tangent bundle and $t_{l,r}$ the natural projection $\Phi_l(V)_{\psi_l(a)} \rightarrow \Phi_r(V)_{\psi_r(a)}$.

Remark 4.9. Let a be a non-singular point of the (σ, D) -variety V . Then $T_{\sigma, D}(V)_a$ is a subgroup of $\mathbb{G}_a^n(K)$, and by the same arguments as above, its prolongation sequence is $(d(t_{l,r})_{\psi_l(a)} : T(\Phi_l(V))_{\psi_l(a)} \rightarrow T(\Phi_r(V))_{\psi_r(a)})_{l \geq r}$.

Lemma 4.10. Let V be a (σ, D) -variety in \mathbb{A}^l and a a non-singular point of V . Then $\mathcal{A}_1 V_a$ is isomorphic to $T(V)_a$. Let \bar{V} be the Zariski closure of $V(\mathcal{U})$ in \mathbb{A}^l and $m \in \mathbb{N}$; then the map given by lemma 4.1 which identifies the fibers of $\mathcal{A}_{m+1} \bar{V}_a \rightarrow \mathcal{A}_m \bar{V}_a$ with $T(\bar{V})_a$ restricts to an isomorphism of the fibers of $\mathcal{A}_{m+1} V_a \rightarrow \mathcal{A}_m V_a$ with $T(V)_a$.

Proof:

As remarked before, we can assume that $\mathcal{A}_1 \bar{V} = T(\bar{V})$. Let $b \in T(\bar{V})_a(K)$. By definition $(a, b) \in \mathcal{A}_1 V(\mathcal{U})$ if and only if $\psi_r(a, b) \in T(\Phi_r(V))(K)$ for all r . As $T(\Phi_r(\bar{V})) = \Phi_r(T(\bar{V}))$, $T(\Phi_r(V))_{\psi_r(a)}$ is an algebraic subvariety of $\Phi_r(T(\bar{V}))$ and $\psi_r(a, b) = (\psi_r(a), \psi_r(b))$. Hence $b \in \mathcal{A}_1 V_a(K)$ if and only if $b \in T(V)_b$ and the first part of the theorem is proved.

Now we look at the map given in 4.1. In particular, if $c \in \mathcal{A}_m V_a(K)$ and $r \geq 0$, by 4.6, $\psi_r(c) \in \mathcal{A}_m \Phi_r(V)_{\psi_r(a)}$ and the following diagram commutes

$$\begin{array}{ccc} (\mathcal{A}_{m+1} \bar{V}_a)_c & \longrightarrow & (\mathcal{A}_{m+1} \Phi_r(\bar{V})_{\psi_r(a)})_{\psi_r(c)} \\ \downarrow & & \downarrow \\ T(\bar{V})_a & \longrightarrow & T(\Phi_r(\bar{V}))_{\psi_r(a)} \end{array}$$

where the horizontal arrows are ψ_r and the vertical arrows are the maps given by 4.1 applied to \bar{V} and $\Phi_r(\bar{V})$. So $(\mathcal{A}_{m+1} V_a)_c$ is identified with $T_{\sigma, D}(V)_a$.

□

Notation and Definition 4.11. In [9] the author proved that given a difference-differential subfield F of K and $a \in K$ there is a numerical polynomial $P_{a/F}(X) \in \mathbb{Q}[X]$ of degree at most 2, such that for sufficiently large $r \in \mathbb{N}$, $P_{a/F}(r) = \text{tr.dg}(\psi_r(a)/F)$. We call the degree of $P_{a/F}$ the (σ, D) -type of a over F , and the leading coefficient of $P_{a/F}$ the typical dimension of a over F , it is denoted $\dim_{\sigma, D}(a/F)$. For a (σ, D) -variety V defined over F we define $P_V = P_{a/F}$ where a is a (σ, D) -generic of V over F . We have that the (σ, D) -type of a over F is 2 if and only if a contains an element which is (σ, D) -transcendental over F .

Let $F = \text{acl}(F) \subset \mathcal{U}$ and let $a \in \mathcal{U}$ and let $p = \text{tp}(a/F)$. We denote by $m(p)$ (or by $m(a/F)$) the (σ, D) -type of a over F and we write $\dim_{\sigma, D}(p)$ for $\dim_{\sigma, D}(a/F)$. If

p' is a non-forking extension of p then $m(p) = m(p')$ and $\dim_{\sigma, D}(p) = \dim_{\sigma, D}(p')$.
 If A is an arbitrary subset of \mathcal{U} we write $m(a/A)$ instead of $m(a/\text{acl}(A))$.

If V is a (σ, D) -variety over K , $m(V)$ denotes the (σ, D) -type of V . Then, if a is a (σ, D) -generic of V , $m(V) = m(\text{qftp}(a/F))$.

Corollary 4.12. . Let V be a (σ, D) -variety in \mathbb{A}^l , and $m \in \mathbb{N}$. Then for $a \in V(K)$ non-singular, the (σ, D) -type of V and $\mathcal{A}_m V_a$ are equal.

Proof:

By 4.6 $\Phi_r(\mathcal{A}_m V_a) = \mathcal{A}_m \Phi_r(V)_{\psi_r(a)}$. But if b is a non-singular point of a variety U , then by 4.10, we have that $\dim(\mathcal{A}_m U_b) = m \dim(U)$.

□

Remark 4.13. By 4.10, if $m = 1$ and for $a \in V(K)$ non-singular, we have $P_V = P_{T(V)_a}$.

Lemma 4.14. Let $F = \text{acl}(F)$. Then

- (1) $m(a, b/F) = \max\{m(a/F), m(b/F)\}$.
- (2) If $m(a/F) = m(b/F)$ then $\dim_{\sigma, D}(a, b/F) = \dim_{\sigma, D}(a/F) + \dim_{\sigma, D}(b/Fa)$.
- (3) If $m(a/F) > m(b/F)$ then $\dim_{\sigma, D}(a, b/F) = \dim_{\sigma, D}(a/F)$.

We borrow the next definitions from [10]

Definition 4.15. (1) Let p be a regular type. We say that p is (σ, D) -type minimal if for any type q , $p \not\perp q$ implies $m(q) \geq m(p)$.
 (2) A (σ, D) -variety V is (σ, D) -type minimal if for every proper (σ, D) -subvariety U , $m(V) < m(U)$.

Lemma 4.16. Let p be a type and let V be the (σ, D) -locus of p over K . If V is (σ, D) -type minimal then p is regular and (σ, D) -type minimal.

Proof:

Let a be a realization of a forking extension of p to some $L = \text{acl}(L) \supset K$. Let b realize a non-forking extension of p to L . Let U be the (σ, D) -locus of (a, b) over L . Then the projection on the second coordinate: $U \rightarrow V$ is dominant, thus $m(a, b/L) \geq m(V)$. Now if $a \not\perp_L b$, then the (σ, D) -locus of b over $\text{acl}(La)$ is a proper subvariety of V and therefore, by 4.14, $m(b/La) < m(V)$; from $m(a/L) < m(V)$, we deduce $m(a, b/L) < m(V)$ which is impossible.

□

Lemma 4.17. If p is a type over K , there is a finite sequence of regular types p_1, \dots, p_k such that $m(p) \geq m(p_i)$ for all i and p is domination-equivalent to $p_1 \times \dots \times p_k$.

Proof:

By 2.12 it suffices to show that given a regular type q , such that $p \not\perp q$, there is a regular type r such that $q \not\perp r$ and $m(r) \leq m(p)$. Let a be a realization of a non-forking extension of p to some L and let b be a realization of a non-forking extension of q to L such that $a \not\perp_L b$. Let $c = Cb(\text{tp}(a/L, b))$. Thus $c \notin \text{acl}(L)$ and $c \in \text{acl}(Lb)$. So $r = \text{tp}(c/L)$ is non-orthogonal to q and regular (because $c \in \text{acl}(Lb)$ and regularity is preserved by algebraicity). On the other hand, by

2.8, there are a_1, \dots, a_l realizations of p such that $c \in \text{dcl}(La_1 \cdots a_l)$. Then, by 4.14, $m(r) \leq m(q)$.

□

Remark 4.18. *By 2.11, in the proof above we can suppose that $SU(c/L) = \omega^i$ for $i \in \{0, 1, 2\}$. Thus, given a type p over K , there is a type q such that it is (σ, D) -type minimal and has SU -rank ω^i .*

Lemma 4.19. *Let G be a (σ, D) -vector group (that is, a (σ, D) -variety which is a subgroup of \mathbb{G}_a^k for some k). Then $T_{\sigma, D}(G)_0$ is definably isomorphic to G . Moreover, if H is a (σ, D) -subgroup of G , then the restriction of this isomorphism to H is an isomorphism between H and $T_{\sigma, D}(H)_0$.*

Proof:

Suppose that G is a (σ, D) -subgroup of \mathbb{G}_a^k . For each $r \in \mathbb{N}$, $\Phi_r(G)$ is a subgroup of $\Phi_r(\mathbb{G}_a^k) = \mathbb{G}_a^{k(r+1)^2}$. Since $\Phi_r(G)$ is an algebraic subgroup of $\mathbb{G}_a^{k(r+1)^2}$, its defining ideal is generated by homogeneous linear polynomials, and thus its tangent space at 0 is defined by the same polynomials. Then the map $z_r : \Phi_r(\mathbb{G}_a^k) \rightarrow T(\Phi_r(\mathbb{G}_a^k))$ defined by $x \mapsto (0, x)$ identifies $\Phi_r(\mathbb{G}_a^k)$ and $T(\Phi_r(\mathbb{G}_a^k))_0$ and it restricts to an isomorphism $\Phi_r(G) \rightarrow T(\Phi_r(G))_0$. Hence $(z_r : r \geq 0)$ identifies the prolongation sequence of G and the prolongation sequence of $T_{\sigma, D}(G)_0$.

For the moreover part, it suffices to note that, by our construction above, the restriction of z_r to $\Phi_r(H)$ is an isomorphism between $\Phi_r(H)$ and $T(\Phi_r(H))_0$.

□

Corollary 4.20. *Let G be a (σ, D) -subgroup of \mathbb{G}_a^k . Suppose that for every proper definable subgroup H of G , $m(H) < m(G)$. Then $m(V) < m(G)$ for any proper (σ, D) -subvariety of G . In particular the generic type of G is regular.*

Proof:

Let V be a (σ, D) -type minimal (σ, D) -subvariety of G such that $m(V) = m(G)$. After possibly replacing V by a translate we may assume that $0 \in V$ and 0 is non-singular. By Remark 4.13, $m(T(V)_0) = m(V) = m(G)$. Since $T(V)_0$ is a subgroup of $T(G)_0 \simeq G$, we obtain $T(V)_0 = T(G)_0$. By 4.12, $P_V = P_{T(V)_0} = P_{T(G)_0} = P_G$. Hence $V = G$. By 4.16, the generic type of G is regular.

□

Lemma 4.21. *Let a, c be tuples of \mathcal{U} . Let V be the (σ, D) -locus of a over K . Assume that $c = \text{Cb}(qftp(a/\text{acl}(Kc)))$. Then there is $m \in \mathbb{N}$ and a tuple d in $\mathcal{A}_m V_a$ such that $c \in K(a, d)_{\sigma, D}$.*

Proof:

Let U be the (σ, D) -locus of a over $\text{acl}(Kc)$. Then $\mathcal{A}_m U_a \subset \mathcal{A}_m V_a$. As $DCFA$ eliminates imaginaries every definable set has a canonical parameter. Then c is interdefinable with the canonical parameter of U which, by 4.7, is interdefinable over $K(a)_{\sigma, D}$ with the sequence of the canonical parameters of $\mathcal{A}_m U_a$ over $K(a)_{\sigma, D}$. By quantifier-free stability $\mathcal{A}_m U_a$ is defined with parameters from $\mathcal{A}_m U_a$.

□

Lemma 4.22. *Let (K, σ, D) be a submodel of (\mathcal{U}, σ, D) . Let V be a (σ, D) -variety defined over K and let $a \in V(\mathcal{U})$ be a non-singular point. Let $b \in \mathcal{A}_m V_a$. Then there are $b_1, b_2, \dots, b_m = b$, such that $b_i \in \text{acl}(Ka, b)$ and each b_i is in some $K \cup \{a, b_{i-1}\}$ -definable principal homogeneous space for $T(V)_a$.*

Proof:

By 4.1 and 4.10 each fiber $\rho_{i+1,i} : \mathcal{A}_{i+1} V_a \rightarrow \mathcal{A}_i V_a$ is a principal homogeneous space for $T(V)_a$. Then set $b_i = \rho_{m,i}(b)$.

□

The following lemmas will state the connections between regular non locally modular types and vector groups.

Lemma 4.23. *Let (K, σ, D) be a submodel of (\mathcal{U}, σ, D) . Let p be a (σ, D) -type minimal regular type over K such that $m(p) = d$. If p is not locally modular, then there are a vector group G and a type q such that:*

- (1) $m(q) = m(G) = d$.
- (2) $(x \in G) \in q$.
- (3) $p \not\perp q$.

Proof:

By 4.18 we may assume that $SU(p) = \omega^j$ where $j \in \{0, 1, 2\}$. By 2.9, enlarging K if necessary, there are tuples a and c , with a a tuple of realizations of p , $tp(c/K)$ p -internal, $c = Cb(a/\text{acl}(Kc))$, $tp(a/Kc)$ p -semi-regular and $c \notin cl_p(Ka)$. Let V be the locus of a over K .

By 4.21, there is a k -tuple d in $\mathcal{A}_m V_a(\mathcal{U})$ such that $c \in \text{acl}(K, a, d)$. For $i = 1, \dots, m$ let $d_i = \rho_{m,i}(d)$. Then for each i , d_i is in some $K(ad_{i-1})$ -definable $T(V)_a^k$ -principal homogeneous space.

Let $m = w_p(c/Ka)$. This means that for any $L = \text{acl}(L) \subset \mathcal{U}$ such that $L \perp_K c$, given a tuple (g_1, \dots, g_m) realizing $p^{(m)}$ we have that $g_i \not\perp_L c$ for all i if and only if $g \subset cl_p(Lc)$.

As $c \in \text{acl}(K, a, d)$, $c \notin cl_p(K, a)$, $tp(c/K)$ is p -internal; there is $j \in \{1, \dots, k\}$ such that $w_p(c/Kad_{j-1}) = m$ and $w_p(c/Kad_j) \leq m - 1$. Let $L = \text{acl}(L) \subset \mathcal{U}$ contain Kad_{j-1} , such that $L \perp_K c$, and (g_1, \dots, g_m) realizing $p^{(m)}$ such that $g_i \not\perp_L c$ for all i . Since $w_p(c/Kad_{j-1}) > w_p(c/Kad_j)$, either there is g_k such that $g_k \not\perp_{Ld_j} c$, or $tp(g_k/Ld_j)$ forks over L . In both cases, d_j and g are dependent over L . Hence $tp(d_j/Kad_{j-1}) \not\perp p$.

Let $q = tp(d_j/Kad_{j-1})$. Then we have $m(q) = m(H) \leq m(T(V)_a) = m(p)$, hence $m(p) = m(q)$.

□

Lemma 4.24. *Let p be a regular (σ, D) -type minimal type. If there are a (σ, D) -vector group G and a type q that satisfy the conclusions of 4.23, then there exists a (σ, D) -vector group whose generic type is regular, (σ, D) -type minimal and non-orthogonal to p .*

Proof:

We order the triplets $ord(G) = \{m(G), \dim_{\sigma, D}(G), SU(G)\}$ with the lexicographical order. We proceed by induction on $ord(G)$.

Claim: We may assume that if H is a proper quantifier-free connected, quantifier-free definable subgroup of G , then $m(H) < m(G)$.

Proof of the claim: Suppose that $m(H) = m(G)$. Let $\mu : G \rightarrow G/H$ be the quotient map. By 4.14, $\text{ord}(G) > \text{ord}(G/H)$. If we replace q by a non-forking extension of q we may assume that H is defined over the domain A of q . Let a be a realization of q with $tp(a/A) \not\perp p$. As $q \not\perp p$, we have either $p \not\perp q_0 = \text{qftp}(\mu(a)/A)$ or $p \not\perp q' = \text{qftp}(a/A\mu(a))$. If $p \not\perp q_0$ then $m(p) \leq m(q_0)$ by 4.15, and since $(x \in G/H) \in q_0$, $m(q_0) \leq m(G/H) \leq m(G) = m(p)$. So $m(q_0) = m(p)$ and we apply the induction hypothesis to p, q_0 and G/H . If $p \not\perp q'$, let b be a realization of $\text{qftp}(a/A\mu(a))$ such that $b \perp_{A\mu(a)} a$. Then $a - b \in H$ and $p \not\perp q'' = \text{qftp}(a - b/Ab)$ and the same argument applies.

By 4.20 and as q is realized in G and $m(p) = m(q) = m(G)$, q is a generic of G , and is regular and (σ, D) -type minimal. \square

Corollary 4.25. *Let p be regular non locally modular type. Then there is a (σ, D) -vector group G whose generic type is (σ, D) -type minimal and non-orthogonal to p .*

Proof:

By 4.17 there is a regular type q of minimal (σ, D) -type which is non-orthogonal to p . By 4.23, q satisfies the hypothesis of 4.24, then there is a (σ, D) -vector group G whose generic type r is nonorthogonal to q ; again by 2.13, then there is such an r which is non-orthogonal to p . \square

Lemma 4.26. *Let G be a (σ, D) -vector group and let p be its generic type. If p is regular there is a definable subgroup of \mathbb{G}_a whose generic type is regular and non-orthogonal to p .*

Proof:

Suppose that $G < \mathbb{G}_a^d$ for some $d \in \mathbb{N}$. One of the projections $\pi : G \rightarrow \mathbb{G}_a$ must have an infinite image in \mathbb{G}_a . Let a realize p , then $\pi(a)$ realizes the generic type of $H = \pi(G)$; this type is $tp(\pi(a)/K)$ which is also regular. Hence H satisfies the conclusion of the lemma. \square

We have now all we need to prove 3.

Proof of Theorem 3:

By 4.25 there is a (σ, D) -vector group G whose generic type q is regular and non-orthogonal to p , by 4.26 there is a definable subgroup H of the additive group whose generic type r is regular and non-orthogonal to q . By transitivity of non-orthogonality on regular types, $p \not\perp r$. \square

5. PARTIAL DIFFERENTIAL FIELDS WITH AN AUTOMORPHISM

As pointed out in [8], the previous work can be generalised to fields with several commuting derivations and a commuting automorphism of characteristic zero. We give the details.

A differential field is a field equipped with a finite set of commuting derivations $\Delta = \{D_1, \dots, D_n\}$. Then the ring of differential polynomials over K is defined as the ring of polynomials on the variables $D_1^{m_1} \cdots D_n^{m_n} X$, we denote it by $K[X]_\Delta$ and we can equip it with structure of a differential ring by extending Δ .

Let \mathcal{L}_n be the language of differential rings with m derivations, and DF_n the theory of differential fields of characteristic zero with n commuting derivations over \mathcal{L}_n . In [11], McGrail showed that this theory has a model companion, the theory of differentially closed fields DCF_n , and proved that it is a complete ω -stable theory which eliminates quantifiers and imaginaries. It has a noetherian topology, defined by zeros of ideals of differential polynomials, known as Δ -topology or Kolchin topology.

A partial differential field with an automorphism is a differential field with n commuting derivations and a commuting automorphism. In [8] the author showed that for the case of characteristic zero, the class of partial differential fields with an automorphism has a model companion. We denote it by D_nCFA .

The axiomatisation is the following (2.1 of [8]):

Fact 5.1. *Let (K, Δ, σ) be a differential-difference field. Then (K, Δ, σ) is existentially closed if and only if*

- (1) (K, Δ) is a model of DCF_n
- (2) Suppose V and W are irreducible Δ -closed sets such that $W \subseteq V \times V^\sigma$ and W projects Δ -dominantly onto both V and V^σ . If O_V and O_W are nonempty Δ -open sets of V and W , respectively, then there is $a \in O_V$ such that $(a, \sigma a) \in O_W$.

It is not known if this is a first order axiomatisation, but 2.3 of [8] gives us one that is first order (using characteristic sets of ideals).

As it is proved in [8], the model theory of D_nCFA is quite similar to the one of DCF_n : it is not complete but its completions are easily described, they eliminate imaginaries, are not stable but are supersimple and quantifier-free stable. Section 2 of [4] can be easily generalised to prove that the SU -rank of a generic of a model of D_nCFA is ω^{n+1} .

Now we want to prove a version of 1.1 for D_nCFA .

Let \mathcal{U} be a saturated model of D_nCFA , let $E = acl(E) \subset \mathcal{U}$, and let G be a connected Δ -group defined over $K = acl(K)$. Let $G^{(n)} = G \times G^\sigma \times \cdots \times G^{\sigma^n}$ and let $q_n : G \rightarrow G^{(n)}$ such that $q(x) = (x, \sigma(x), \dots, \sigma^n(x))$.

Let g be a generic of G such that $q_n(g)$ is Δ -independent over K . Then $q_n(g)$ is a (Δ) -generic of $G^{(n)}$ and thus $q_n(G)$ is Δ -dense in $G^{(n)}$. This implies that $G^{(n)}$ is connected.

Let H be a definable subgroup of G and let $H^{(n)}$ be the Δ -Zariski closure of $q_n(H)$ in $G^{(n)}$.

Let $\tilde{H}^{(n)} = \{x \in G : q_n(x) \in H^{(n)}\}$. These groups form a decreasing sequence of quantifier-free definable subgroups of G containing H . By Noetherianity this sequence is finite, so there is N such that $\tilde{H}^{(N)} = \bigcap_n \tilde{H}^{(n)}$. Thus this is the Zariski closure of H in G .

As in [4], we can prove that $[\tilde{H} : H] < \infty$.

Let \mathcal{C}_i be the field of constants of D_i . The field of total constants is $\mathcal{C} = \bigcap \mathcal{C}_i$. Let $\text{Fix } \sigma$ be the fixed field. Since every algebraic subgroup of a vector group is defined by linear equations, each \tilde{H}^n is defined by linear differential equations, and thus H

is defined by linear (σ, Δ) -equations and this implies that it is a $(\text{Fix } \sigma \cap \mathcal{C})$ -vector space. Thus, following [4], we can prove the following lemma.

Lemma 5.2.

- (1) *Let H be a quantifier-free definable subgroup of \mathbb{G}_a^n . Then H is a $(\text{Fix } \sigma \cap \mathcal{C})$ -vector space, so it is divisible and has therefore no proper subgroup of finite index. This implies that every definable subgroup of \mathbb{G}_a^n is quantifier-free definable.*
- (2) *Let G be a definable subgroup of \mathbb{G}_a^n , and H a definable subgroup of G . Then G/H is definably isomorphic to a subgroup of \mathbb{G}_a^l for some l .*

We can apply the above lemma to definable subgroups of the additive group, and following [4] again, obtain the following proposition.

Proposition 5.3. *Let G be a definable subgroup of \mathbb{G}_a^n . If G has infinite dimension then $SU(G) \geq \omega$.*

In [8], the author proved the dichotomy for finite dimensional types.

Fact 5.4. *Let (\mathcal{U}, σ, D) be a saturated model of $D_n\text{CFA}$ and let $K = \text{acl}(K) \subset \mathcal{U}$. Let $p \in S(K)$ be a finite dimensional type of SU -rank 1. Then p is either 1-based or non orthogonal to $\text{Fix } \sigma \cap \mathcal{C}$*

Now in order to define arc spaces for a (σ, Δ) -variety we must define the differential prolongation of an algebraic variety. We use in this case the approach of [10] (section 2).

Let $(K, \Delta = (D_1, \dots, D_n))$ be a differential field. Define $K_m = K[\eta_1, \dots, \eta_n]/(\eta_1, \dots, \eta_n)^{m+1}$. Let $E : K \rightarrow K_m$ defined by:

$$a \mapsto \sum_{0 \leq \alpha_1 + \dots + \alpha_n} \frac{1}{\alpha_1! \dots \alpha_n!} D_1^{\alpha_1} \dots D_n^{\alpha_n}(a) \eta_1^{\alpha_1} \dots \eta_n^{\alpha_n}$$

If V is an algebraic variety defined over K , the m -th Δ -prolongation $\tau_m V$ of V is the Weil restriction of $V \otimes_E K_m$ from $\text{Spec}(K_m)$ to $\text{Spec}(K)$.

The reduction maps $K_l \rightarrow K_m$ for $l \geq m$ imply that the prolongations form a projective system $\pi_{l,m} : \tau_l \rightarrow \tau_m$. If we identify τ_0 with the identity and denote $\pi_m, 0$ as π_m , we obtain the projection $\pi_m : \tau_m V \rightarrow V$. The map $\nabla_m : V \rightarrow \tau_m V$ defined by

$$x \mapsto \sum_{0 \leq \alpha_1 + \dots + \alpha_n} \frac{1}{\alpha_1! \dots \alpha_n!} D_1^{\alpha_1} \dots D_n^{\alpha_n}(x) \eta_1^{\alpha_1} \dots \eta_n^{\alpha_n}$$

is a section of π_m and $\nabla_m(V)$ is Δ -dense in $\tau_m V$.

Using the definition of q_m and $S_m(V)$ from the previous section we define $\Phi(V) = \tau_m(S_m(V)) (= S_m(\tau_m V))$ and $\varphi = \nabla_m \circ q_m$.

Now we can define arc spaces of (σ, Δ) -varieties as before:

We extend σ and D_i to $K^{(m)}$ by defining $\sigma(\eta_j) = \eta_j$ and $D_i \eta_j = 0$. Since σ and D_i commute, we can identify $\mathcal{A}_r(S_m(V))(K)$ with $S_m(\mathcal{A}_r(V))(K)$ and we can assume that $\mathcal{A}_r(\Phi_m(V))(K) = \Phi_m(\mathcal{A}_m(V))(K)$.

If X is a (σ, Δ) -variety given as a (σ, Δ) -closed subset of an algebraic variety \bar{X} , we define $\Phi_m(X)$ as the Zariski closure of $\psi_m(X)$ in $\Phi_m(\bar{X})$. X is determined by the prolongation sequence $\{t_{l,m} : \Phi_l(X) \rightarrow \Phi_m(X) : l \geq m\}$.

We extend the previous notion of non-singular point, and as before we can show that if V be a (σ, D) -variety given as a closed subvariety of an algebraic variety \bar{V} . If $m \in \mathbb{N}$ and $a \in V(K)$ is a non-singular point. Then $\{\mathcal{A}_m(t_{r,s}) : \mathcal{A}_m\Phi_r(V)_{\psi_r(a)} \rightarrow \mathcal{A}_m\Phi_s(V)_{\psi_s(a)}, r \geq s\}$ form the (σ, D) -prolongation sequence of a (σ, D) -subvariety of $\mathcal{A}_m\bar{V}_a$.

So we define the m -th arc space of V at a , \mathcal{A}_mV_a , to be this subvariety.

Now we remark that in [9], the author proved his theorem for a field with n derivations and k automorphisms, so for our case subfield F of K and $a \in K$ there is a numerical polynomial $P_{a/F}(X) \in \mathbb{Q}[X]$ of degree at most $n + 1$, such that for sufficiently large $r \in \mathbb{N}$, $P_{a/F}(r) = \text{tr.dg}(\psi_r(a)/F)$. This implies that the notion of (σ, Δ) -type if a (σ, Δ) -variety is well defined and it is less or equal than $n + 1$.

If p is a type over K , there is a finite sequence of regular types p_1, \dots, p_k such that $m(p) \geq m(p_i)$ for all i and p is domination-equivalent to $p_1 \times \dots \times p_k$. As in the previous section, we can prove that given a type p over K and a regular type q , such that $p \not\leq q$, there are c and L such that the type $r = tp(c/L)$ is regular, $q \not\leq r$, $m(r) \leq m(p)$ and $SU(r) = \omega^i$ for $i \in \{0, 1, \dots, n + 1\}$. This is essentially what we need to prove 4.23.

The other propositions can be translated almost verbatim to the case of several derivatives, thus obtaining the following version of 4.26.

Lemma 5.5. *Let G be a (σ, Δ) -vector group and let p be its generic type. If p is regular there is a definable subgroup of \mathbb{G}_a whose generic type is regular and non-orthogonal to p .*

Putting 5.3, 5.4 and 5.5 together we obtain Zilber's dichotomy.

Theorem 4. *Let $(\mathcal{U}, \sigma, \Delta)$ be a saturated model of D_nCFA and \mathcal{C} its field of total constants. Let $K = \text{acl}(K) \subset \mathcal{U}$ and let p be a type over K . If p is a type of SU -rank 1, then it is either 1-based or non orthogonal to $\text{Fix } \sigma \cap \mathcal{C}$.*

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