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On twin peak quasi-periodic oscillations resulting from the interaction between discoseismic modes and turbulence in accretion discs around black holes

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ABSTRACT

Given the peculiar and (in spite of many efforts) unexplained quasi-periodic oscillation (QPO) twin peak phenomena in accretion disc PSD observations, the present exploratory analytical article tries to inquire deeper into the relationship between discoseismic modes and the underlying driving turbulence in order to assess its importance. We employ a toy model in the form of a Gaussian white noise driven damped harmonic oscillator *with stochastic frequency*. This oscillator represents the discoseismic mode. (Stochastic *damping* was also considered, but interestingly found to be less relevant for the case at hand.) In the context of this model, we find that turbulence interacts with disc oscillations in interesting ways. In particular, the stochastic part in the oscillator frequency behaves as a separate driving agent. This gives rise to 3:2 twin peaks for some values of the physical parameters, which we find. We conclude with the suggestion that the study of turbulence be brought to the forefront of disc oscillation dynamics, as opposed to being a mere background feature. This change of perspective carries immediate observable consequences, such as considerably shifting the values of the (discoseismic) oscillator frequencies.

Key words: accretion, accretion discs – black hole physics – X-rays: binaries

1 INTRODUCTION

After two decades, and in spite of active research, the remarkable structure in the power spectra of high frequency (40-450 Hz) quasi-periodic oscillations (HFQPO) in several X-ray binaries remains an intriguing puzzle. This structure often consists of power spectra twin peaks in a 3:2 frequency ratio (Abramowicz & Kluźniak 2001; see also Remillard et al. 2002; Homan et al. 2005; Belloni et al. 2006; for a full list, refer to Ortega-Rodríguez et al. 2014 and references therein).

An understanding of HFQPOs may allow us to obtain important information about the corresponding black hole's mass and spin, and the behavior of inner-disc accretion flows.

The physics of HFQPOs is not completely understood. Since the observed 3:2 frequency ratio suggests the presence of non-linear physics, resonant models have been proposed (see e.g. Kluźniak & Abramowicz 2001; Stuchlík, Kotrlová & Török 2013; a detailed discussion of models can be seen in Török et al. 2011).

We wish to consider here a different line of inquiry, which does not exclude non-linear considerations but rather could work in tandem with them.

The main objective of this paper is to explore the question of what happens when the MRI-generated turbulent background in an accretion disc interacts with a discoseismic fundamental g-mode, or any type of sufficiently stable oscillation for that matter (we discuss below what has been researched so far). For a review of discoseismology, see Wagoner (2008); it is important to mention that while discoseismic modes were reported to be present in hydrodynamic simulations (Reynolds & Miller 2009), they have yet to be seen in MHD simulations.

We implement this interaction in the form of a toy model. When devising a toy model for the disc's complicated dynamics, our aim was to propose the simplest mathematical expression which includes the main physics: turbulence, mode oscillation, plus the turbulence-induced frequency variability of this mode oscillation. (For completeness, damping variability was also included.) We aimed thus for a single differential equation:

$$\ddot{x} + \alpha(t)\dot{x} + \omega^2(t)x = f(t), \tag{1}$$

where the driving term $f(t)$, representing turbulence, is (within our toy framework) Gaussian white noise (see, how-

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ever, below) with zero mean and

$$\langle f(t)f(t')\rangle = \text{constant} \cdot \delta(t-t') \tag{2}$$

(brackets standing for ensemble average), whereas $\omega(t)$, $\alpha(t)$, representing the mode frequency and damping coefficient, are randomly varying functions of time (in a way specified below).

At this point it is sensible to ask whether the single equation (1) is by itself enough to capture the necessary insights to understand this physical system.

We believe that, when modeling the system, the mental picture one should have in mind is less a set of coupled oscillators (one for the mode, one for turbulence) exchanging energy back and forth, and more one of a mostly unidirectional transfer of energy in which turbulence acts effectively as a reservoir whose energy is fed by a (presumably MRI driven) cascading Kolmogorov process from larger to smaller scales.

Furthermore, note that the effects of turbulence enter in two places in equation (1): in the RHS, via $f(t)$, and in the LHS, via the variability in ω and α, and although $f(t)$ is white, the LHS terms effectively couple the modes only to certain frequencies in the turbulence, as discussed below. (One might ask why the need of $f(t)$ in the first place, and the reason is that it helps with stability of the system; see e.g. Frisch 1968.)

With these considerations, we will find that turbulence interacts with randomly-variable oscillations in interesting ways, and this gives a possible explanation for twin peak QPOs. We conclude that turbulence might be important in these dynamics and suggest that its study be brought to the forefront, as opposed to being a mere background feature. This change of perspective brings immediate observable consequences, such as considerably shifting the values of the (discoseismic) oscillator frequencies.

It is important to compare our approach with that of Vio et al. (2006). Their toy model uses *two* coupled nonlinear oscillator equations of *constant* (or non-stochastically variable) frequency with a stochastic source for one of them. Their results are interesting because they find that such a turbulent source does enhance both oscillations, and that there is a window of opportunity (too much or too little turbulence result in no enhancing). As these findings are also results of the present paper, both papers taken together do make a strong case for the importance of turbulence in these astrophysical systems. The main difference between the articles is that we only assume a single oscillator, so the appearance of the second frequency in the spectrum is more intriguing and meaningful.

2 THE MODEL

2.1 Parametrizing stochasticity

The approach starts from oscillator equation (1), which is a generalization of the one developed by Bourret, Frisch & Pouquet (1973, hereafter BFP) in that it allows for a time dependent $\alpha(t)$. The rationale is that turbulence is as likely to modify α as to modify ω.

The stochasticity of ω is parametrized thus:

$$\omega^2(t) = \omega_0^2[1+\varepsilon m(t)]\,, \tag{3}$$

where ω_0 is a constant, ε is (without loss of generality) a positive number and $m(t)$ is a two-valued Markov process variable taking the values ± 1 (central limit theorem considerations render this approach less toyish than it might seem at first sight). Concerning $m(t)$, its first important property is

$$\langle m(t)\rangle = 0\,. \tag{4}$$

Brackets stand for ensemble average. The ensemble average of a quantity W which depends on a set of two-valued variables m_i [so that $W(m_i)$ stands for $W(m_1, m_2, ...)$] is defined here as

$$\langle W(m_i)\rangle \equiv \sum P(m_k)W(m_k)\,, \tag{5}$$

where the sum runs over all specific combinations of the m_i set, and $P(m_k)$ is the probability of the specific combination of m_k values.

With this definition, equation (4) follows trivially. For averages involving variables at different times, we will need to introduce the corresponding conditional probabilities: $(s, \nu > 0)$

$$\text{Prob}\{m(t+s) = \pm 1 \,|\, m(t) = \pm 1\} = \tfrac{1}{2}(1+e^{-\nu s})\,, \tag{6}$$

$$\text{Prob}\{m(t+s) = \pm 1 \,|\, m(t) = \mp 1\} = \tfrac{1}{2}(1-e^{-\nu s})\,. \tag{7}$$

These equations can be thought of as defining ν. We can now obtain the autocorrelation

$$\langle m(t+s)\, m(t)\rangle = e^{-\nu|s|} \tag{8}$$

(the absolute value on s allows it to take negative values).

For the damping term, we proceed analogously:

$$\alpha(t) = \alpha_0[1+\delta n(t)]\,, \tag{9}$$

where α_0 is a constant, δ is a positive number and $n(t)$ is a two-valued Markov process variable with the following properties:

$$\langle n(t)\rangle = 0\,, \quad \langle n(t+s)\, n(t)\rangle = e^{-\bar{\nu}|s|}\,. \tag{10}$$

In a spirit of simplicity, we will work here the case in which $m(t)$ and $n(t)$ are uncorrelated, i.e. $\langle m(t)\, n(t')\rangle = 0$.

2.2 Physical meaning of the parameters

We will now make explicit the correspondence 'dictionary' relating the toy-model parameters to their physical counterparts in the accretion disc system.

(i) ω_0 refers to the mode frequency; this is not the same as the *observed* frequency, which is given by the effective frequency $\omega_{\rm eff}$, as explained below (as usual, all frequency values in the present article refer to those detected far away from the black hole).

(ii) $\alpha_0/\omega_0 \approx 0.1$ is the inverse QPO quality factor Q; note that this parameter and the previous one can be read off, in an approximate fashion, directly from a power spectral density (PSD) graph (the numerical value 0.1 is taken from typical observational data); the reading is only approximate because ε can affect the reading of α_0, making it appear slightly larger than it is, and turbulence effectively shifts the frequency, as explained below.

(iii) the white noise $f(t)$ refers to the underlying turbulence; its mathematical properies are defined by equation (2).

(iv) ε, defined as positive without loss of generality, measures the variability of ω; we will place it in the range $0 < \varepsilon < 1$ because larger values of ε are incompatible with observations as they would force broad PSD peaks with low values of Q.

(v) δ, also defined positive, measures the variability of α; we also set it in the range $0 < \delta < 1$ because its physical origin is presumably the same as that of ε.

(vi) ν and $\bar{\nu}$ measure the time correlations of the time-changing variables (frequency and damping coefficient, respectively); scaling Kolmogorov considerations (Kolmogorov 1941; Cho, Lazarian & Vishniac 2003) indicate that the coupling between oscillations such as fundamental discoseismic g-modes and turbulence is strong only when ν, $\bar{\nu} \sim \omega_0$ or slightly smaller.

To elaborate on the last point, Kolmogorov scaling considerations assert that turbulent eddy frequencies ($\omega_{\rm tur}$) and length scales (λ) satisfy $\omega_{\rm tur} \propto \lambda^{-2/3}$ under broad conditions. The constant of proportionality is $u_* L_*^{-1/3}$, where u_* and L_* are the typical speed and length scales of the turbulent physical system. For thin disks, $u_* \sim h\Omega$ and $L_* \sim h$, where h is the typical disc thickness and Ω stands for the typical angular frequency. The typical angular frequency is also (approximately) the g-mode frequency, i.e. $\Omega \approx \omega_0$. This all means that setting λ = size of fundamental g-mode $\sim \sqrt{hR}$, where R is the typical radial scale, one has $\omega_{\rm tur} \sim \omega_0\,(h/R)^{1/3}$, which places the frequency of g-mode-sized turbulent eddies somewhat below ω_0 (for $h/R = 0.1$, $\omega_{\rm tur} \approx 0.5\,\omega_0$). These are the eddies that have a greater effect on discoseismic g-modes.

3 RESULTS AND DISCUSSION

3.1 Frequency shift

We then follow through the analytical method devised by BFP, expanding it to include variable damping. Since the approach is straightforward and grounded in well established mathematics, we merely give the results, leaving details for Appendix A.

We first look at what happens to the *observed* frequency after the effects of stochasticity have been introduced. This effective frequency is given by

$$\omega_{\rm eff}^2 \equiv \frac{\langle \dot{x}^2 \rangle_{\rm eq}}{\langle x^2 \rangle_{\rm eq}}, \tag{11}$$

where brackets indicate ensemble averages and 'eq' indicates that the system has been allowed to reach stationarity (and we therefore assume that it does).

For the special case of stochastic frequency and constant damping ($\delta = 0$), we obtain

$$\omega_{\rm eff}^2 = \omega_0^2 - \frac{2\varepsilon^2\omega_0^4(2\alpha_0 + \nu)}{(\alpha_0 + \nu)\left(2\alpha_0\nu + \nu^2 + 4\omega_0^2\right)}, \tag{12}$$

which reproduces equation (3.25) in BFP.

In order of magnitude, taking $\nu \sim \omega_0$ and $\alpha_0 \sim 0.1\,\omega_0$, one obtains the following:

$$\omega_{\rm eff}^2 = \omega_0^2\,[1 + O(\varepsilon^2)]\,. \tag{13}$$

This is an important result since $\omega_{\rm eff}$ is the frequency that detectors measure, and it can differ substantially from the theoretical one, ω_0 (calculated, for example, from discoseismology), given that ε is not much smaller than 1. (When substituting the relevant numbers, one finds that the shift in frequency ranges from a few percent to $\approx 40\%$.)

We now turn to the special case of constant frequency ($\varepsilon = 0$) and stochastic damping ($\delta \neq 0$), for which one obtains

$$\omega_{\rm eff}^2 = \omega_0^2 - \frac{2\alpha_0^2\delta^2\bar{\nu}\omega_0^2}{\bar{\nu}\left[2\alpha_0^2\left(\delta^2 + 1\right) + 3\alpha_0\bar{\nu} + \bar{\nu}^2\right] + 4\omega_0^2(\alpha_0 + \bar{\nu})}\,. \tag{14}$$

In order of magnitude, we have

$$\omega_{\rm eff} = \omega_0^2\,[1 + O\{\delta^2(\alpha_0/\omega_0)^2\}]\,. \tag{15}$$

In this way, the variability of the damping coefficient is less important, by two orders of magnitude, than the variability of the frequency. This points to a scenario in which the physics of stochasticity is contained already when considering just variable frequency. For this reason, from now on (unless otherwise indicated) we consider the $\delta = 0$ case only.

3.2 Stationarity and stability

As BFP correctly point out, procedures such as the one carried out in the present paper *assume* that there exist stationary solutions. With this assumption one can obtain results like this one: ($\delta = 0$ case)

$$\langle x^2 \rangle_{\rm eq} = \frac{S}{2\omega_0^2\alpha_{\rm eff}}, \tag{16}$$

where

$$\alpha_{\rm eff} = \alpha_0 - \frac{\varepsilon^2\omega_0^2(\nu + 2\alpha_0)^2}{(\nu + \alpha_0)[4\omega_0^2 + \nu(\nu + 2\alpha_0)]} \tag{17}$$

(see Appendix A for details).

Self-consistency must be checked though. This result for $\langle x^2 \rangle$ does not make physical sense for some values of the parameters; in an obvious fashion, for those which render $\alpha_{\rm eff}$ negative, and it is not clear that all solutions with positive $\alpha_{\rm eff}$ are stable. What one needs is a proof of stability. BFP actually prove that all solutions for which $\alpha_{\rm eff}$ is positive are indeed stable (see Appendix B for a discussion).

We see then that for large enough values of ε the system becomes unstable and therefore non-stationary. This happens for the critical value ε_c (for which $\alpha_{\rm eff} = 0$) such that

$$\varepsilon_c^2 = \frac{\alpha_0(\alpha_0 + \nu)[4\omega_0^2 + \nu(\nu + 2\alpha_0)]}{\omega_0^2(\nu + 2\alpha_0)^2}\,. \tag{18}$$

3.3 Twin peaks

There are two peaks in the PSD of the stationary solution of equation (1) for some values of the parameters. The way to obtain the formula for the PSD is explicated in Appendix A, and the results are given by equations (A21) and (A16). Peak positions are obtained analytically by calculating the complex zeros of the function B defined in equation (A18).

Furthermore, while peak frequencies can be in any ratio, for some values of the parameters this will be a 3:2 ratio. Fig. 1 shows the two peaks in a 3:2 ratio for values of the parameters in the ranges discussed above.

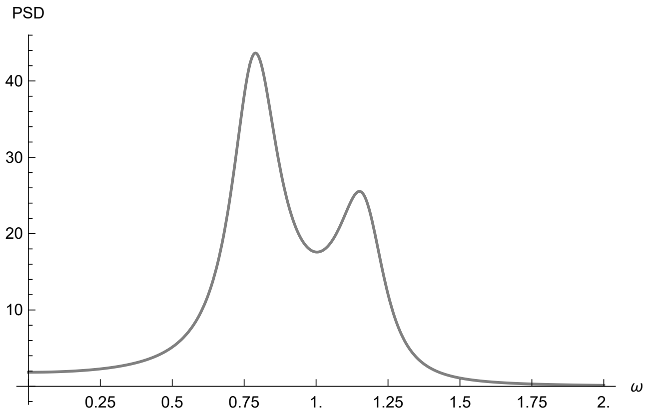

Figure 1. Power spectral density (arbitrary units) as a function of the frequency ω (in units of ω_0) corresponding to the stationary stable solution of equation (1) for the following choice of parameters: $\varepsilon = 0.4$, $\alpha_0 = \nu = 0.1\,\omega_0$, $\delta = 0$. The peak on the right arises purely as a result of the stochasticity of the system.

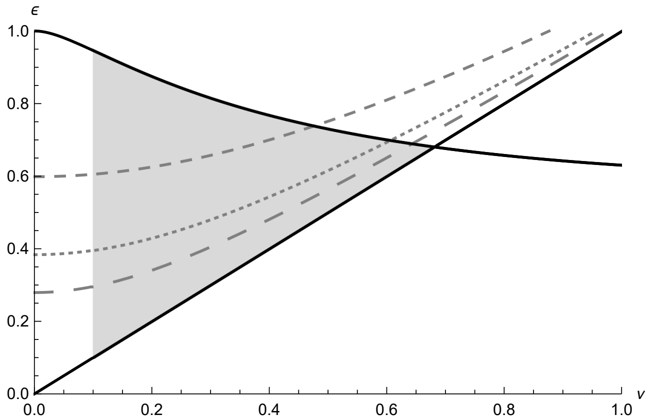

Figure 2. Values of ε and ν (in units of ω_0) that produce peaks in a 3:2 frequency ratio (dotted line). Also shown are the cases of 2:1 ratio (short dash) and 4:3 ratio (long dash). The shaded region corresponds to stationary, two-peaked solutions with values of ν that allow mode-turbulence coupling. The region with low values of ν is excluded in agreement with the qualitative discussion of item 2.2 (vi).

As we mentioned, one way to understand the physical origin of the higher-frequency peak (the one not arising from ω_0) is by noting that the term $\varepsilon m(t)\omega_0^2 x$ on the LHS of equation (1) has, by virtue of its stochasticity, a behavior which is so different from the other one ($\omega_0^2 x$) that it behaves as a separate driving agent from $f(t)$. Of course one needs the presence of the (non-stochastic) oscillator on the LHS of equation (1) for this high-frequency peak to appear, no less than for the low-frequency one. Favoring this separate-agent interpretation is the fact that two-peakedness disappears for low enough ε, in which case the stochastic term is 'assimilated' by the dominant $\omega_0^2 x$ term.

The question might arise of how one knows that it is the *lower* (and not the upper) frequency peak the one that corresponds to the ω_0 term in equation (1). The answer is that, as one computes the PSD graph by scanning all possible values of the variables ε, ν and α_0, one can see the appearance of the higher frequency peak branching off to the right side, and always from the bottom part, of the main one (which is always present). By such a parameter scanning one can also appreciate that the lower frequency peak is always larger in amplitude than the higher frequency one; this makes sense as the lower frequency peak corresponds to the main ω_0 oscillator in equation (1).

Fig. 2 shows the possible values of the parameters for which the frequencies of the PSD peaks are in a 3:2 ratio. The vertical axis is ε; the horizontal axis is ν/ω_0; α_0 is set to $0.1\,\omega_0$ but the results are rather insensitive to its value. We are interested in solutions lying in the shaded region. All points which are above the solid straight line correspond to two-peak solutions (as opposed to just one-peak), and all points below the solid non-straight curve, which is given by $\varepsilon = \varepsilon_c$ of equation (18), correspond to *stationary* solutions. Thus, the shaded region corresponds to stationary, two-peaked solutions. In order to be consistent with the discussion in item 2.2 (vi), we have left out as well the leftmost part of the region between the solid lines, say $\nu/\omega_0 < 0.1$; turbulence is not an effective agent there.

The solutions in Fig. 2 corresponding to peaks in a 3:2 frequency ratio are given by the dotted curve (between the two dashed curves). Also shown are solutions corresponding to other ratios. It is important to emphasize that being in the shaded region is only a necessary condition for the existence of the twin peaks and by all means not a sufficient one. The peaks still need a mechanism to grow and become visible. This mechanism might be given, for example, by a non-linear resonance, which is more effective for lower values of a, b in an $a : b$ frequency ratio (see the references in the first section of this article).

The above considerations give the 3:2 frequency ratio an optimal condition of having low integers while at the same time being compatible with larger values of ν (which imply a stronger coupling between modes and turbulence), and being not too close to the bold straight line in Fig. 2 (so as to allow a clear valley between the peaks).

Our considerations allow us to make the following prediction. As better X-ray timing observations become available, the lower frequency QPO will show a larger amplitude than the other twin. Furthermore, the higher frequency QPO is the one that will intermittently appear and disappear, as it depends on favorable conditions of the variable turbulent dynamics.

Future work could attempt to explore the physical system using an equation or set of equations having more structure than equation (1), for example one based on hydrodynamical considerations. Numerical work can also complement this line of research.

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APPENDIX A: DERIVATION OF FORMULAS FOR THE EFFECTIVE FREQUENCY AND THE PSD

We follow, and generalize, the solution of BFP to equation (1) under the conditions of stationarity of the solutions.

The starting point is recasting equation (1) in matrix form:

$$\frac{dX}{dt} = M(t)\,X + F(t), \tag{A1}$$

where

$$X \equiv \begin{pmatrix} x \\ \dot{x} \end{pmatrix}, \quad M(t) \equiv \begin{pmatrix} 0 & 1 \\ -\omega^2(t) & -\alpha(t) \end{pmatrix}, \quad F(t) \equiv \begin{pmatrix} 0 \\ f(t) \end{pmatrix} \tag{A2}$$

(all variables were defined in the main text). Unlike BFP, we allow $\alpha(t)$ to depend on time.

The quantities of interest take the form:

$$\Gamma_{ij}(t_1, t_2) \equiv \langle X_i(t_1)\, X_j(t_2)\rangle\,, \tag{A3}$$

and we note that in stationary regime $\Gamma_{ij}(t_1, t_2)$ by definition will depend only on the difference $t_2 - t_1 \equiv s$. Thus, from now on, we will write $\Gamma_{ij}(s)$.

We are interested in particular in $\Gamma_{11}(0)$ and $\Gamma_{22}(0)$ (section 3.1) and in the Laplace transform of $\Gamma_{11}(s)$ (section 3.3).

We will use the Green function of equation (A1), defined as the matrix which satisfies

$$\frac{dG(t_1, t_2)}{dt_1} = M(t_1)\, G(t_1, t_2), \quad G(t_1, t_1) = I\,, \tag{A4}$$

where I is the 2×2 identity matrix. The function $\Gamma_{ij}(s)$ can now be expressed in terms of $G(t_1, t_2)$:

$$\Gamma_{ij}(s) = S \int_0^t dt' \langle G_{i2}(t, t')\, G_{j2}(t + s, t')\rangle\,, \tag{A5}$$

where S is the constant that appears in equation (2), and the limit $t \to \infty$ is implied so as to ensure that the transient terms have vanished. Equation (A5) can be put in a cleaner form noting that $G(t_1, t_2)$, being stationary, is invariant under time shifts. Subtracting then t' from all time arguments and introducing the time variable $T \equiv t - t'$, one obtains

$$\Gamma_{ij}(s) = S \int_0^\infty dT \langle G_{i2}(T, 0)\, G_{j2}(T + s, 0)\rangle\,. \tag{A6}$$

Using this notation, we follow BFP and express everything in terms of the 'vector' quantity $Z(t)$:

$$\frac{dZ(t)}{dt} = [L + m(t)L' + n(t)L'']\, Z(t), \quad Z(0) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \tag{A7}$$

where

$$Z(t) \equiv \begin{pmatrix} [G_{12}(t,0)]^2 \\ [G_{22}(t,0)]^2 \\ G_{12}(t,0)\, G_{22}(t,0) \end{pmatrix}, \tag{A8}$$

$$L \equiv \begin{pmatrix} 0 & 0 & 2 \\ 0 & -2\alpha_0 & -2\omega_0^2 \\ -\omega_0^2 & 1 & -\alpha_0 \end{pmatrix}, \tag{A9}$$

$$L' \equiv -\varepsilon\omega_0^2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 1 & 0 & 0 \end{pmatrix}, \quad L'' \equiv -\delta\alpha_0 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{A10}$$

(We note in passing that there is a typographical error in the central term of L in the first matrix of equation (3.10) in BFP; also note the difference in notation between their paper and ours.)

We solve the differential equation in Laplace space. The solution is

$$\langle \tilde{Z}(p)\rangle_i = [p - L - L'(p+\nu-L)^{-1}L' - L''(p+\bar{\nu}-L)^{-1}L'']^{-1}_{i2} \tag{A11}$$

(recall that brackets refer to ensemble averaging), where

$$\tilde{Z}(p) \equiv \int_0^\infty Z(t)\, e^{-pt}\, dt\,. \tag{A12}$$

We can finally write:

$$\langle x^2\rangle_{\rm eq} = \Gamma_{11}(0) = S \int_0^\infty \langle Z_1(t)\rangle\, dt = S\,\langle \tilde{Z}_1(0)\rangle\,, \tag{A13}$$

$$\langle \dot{x}^2\rangle_{\rm eq} = \Gamma_{22}(0) = S \int_0^\infty \langle Z_2(t)\rangle\, dt = S\,\langle \tilde{Z}_2(0)\rangle\,, \tag{A14}$$

which give us straightforwardly equations (12) and (14).

The other quantity of interest is the Laplace transform of $\Gamma_{11}(s)$. To obtain it, one starts form equation (A6) and (using the properties of the Green function) reexpresses it thus:

$$\Gamma_{11}(s) = S \int_0^\infty [\langle G_{11}(t+s, t)\, Z_1(t)\rangle + \langle G_{12}(t+s, t)\, Z_3(t)\rangle]\, dt\,. \tag{A15}$$

Working once more in Laplace space, and following a procedure analogous to the one of the preceding paragraphs, one obtains

$$\tilde{\Gamma}_{11}(p) = \frac{A}{B}(C + D)\,, \tag{A16}$$

where

$$A(\alpha, \nu, \omega_0, \varepsilon) \equiv \frac{S}{2\omega_0^2}\left(\alpha - \frac{\varepsilon^2\omega_0^2(\nu + 2\alpha)^2}{(\nu + \alpha)[4\omega_0^2 + \nu(\nu + 2\alpha)]}\right)^{-1}, \tag{A17}$$

$$B(p,\alpha,\nu,\omega_0,\varepsilon) \equiv p^4 + 2(\nu+\alpha)p^3 + (\nu^2+3\alpha\nu+\alpha^2+2\omega_0^2)p^2 + (\nu+\alpha)(\nu\alpha+2\omega_0^2)p + \omega_0^4(1-\varepsilon^2) + \nu(\nu+\alpha)\omega_0^2 , \quad (A18)$$

$$C(p,\alpha,\nu,\omega_0) \equiv (p+\alpha)(p+\nu+\alpha)(p+\nu) + \omega_0^2(p+\alpha) , \quad (A19)$$

$$D(p,\alpha,\nu,\omega_0,\varepsilon) \equiv \frac{\varepsilon^2\omega_0^4(\nu+2\alpha)(2p+2\alpha+3\nu)}{(\nu+\alpha)(\nu^2+2\alpha\nu+4\omega_0^2)} . \quad (A20)$$

We have worked here the $\delta = 0$ case, and thus α actually stands for α_0 in the last four equations. Solving the $\varepsilon = 0$, $\delta \neq 0$ case is straightforward but those results are astrophysically less relevant as they have no measurable consequences, as explained in the main text. (This means that all which came after equation (A15) is actually a restatement of BFP, and is included here for completeness.)

The above formalism allows us to express the PSD in terms of the Laplace transform of Γ_{11}. The PSD is the square modulus of $F(\omega)$, the Fourier transform of $x(t)$, i.e.

$$\text{PSD} \equiv |F(\omega)|^2 = \int_{-\infty}^{+\infty} e^{i\omega s} \langle x(t)\, x(t+s)\rangle_{\rm eq}\, ds = 2\,\tilde{\Gamma}_{11}(-i\omega) . \quad (A21)$$

Note that peak location is especially sensitive to the B function as it appears in the denominator in equation (A16).

APPENDIX B: PROOF OF STABILITY

It is important to prove that the allegedly stationary solutions are indeed stable.

Rather than repeating the proof of stability for the stationary solutions of the $\varepsilon \neq 0$, $\delta = 0$ case, which is described in detail in BFP, section 4, we offer here the proof for the $\varepsilon = 0$, $\delta \neq 0$ case, which follows the same argument. Future work includes proving the general case for which $\varepsilon \neq 0$ and $\delta \neq 0$.

For the $\varepsilon = 0$, $\delta \neq 0$ case, (16) holds but the analogous to (17) is now

$$\alpha_{\rm eff} = \alpha_0 - \frac{2\alpha_0^2\delta^2(2\alpha_0\bar{\nu}+\bar{\nu}^2+2\omega_0^2)}{\bar{\nu}[2\alpha_0^2(1+\delta^2)+3\alpha_0\bar{\nu}+\bar{\nu}^2]+4(\alpha_0+\bar{\nu})\omega_0^2} , \quad (B1)$$

and the critical value of δ (at which $\alpha_{\rm eff}$ changes sign) is given by

$$\delta_c^2 = \frac{(\alpha_0+\bar{\nu})(2\alpha_0\bar{\nu}+\bar{\nu}^2+4\omega_0^2)}{2\alpha_0\bar{\nu}(\alpha_0+\bar{\nu})+4\alpha_0\omega_0^2} . \quad (B2)$$

The proof of stability for $\delta < \delta_c$ is as follows. The starting point is the fact that the necessary and sufficient condition for stability is that the matrix that appears in (A11), with $\varepsilon = 0$, i.e.

$$[p - L - L''(p+\bar{\nu}-L)^{-1}L'']^{-1} , \quad (B3)$$

does not have any singularities for $\text{Re}(p) > 0$ (otherwise the inversion of the Laplace transform meets divergences). The singularities occur for those values of p which make the determinant of this matrix vanish. This determinantal equation takes the explicit form

$$H(p) - \alpha_0^2\delta^2 J(p) + \delta^4 K(p) = 0 , \quad (B4)$$

where

$$H(p) = (\alpha_0+p)(\alpha_0+p+\bar{\nu})\left(2\alpha_0 p + p^2 + 4\omega_0^2\right) \times \left[(p+\bar{\nu})(2\alpha_0+p+\bar{\nu}) + 4\omega_0^2\right] , \quad (B5)$$

$$J(p) = p(p+\bar{\nu})\left[8\alpha_0^2 + 6\alpha_0(2p+\bar{\nu}) + 5p(p+\bar{\nu})\right] + 8\omega_0^2\left[\bar{\nu}(\alpha_0+p) + p(2\alpha_0+p) + \bar{\nu}^2\right] + 16\omega_0^4 , \quad (B6)$$

$$K(p) = 4\alpha_0^4 p(p+\bar{\nu}) . \quad (B7)$$

The argument then is this: Firstly, prove that there are no singularities with $\text{Re}(p) > 0$ for sufficiently small δ. Secondly, find the value of δ where the system becomes unstable by setting $p = 0$ in equation (B4). We do these steps in turn.

First, for the no singularities part, proceed by *reductio ad absurdum*. Assume there is a solution of equation (B4) such that $\text{Re}(p) > 0$. This solution must satisfy $\text{Im}(p) = 0$ because otherwise quantities such as $\langle x^2\rangle$ would take negative values. But it is impossible for p to always take a real positive value for the following reason. As $\delta \to 0^+$, the term proportional to δ^4 becomes unimportant, and note that $H(p)/J(p)$ has a lower positive bound as it is a ratio of positive polynomials and tends to infinity as $p \to \infty$. We have thus reached a contradiction which means that the system is stable for sufficiently small δ.

Having established the first part of the argument, we now want to obtain the value of δ at which the system becomes unstable. Any onset of instability must to go through $p = 0$ [again, $\text{Im}(p)$ must vanish]. We set $p = 0$ in (B4) and find that there is only one solution. Reassuringly, this solution is exactly the one given by (B2). The system is thus stable for $\delta < \delta_c$.

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