Taking advantage of the different types of mathematical languages to promote students' meaningful learning

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The low performance in mathematics of non-mathematics majors has forced institutions of higher education to implement different measures to address the problem. Many of these measures have focused on curriculum modifications. This study presents a methodological way of approaching the problem, using written exercises that combine the symbolic, natural and pictorial language to improve the mathematical learning of the students. These exercises promote the development of essential mathematical skills to achieve successful mathematical learning. In this paper, I describe one exercise and analyze the solutions of 28 students of Calculus 1 course, at the University of Costa Rica. The results suggest that the exercises allow to explore the benefits of the different mathematical languages, so that the students can make connections between knowledge and theoretical concepts.

Keywords: mathematical languages, university mathematics education, languaging exercises.

Introduction

During the last few years, improving the mathematics performance of university students has been an important issue. Special attention has been paid to students of non-mathematics majors in the transition process from school to university (Goodchild & Rønning, 2014), since the students' mathematical background is not strong enough when they enter university. They may reach levels of reproduction of procedures, but without understanding the mathematical significance of the contents involved (Winsløw et al., 2018). Therefore, the students do not have the level of mathematical reasoning, abstract thinking and rigor required at university level (Gruenwald, Klymchuk & Jovanoski, 2004; Luk, 2005). This situation is reflected in the alarming failure and dropout rates presented in the initial math courses, from a large number of students who have mathematics in their academic programs (Biza et al., 2016).

This gap in students' mathematical knowledge has led to the implementation of several measures taken by the universities to address it. For instance, peer work, bridging courses, mathematical support centers, interactive lectures, videos, digital assessment, among others (Mustoe & Lawson, 2002). White-Fredette (2009) highlights that the actions taken for facing this situation should consider the instructional level, and Gruenwald et al. (2004) suggest that teachers should look for effective ways to help students to "understand the abstract concepts, master the formal language, follow rigorous reasoning, get a good feeling for the mathematical objects and acquire so-called mathematical maturity" (p.12). In a nutshell, attention should be paid to the students' understanding of the mathematical concepts and the need to develop their mathematical thinking.

Considering this need, I present the written languaging exercises as a teaching resource to improve the understanding of mathematical concepts by students, through the use of different languages. The exercises ask students to provide written explanations or justifications using symbols, drawings or their own words.

In this way, they have to organize their thoughts and review the reasoning that led to their solution, being more aware of the knowledge and concepts used, and the connection between them. The languaging exercises have been applied in university engineering mathematics (Joutsenlahti, Ali-Löytty, & Pohjolainen, 2016) and honor mathematics (Sillius et al., 2011), in Finish universities showing promising results. In this paper, I present the results of an application in a Calculus I course for non-mathematics majors in Costa Rica.

Theoretical Background

As literature indicates, it is necessary to promote conceptual understanding in students (Engelbrecht and Harding, 2015), teach them how to make connections between concepts (Nardi, 1996) and how to deal with the abstract nature of Mathematical concepts and the complexity of Mathematical thinking (Biza et al., 2016) required at the university level; if we want them to experience a successful learning of mathematics and facilitate a process of transition from school to university without problems. The Mathematical proficiency theory offered by Kilpatrick et al. (2001) proposes the development of key mathematical skills that help in this purpose. The theory suggests five main competences that are necessary to accomplish effective mathematics learning: conceptual understanding, procedural fluency, strategic competence, adaptive reasoning and productive disposition. These competences promote, among other, the ability to identify connections between concepts; to understand and provide justifications and reasons for procedures; to perform procedures flexibly, accurately and efficiently, knowing how, when and why to do it; to think logically, to represent, formulate and solve mathematical problems in different context; and to consider different strategies of solution. Therefore, all the strands are interwoven and should be practiced equivalently.

The mathematical proficiency competences can be developed by means of languaging exercises, which are designed based on the Languaging theory developed in Finland by Joutsenlahti (2009). Languaging is defined as the students' expression of their mathematical thinking using different languages (Joutsenlahti et al., 2016) and considers three languages: mathematical symbolic language (SL), natural language (NL) and pictorial language (PL).In this way, the languaging written exercises combine models and tasks, which aim to promote different mathematics competencies, as well as the use of different languages to access the characteristics of the mathematical objects and students' mathematical thinking.

Languaging theory rationalizes the use of different languages based on the multimodal approach. Languages play an important role in mathematics communication. Considering that most of the mathematics entities are abstract and can only be accessed through the use of languages (goo, 2006), it is important to address and recognize the multimodal nature of mathematics communication (Morgan et al., 2014). Having different representations to refer to the same element makes the "meaning making process" more meaningful. In addition, it shows more properties of a mathematical object than using only one (Dreher, Kuntze & Lerman, 2016), because each language shows specific features and connotations (O' Halloran, 2015).

In addition, the choice of written exercises is based on research that suggest that by writing, students have to organize their thoughts, review and clarify the mental processes they went through in the solution of a task. Furthermore, they have to try to express it in a clear and concrete way, so that readers can understand their mathematical thinking (Morgan, 2002). According to Kline and Ishii (2008), this process improves students' understanding.

Context and method

Due to the high rates of failure of non-mathematics majors in Calculus 1 course in the University of Costa Rica, the School of Mathematics decided to introduce the pre-calculus course, in order to provide students with the necessary knowledge for studying mathematics at university level. However, the high failure rates just moved to this new course, and the problem remains unsolved. Therefore, in this study, I suggest a different way of approaching the problem, with a resource that can be introduced in classes for students to have more meaningful learning by analyzing their solution processes when they have to write or explain them.

These languaging exercises were applied to 28 voluntary participants of non-mathematics major taking Calculus 1 course at the University of Costa Rica. There were 17 languaging exercises (see Alfaro (2018) for details) that were used in class or as homework during the period in which the subject of derivatives was studied. The exercises were designed combining different tasks as explain with your own words, complete missing steps, identify mistakes, argumentation of the solution, organizing solution steps, follow given solutions and various models to combine the use of the three languages. The exercises aim to promote the different competences of the mathematical proficiency theory, especially procedural proficiency, conceptual understanding and adapting reasoning; and to allow students to experience the use of different languages to express their thoughts.

In this paper, I will describe the exercises number 3 which exemplifies the use of symbolic, natural and pictorial language, to make different representations of a mathematical knowledge and the analysis of students' solutions. The aim is to provide evidence of the languaging exercises as an effective teaching resource for improving students understanding of mathematical concepts.

Description of the exercise

Exercise 3 (Figure 1) consists of a table that presents three cases in which a function is not derivable. Each case is exemplified with one language: symbolic, natural or pictorial, and the students have to complete the empty boxes with examples in the missing languages respectively, as shown in Figure 1. The use of the three languages allows students to explore different characteristics and properties of each case. Case I is described in NL with the phrase "At points where the curve presents peaks, since the lateral derivatives would be different." This statement has several characteristics. First, it does not refer to a particular function; therefore, students are not limited to the examples they can provide. In addition, it makes a suggestion to the graphical form (sharp points) of the function where the derivability requirement is violated, emphasizing the pictorial features. Finally, it refers to the theoretical aspect that fails (the lateral derivatives are different as shown in Figure 1).

For the case II, the example is given in SL, and refers to the situation in which the function has a vertical tangent line, at a point. This example refers to a specific function. However, students have to interpret from the symbolic expression what case it refers to. It means that the student has to recall the definition of derivability to identify the characteristic that makes the function not derivable.

Finally, case III shows a graph of a function that presents a discontinuity in the point x_0. As in the previous case, the student has to identify which case is presented in order to express it in NL.

The objective of the exercise is to observe if the students have clarity about the concepts and rules involved, in such a way that they are able to interpret them from any of the given representations and can express them in different ways.

What are the possible cases in which a function is not derivable?
Give examples of each of them using the three types of language.

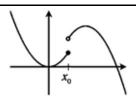
	Mathematical symbolic: numbers, symbols.	**Natural Language: written words.**	**Pictorial Language: drawings, graphs, etc.**
I		At points where the curve presents peaks, since the lateral derivatives would be different.	
II	$f(x) = \sqrt[3]{x}, in\ x = 0$		
III			

Figure 1: Languaging exercise #3

In the next section, I will present some excerpts of the students' solutions, as evidence of the different uses of the languages they made, the different ways in which the students expressed the cases in their own words and some errors of interpretation and formality.

Results

From the analysis of the solutions of 28 participant students, some observations can be made about the students' understanding of the mathematical concepts involved from the different representations. I will discuss them case by case.

Case I: statement in natural language

For this case, students have to offer examples in symbolical and pictorial language. In the column of SL, they wrote different function samples such as absolute value and piecewise functions, with criteria of minor and greater complexity (Figure 2). As well, they included more general expression like $f'_-(c) \neq f'_+(c)$, and there were students that provided more complete answers by giving an example of a function and the values of the lateral derivatives.

A) $|x-2| = \begin{cases} x-2 & si\ x-2 \geq 0 \\ 2-x & si\ x-2 < 0 \end{cases}$ en $x = 2$	B) $f(x) = \begin{cases} \frac{1}{x} & si\ x \leq -1 \\ x & si\ -1 < x \end{cases}$ $f'_-(-1) = \lim_{x \to -1^-} = -1$ $f'_+(-1) = \lim_{x \to -1^+} = 1$

Figure 2: Students' examples in SL

In most cases, in both types of examples, students (n=10) did not make explicit in which point of that function the derivative does not exist. This situation raises the question of whether the students are aware of what is important in their example is the specific point where the derivative does not exist and what happens in it. In the solutions in which it was possible to associate the example in SL with that of PL, it can be verified if the student knew in what point the function described was not derivable by referencing the drawing; however, in the others, it was not clear. For instance, there were students giving an example in which the function presented two cases where the function was not derivable, and if they did not mark the point, one cannot know if they understood the case under discussion.

It is important to note that all the examples chosen by the students represented the given case, which means that they were able to correctly interpret the sentence in NL. In addition, by combining the SL and PL columns, it was possible to evaluate the students' abilities to correctly graph functions, pointing out asymptotes and points of intersection, as shown in Figure 3.

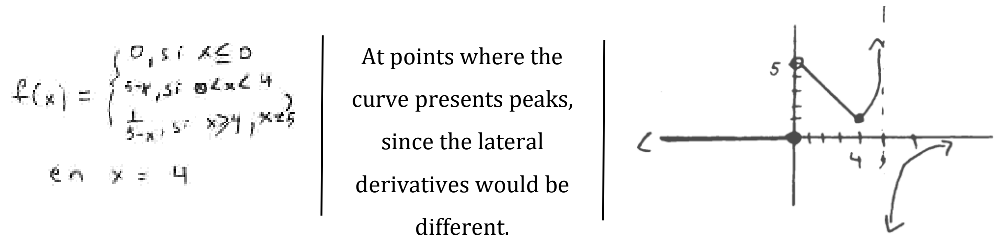


Figure 3: Function with two cases

Finally, some errors of rigor can be observed when writing in SL, as in example B of Figure 2, where the student writes the limit without indicating the function involved.

Case II: example of the criteria of a function in SL

In this case, the most interesting results were presented in the NL column. In the PL column, most of the students drew the plot of the given function and a few drew the tangent line. However, the results in NL, show that the students are not clear how to explain this case. Among its expressions, we found students (n=11) who could not even identify what was happening, arguing that the function was indefinite at that point, was discontinuous, had vertical asymptote or was constant. Nevertheless, there were cases (n=8) in which students seem unable to express their ideas in a mathematically correct way. For example, when referring to a vertical line as the function: “where the function is a vertical line” or when associating the derivative with the line instead of the slope: “where the derivative is vertical.”

Within the phrases they used to explain the phenomenon in case two, we can identify different connections between concepts that students used to justify their claims. Some made reference to the calculation of the limit of the derivative at that point and others properly to the relationship between the derivative and the slope. Examples can be found in Table 1. Although in these sentences one can identify some conceptual errors, such as the idea of a “vertical function” that must be corrected so that they do not recur, they show that the students had an idea of what was happening in the given case.

S16	*When the tangent of the point is vertical, as in this case, it is considered that the function is not derivable at that point.*
S4	*In the points where there are vertical lines, because this has no slope and therefore has no derivative.*
S13	*In the points where the derivative tends to* $+\infty$*, since this would mean a perpendicular tangent line, which does not exist.*
S5	*When solving the limit results in* $\frac{x}{0}$*, then it is a vertical function.*

Table 1: Students' answers in NL

Case III: the graphic of a discontinuous function

In this case, the students had to complete the SL and NL columns. In the SL column, responses with different characteristics were presented. Some students wrote piecewise functions in which, as in case 1, they forgot to point out the point of discontinuity. Though, in other cases, the students, in addition to the criterion of the function and the point where the function was not derivable, also added the calculations of the conditions of continuity: lateral limits and the value of image in the point (Figure 4A). This was also evidenced in answers like the one in figure 5B. Though, in this case they used a more general form. In this way they evidenced the theoretical knowledge about the topic.

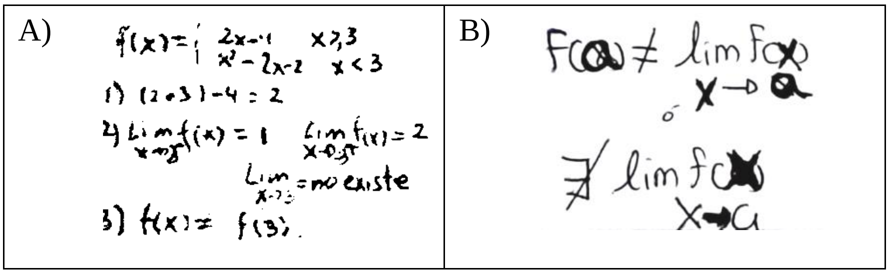


Figure 4: Students' examples in SL

Regarding the answers in NL, there were statements that highlighted the graphical features as in the jumps or the breaks, and others that considered the lateral limits, the continuity conditions and the fact that if a function is not continuous in x = a, then it is not derivable in x = a.

Final considerations

As stated at the beginning, the students coming to the universities lack of mathematical skill and knowledge required for university level. I suggest languages exercises as a methodological tools useful to address those issues and promote students' meaningful learning. The written languaging exercises, offer an option for boosting students' understanding of mathematics concepts, noticing the connections between concepts, the rules and properties that justify the procedures and the different representations. The use of the three languages reinforces different strands of mathematical proficiency, such as conceptual understanding, adaptive reasoning and strategic competence. For example, from the results it is possible to conclude that to solve the exercise students must understand, identify and verbalize connections between concepts, as well as represent mathematical situations in different situations, actions associated with conceptual understanding. It is important to highlight that in each case, in order to complete the empty boxes, the students had to interpret the given example, from that point they were already doing connections between representations. The adaptive reasoning is evidenced in the justifications and explanations that the students provide in natural language, to explain each case, and the strategic competence is present since the task presented to the students is not common for them, so they must show a flexible approach to solve this novel situation. All of this actions help them to experience a meaningful learning and to think about the concepts involved in the exercises instead of solving mechanically.

The different languages allow to study different characteristics of the mathematical concepts involved: a) NL evidenced the theoretical knowledge and, how the connections between the ideas are made, b) SL showed aspects related to the correct use of the symbols and specificities in relation to the examples, as in the mention of the point where the derivability was violated and c) PL all the features were combined and represented. In addition, the use of different languages makes it possible to observe the gaps in knowledge, misconceptions or difficulties that students have, that if only one language was used, some of them would not be evident. As evidenced in case 2, where most of the students could make the graph from the interpretation of the symbolic expression, but they face problems when trying to explain the situation in natural language, for which deeper knowledge of the subject was required.

This experience shows the potential of the use of different languages to improve the mathematical learning of the students and to promote their competences to become mathematically proficient. Through the results, I hope to motivate the exploration of the benefits of using different mathematical languages in the classrooms.

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