

# On summability of distributions and spectral geometry

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## Abstract

Modulo the moment asymptotic expansion, the Cesàro and parametric behaviours of distributions at infinity are equivalent. On the strength of this result, we construct the asymptotic analysis for spectral densities arising from elliptic pseudodifferential operators. We show how Cesàro developments lead to efficient calculations of the expansion coefficients of counting number functionals and Green functions. The bosonic action functional proposed by Chamseddine and Connes can more generally be validated as a Cesàro asymptotic development.

## 1 Introduction

Most approaches to spectral geometry rely on the asymptotic expansion of the heat kernel and Tauberian theorems. In this work, motivated by a string of recent papers by Connes, we develop spectral geometry from a more fundamental object. According to a deep statement by Connes [10], there is a one-to-one correspondence between Riemannian spin geometries and commutative real  $K$ -cycles, the dynamics of the latter being governed by the spectral properties of its defining Dirac operator. On ordinary manifolds, gravity (of the Einstein and the Weyl variety) is the only interaction naturally described by the  $K$ -cycle [1, 27, 28].

That is to say, in noncommutative geometry, existence of gauge fields requires the presence of a noncommutative manifold structure, whose “diffeomorphisms” incorporate the gauge transformations. Connes’ new gauge principle points thus to an intrinsic coupling between gravity and the other fundamental interactions. In a recent formulation [7], the Yang–Mills action functional is replaced by a “universal” bosonic functional of the form:

$$B_\phi[D] = \text{Tr } \phi(D^2),$$

with  $\phi$  being an “arbitrary” positive function of the Dirac operator  $D$ .

Chamseddine and Connes' work on the universal bosonic functional has two main parts. In the first one, they argue that  $B_\phi$  has the following asymptotic development:

$$B_\phi[D/\Lambda] \sim \sum_{n=0}^{\infty} f_n \Lambda^{4-2n} a_n(D^2) \quad \text{as } \Lambda \rightarrow \infty, \quad (1.1)$$

where the  $a_n$  are the coefficients of the heat kernel expansion [19] for  $D^2$  and  $f_0 = \int_0^\infty x\phi(x) dx$ ,  $f_1 = \int_0^\infty \phi(x) dx$ ,  $f_2 = \phi(0)$ ,  $f_3 = -\phi'(0)$ , and so on. Then they proceed to compute the development for the  $K$ -cycle currently [9, 32] associated to the Standard Model, indeed obtaining all terms in the bosonic part of the action for the Standard Model, plus gravity, plus some new ones. Their approach gives *prima facie* relations between the parameters of the Standard Model in terms of the cutoff parameter  $\Lambda$ , falling rather wide of the empirical mark. In the second part of their paper, they enterprise to improve the situation by use of the renormalization group flow equations [2]. This need not concern us here.

Formula (1.1) can be given a quick derivation, by assuming that  $\phi$  is a Laplace transform. This condition, however, will almost never be met in practice. In order to see that the asymptotic development of  $B_\phi$  cannot be taken for granted, let us consider, as Kastler and coworkers have done [6, 26] the characteristic functions  $\phi_\Lambda := \chi_{[0,\Lambda]}$ . This looks harmless enough, giving nothing but  $N_{D^2}(\Lambda^2)$ , the counting number of eigenvalues of  $D^2$  below the level  $\Lambda^2$ . However, it has been known for a long time – see for instance [24] – that there is no asymptotic development for the counting functional beyond the first term. Therefore equation (1.1), as it stands, is not applicable to that situation.

One of our aims in this paper is to decrypt the meaning of “arbitrary functional”; a related one is to put on a firm footing the development (1.1). Our contribution turns around the Cesàro behaviour of distributions, and its relation with asymptotic analysis. Most results are new, or seem ignored in the literature; the paper is written with a pedagogical bent.

The article is organized as follows. Section 2 is the backbone of the paper; there the Cesàro behaviour of distributions and Cesàro summability of evaluations are examined. The distributional theory of asymptotic expansions [15] is summarized. The latter is brought to bear by finding the essential equivalence between the Cesàro behaviour and the parametric behaviour of distributions at infinity. Also we prove that a distribution satisfies the moment asymptotic expansion iff it belongs to  $\mathcal{K}'$ , the dual of the space of Grossmann–Loupias–Stein operator symbols [20]. These results are new, having been obtained very recently by one of us [12]. We try to enliven this somewhat technical section with pertinent examples.

Next we consider elliptic, positive pseudodifferential operators; let  $H$  be one of those; the functional calculus for  $H$  can be based on the **spectral density**, formally written as  $\delta(\lambda - H)$ . This is arguably a more basic object than the heat kernel, and its study is very rewarding. In Section 3, we show that  $\delta(\lambda - H)$  is an operator-valued distribution in  $\mathcal{K}'$ . With that in hand, one can proceed to give a meaning to the universal bosonic action for a very wide class of functionals. Following some old ideas by Fulling [17], insufficiently exploited up to now, we emphasize that the Cesàro behaviour of the spectral density for differential operators is local, i.e., independent of the boundary conditions. This is practical for computational purposes, as it sometimes allows to replace an operator in question by a more convenient local model.

In Section 4, we reach the heart of the matter: let  $d_H(x, y; \lambda)$  denote the distributional kernel of  $\delta(\lambda - H)$ ; a formula for  $d_H$  is given and immediately applied to compute the coefficients of

its asymptotic expansion on the diagonal, in terms of the noncommutative residues [38] of certain powers of  $H$ . We hope to have clarified in the paper that the identification of the higher Wodzicki terms is essentially a “finite-part” calculation. The spectral density is actually a less singular object for operators with continuous spectra than for operators with discrete spectra, and all of the above applies to operators associated to noncompact manifolds: for that purpose, taking account of locality, we work with densities of noncommutative residues throughout. We go on to extend Connes’ trace theorem [8] to noncompact  $K$ -cycles. The case of generalized Laplacians is then treated within our procedure.

In the light of the preceding, the last two sections of the paper are concerned, respectively, with the counting number and the heat kernel expansions. The counting functional  $N_H(\lambda)$  is treated mainly by way of example. Then we reexamine the status of arbitrary smoothing asymptotic expansions, in particular the Laplace-type expansions like the Chamseddine–Connes Ansatz. We point out conditions for the expansions to be valid without qualification, and to be valid only in the Cesàro sense. Also we exemplify circumstances under which the formal Laplace-type expansion does not say anything about the true asymptotic development. The Chamseddine–Connes expansion is derived and reinterpreted.

## 2 Cesàro computability of distributions

Besides the standard spaces of test functions and distributions, the space  $\mathcal{K}$  first introduced in [20] and its dual  $\mathcal{K}'$  play a central role in our considerations. Familiarity with the properties of  $\mathcal{K}$  and  $\mathcal{K}'$  and with some of their elements will be convenient. For all general matters in distribution theory, we refer to [18].

As our interest is mainly in spectral theory, we consider Grossmann–Loupias–Stein symbols in one variable, almost exclusively. A smooth function  $\phi$  of a real variable belongs to  $\mathcal{K}_\gamma$  for a real constant  $\gamma$  if  $\phi^{(k)}(x) = O(|x|^{\gamma-k})$  as  $|x| \rightarrow \infty$ , for each  $k \in \mathbb{N}$ . A topology for  $\mathcal{K}_\gamma$  is generated by seminorms  $\|\phi\|_{k,\gamma} = \sup_{x \in \mathbb{R}} \{ \max(1, |x|^{k-\gamma}) |\phi^{(k)}(x)| \}$ , and so  $\mathcal{K}_\gamma \hookrightarrow \mathcal{K}_{\gamma'}$  if  $\gamma \leq \gamma'$ . Notice that  $\phi^{(k)} \in \mathcal{K}_{\gamma-k}$  if  $\phi \in \mathcal{K}_\gamma$ . The space  $\mathcal{K}$  is the inductive limit of the spaces  $\mathcal{K}_\gamma$  as  $\gamma \rightarrow \infty$ .

Since every polynomial is in  $\mathcal{K}$ , a distribution  $f \in \mathcal{K}'$  has *moments*

$$\mu_n := \langle f(x), x^n \rangle, \quad n \in \mathbb{N}$$

of all orders; this is an indication that  $f$  decays rapidly at infinity in some sense.

Denote by  $\mathcal{D}'_0(\mathbb{T})$  the space of periodic distributions with zero mean. They constitute a first class of examples: if  $f \in \mathcal{D}'_0(\mathbb{T})$ , then, for  $n$  suitably large, the periodic primitive with zero mean  $f_n$  of  $f$  of order  $n$  is continuous and defines the evaluation of  $f$  at  $\phi \in \mathcal{K}$  by a convergent integral:

$$\langle f(x), \phi(x) \rangle = (-1)^n \langle f_n(x), \phi^{(n)}(x) \rangle.$$

Note that in this case all the moments are zero.

The algebra  $\mathcal{K}$  is normal (i.e.,  $\mathcal{S}$  is dense in  $\mathcal{K}$ ) and is a subalgebra of the multiplier algebras  $\mathcal{O}_M, \mathcal{M}$  of  $\mathcal{S}$ , respectively for the ordinary product and the Moyal star product [16]. Other properties of  $\mathcal{K}$  and  $\mathcal{K}'$  will be invoked opportunely. The usefulness of  $\mathcal{K}$  in phase-space quantum mechanics lies in the similitude of behaviour of the ordinary and the Moyal product, when applied to elements of  $\mathcal{K}$ . The link between both appearances of  $\mathcal{K}$  is still mysterious to us.

► The natural method of studying generalized functions at infinity is by considering the parametric behaviour. The *moment asymptotic expansion* of a distribution [15] is given by

$$f(\lambda x) \sim \sum_{k=0}^{\infty} \frac{(-1)^k \mu_k \delta^{(k)}(x)}{k! \lambda^{k+1}} \quad \text{as } \lambda \rightarrow \infty. \quad (2.1)$$

The interpretation of this formula is in the distributional sense, to wit:

$$\langle f(\lambda x), \phi(x) \rangle = \sum_{k=0}^N \frac{\mu_k \phi^{(k)}(0)}{k! \lambda^{k+1}} + O\left(\frac{1}{\lambda^{N+2}}\right) \quad \text{as } \lambda \rightarrow \infty,$$

for each  $\phi$  in an appropriate space of test functions. Such an expansion holds only for distributions that decay rapidly at infinity, in a sense soon to be made completely precise; it certainly does not hold for all tempered distributions, as their moments generally do not exist. Distributions endowed with moment asymptotic expansions are said to be “distributionally small at infinity”. We are not happy with this terminology and invite suggestions to improve it.

On the other hand, the classical analysis [23] notion of Cesàro or Riesz means of series and integrals admits a generalization to the theory of distributions, that we intend to exploit in this paper. It turns out that Cesàro limits and “distributional” ones are essentially equivalent; this will enable us to apply the simpler ideas of parametric analysis to complicated averaging schemes.

► We now begin in earnest by introducing the basic concept of Cesàro behaviour of the distributions; justification will follow shortly. Assume  $f \in \mathcal{D}'(\mathbb{R})$ ,  $\beta \in \mathbb{R} \setminus \{-1, -2, \dots\}$ .

**Definition 2.1.** We say that  $f$  is of order  $x^\beta$  at infinity, in the Cesàro sense, and write

$$f(x) = O(x^\beta) \quad (C) \quad \text{as } x \rightarrow \infty,$$

if there exists  $N \in \mathbb{N}$ , a primitive  $f_N$  of  $f$  of order  $N$ , and a polynomial  $p$  of degree at most  $N - 1$ , such that  $f_N$  is locally integrable for  $x$  large and the relation

$$f_N(x) = p(x) + O(x^{N+\beta}) \quad \text{as } x \rightarrow \infty \quad (2.2)$$

holds in the ordinary sense.

The relation  $f(x) = o(x^\beta) \quad (C)$  is defined similarly. The notation  $(C, N)$  can be used if one needs to be more specific; if an order relation holds  $(C, N)$  for some  $N$ , it also holds  $(C, M)$  for all  $M > N$ . The assumption  $\beta \neq -1, -2, \dots$  is provisionally made in order to avoid dealing with the primitives of  $x^{-1}$ ,  $x^{-2}$  and such (see Section 6 for the general case). If  $\beta > -1$ , the polynomial  $p$  is arbitrary and thus irrelevant. We shall suppose when needed that our distributions have bounded support, say, on the left. In that case, we denote by  $I[f]$  the first order primitive of  $f$  with support bounded on the left. When  $f$  is locally integrable, then,

$$I[f](x) = \int_{-\infty}^x f(t) dt.$$

The notation

$$f(x) = o(x^{-\infty}) \quad (C) \quad \text{as } x \rightarrow \infty$$

will mean  $f(x) = O(x^\beta) \quad (C)$  for every  $\beta$ .

For the proof of the following workhorse proposition we refer to [12].

**Lemma 2.1.** (a) Let  $f \in \mathcal{D}'$  such that  $f(x) = O(x^\beta) (C, N)$  as  $x \rightarrow \infty$ . Then for each  $k = 1, 2, 3, \dots$  there holds:

$$f^{(k)}(x) = O(x^{\beta-k}) (C, N+k) \text{ as } x \rightarrow \infty.$$

(b) Let  $f \in \mathcal{D}'$  such that  $f(x) = O(x^\beta) (C)$  as  $x \rightarrow \infty$ , and let  $\alpha \in \mathbb{R}$ . Provided that  $\alpha + \beta$  is not a negative integer, there holds

$$x^\alpha f(x) = O(x^{\alpha+\beta}) (C) \text{ as } x \rightarrow \infty. \quad \square$$

**Definition 2.2.** We write  $\lim_{x \rightarrow \infty} f(x) = L (C)$  when  $f(x) = L + o(1) (C)$  as  $x \rightarrow \infty$ . That is,  $\lim_{x \rightarrow \infty} f(x) = L (C, k)$  when  $f_k(x) k! / x^k = L + o(1)$ , for  $f_k$  a primitive of order  $k$  of  $f$ .

For example, if  $f$  is periodic with zero mean value, there exists  $n \in \mathbb{N}$  and a continuous (thus bounded) periodic function  $f_n$  with zero mean value such that  $f_n^{(n)} = f$ ; then clearly

$$f(x) = o(x^{-\infty}) (C) \text{ as } x \rightarrow \infty,$$

a fact that yields, for  $f$  periodic with mean value  $a_0$ :

$$\lim_{x \rightarrow \infty} f(x) = a_0 (C).$$

Let  $f \in \mathcal{D}'$  be a distribution with support bounded on the left and let  $\phi$  be a smooth function. The following is a key concept of the theory.

**Definition 2.3.** We say that the  $\langle f(x), \phi(x) \rangle$  has the value  $L$  in the Cesàro sense, and write

$$\langle f(x), \phi(x) \rangle = L (C),$$

if there is a primitive  $I[g]$  for the distribution  $g(x) = f(x)\phi(x)$ , satisfying

$$\lim_{x \rightarrow \infty} I[g](x) = L (C) \text{ as } x \rightarrow \infty.$$

A similar definition applies when  $f$  has support bounded on the right. If  $f$  is an arbitrary distribution, let  $f = f_1 + f_2$  be a decomposition of  $f$ , where  $f_1$  has support bounded on the left and  $f_2$  has support bounded on the right. Then we say that  $\langle f(x), \phi(x) \rangle = L (C)$  if both  $\langle f_i(x), \phi(x) \rangle = L_i (C)$  exist for  $i = 1, 2$  and  $L = L_1 + L_2$ : this definition is seen to be independent of the decomposition.

For instance, let  $f$  be a periodic distribution of zero mean and let  $f_1, f_2, \dots, f_{n+1}$  denote the periodic primitives with zero mean of  $f$ , up to the order  $n + 1$ . Then

$$x^n f_1(x) - n x^{n-1} f_2(x) + n(n-1) x^{n-2} f_3(x) - \dots + (-1)^n n! f_{n+1}(x)$$

is a first order primitive of  $x^n f(x)$ , and since  $f_i(x) = o(x^{-\infty}) (C)$  for  $i = 1, \dots, n$  as  $x \rightarrow \infty$ , it follows that

$$\langle f(x), x^k \rangle = 0 (C) \text{ for all } k \in \mathbb{N}.$$

► To perceive the point of our hitherto abstract definitions, it is worthwhile to recall here briefly the classical theory [23]. Let  $\{a_n\}_{n=1}^{\infty}$  be a sequence of real or complex numbers. Often it has no limit, but the sequence of averages  $H_n^{(1)} := (a_1 + \cdots + a_n)/n$  does. Then people write

$$\lim_{n \rightarrow \infty} a_n = L \quad (C, 1).$$

If  $H_n^{(1)}$  still does not have a limit, then one may apply the averaging procedure again and again, hoping that eventually a limit will be obtained. There are two main procedures to perform such higher order averages: the Hölder means and the Cesàro means. The *Hölder means* are single-mindedly constructed as

$$H_n^{(k)} := \frac{H_1^{(k-1)} + \cdots + H_n^{(k-1)}}{n}$$

and  $\lim_{n \rightarrow \infty} H_n^{(k)} = L$  is written

$$\lim_{n \rightarrow \infty} a_n = L \quad (H, k).$$

The properly named *Cesàro means* are defined as follows: let  $A_n^{(0)} := a_n$  and define recursively  $A_n^{(k)} = A_1^{(k-1)} + \cdots + A_n^{(k-1)}$ . If  $\lim_{n \rightarrow \infty} k! A_n^{(k)}/n^k = L$ , we write

$$\lim_{n \rightarrow \infty} a_n = L \quad (C, k),$$

so that the  $(C, 1)$  and the  $(H, 1)$  notions are identical. The Cesàro limits have nicer analytical properties. The good news, at any rate, is that both procedures are equivalent:

$$\lim_{n \rightarrow \infty} a_n = L \quad (C, k) \iff \lim_{n \rightarrow \infty} a_n = L \quad (H, k).$$

One uses the simpler notation  $\lim_{n \rightarrow \infty} a_n = L \quad (C)$  if  $\lim_{n \rightarrow \infty} a_n = L \quad (C, k)$  for some  $k \in \mathbb{N}$ .

A third averaging procedure is equivalent to Cesàro's, the so-called *Riesz typical means*. For real  $\mu$ , one writes

$$\lim_{n \rightarrow \infty} a_n = L \quad (R, k, n) \quad \text{if} \quad \lim_{\mu \rightarrow \infty} \frac{1}{\mu} \sum_{n \leq \mu} \left(1 - \frac{n}{\mu}\right)^{k-1} a_n = L.$$

Riesz originally studied this formula for integral  $\mu$ , but the means have more desirable properties with  $\mu$  real. Now, one may study the summability of a series  $\sum_{n=1}^{\infty} a_n$  by studying the generalized function of a real variable  $f(x) = \sum_{n=1}^{\infty} a_n \delta(x - n)$ . The definition of Cesàro limits of distributions is tailored in such a way that  $\langle f, 1 \rangle \quad (C)$  and  $\sum_{n=1}^{\infty} a_n \quad (C)$  coincide: a primitive of order  $k$  of  $\sum_{n=1}^{\infty} a_n \delta(x - n)$  is given by  $f_k(x) = \sum_{n \leq x} (x - n)^{k-1} a_n / (k - 1)!$  Note that one could consider distributions of the form  $h(x) = \sum_{n=1}^{\infty} a_n \delta(x - p_n)$ , with  $p_n \uparrow \infty$ ; this gives rise to the  $(R, k, p_n)$  means.

In summary, we have demonstrated the following equivalence.

**Theorem 2.2.** *The evaluation*

$$\left\langle \sum_{n=1}^{\infty} a_n \delta(x - n), \phi(x) \right\rangle = L \quad (C)$$

holds iff  $\sum_{n=1}^{\infty} a_n \phi(n) = L$  in the Cesàro sense of the theory of summability of series. □

In the same vein:

**Theorem 2.3.** *If  $f$  is locally integrable and supported in  $(a, \infty)$ , then*

$$\langle f(x), \phi(x) \rangle = L \quad (C)$$

if and only if

$$\int_a^\infty f(x)\phi(x) dx = L$$

in the Cesàro sense of the theory of summability of integrals.  $\square$

As shown below, if  $f \in \mathcal{K}'$  and  $\phi \in \mathcal{K}$ , then the evaluation  $\langle f(x), \phi(x) \rangle$  is always (C)-summable. We pause an instant to show by the example just how useful is the concept of Cesàro computability of evaluations. An interesting periodic distribution is the Dirac comb  $\sum_{n=-\infty}^\infty \delta(x - n)$ . Its mean value is 1; therefore

$$\sum_{n=-\infty}^\infty \delta(x - n) = 1 + f(x), \quad \text{with } f \in \mathcal{D}'_0(\mathbb{T}). \quad (2.3)$$

The distributions

$$\sum_{n=1}^\infty \delta(x - n) - H(x - 1), \quad \sum_{n=1}^\infty \delta(x - n) - H(x),$$

where  $H$  is the Heaviside function, belong to  $\mathcal{K}'$ . In effect, take a function  $\phi_1 \in \mathcal{K}$  such that  $\phi_1(x) = 1$  for  $x > \frac{1}{2}$  and  $\phi_1(x) = 0$  for  $x < \frac{1}{4}$ . Then  $\phi_1(x)(\sum_{n=-\infty}^\infty \delta(x - n) - 1)$  only differs from  $\sum_{n=1}^\infty \delta(x - n) - H(x - 1)$  or  $\sum_{n=1}^\infty \delta(x - n) - H(x)$  by a distribution of compact support.

It follows that the evaluation

$$\left\langle \sum_{n=1}^\infty \delta(x - n) - H(x - 1), \phi(x) \right\rangle = \sum_{n=1}^\infty \phi(n) - \int_1^\infty \phi(x) dx$$

is Cesàro summable whenever  $\phi \in \mathcal{K}$ . Now,  $x^\alpha$  does not belong to  $\mathcal{K}$  unless  $\alpha \in \mathbb{N}$ , but the previous argument, using  $\phi_\alpha(x) = \phi_1(x) x^\alpha$ , allows us to conclude that the evaluation

$$Z(\alpha) := \left\langle \sum_{n=1}^\infty \delta(x - n) - H(x - 1), x^\alpha \right\rangle$$

is (C)-summable for any  $\alpha \in \mathbb{C}$ . Also,  $Z(\alpha)$  is an entire function of  $\alpha$ , since  $\phi_\alpha$  is. We find a formula for  $Z(\alpha)$  by observing that if  $\Re \alpha < -1$  then the evaluation is given by the difference of a series and an integral, so that

$$Z(\alpha) = \sum_{n=1}^\infty n^\alpha - \int_1^\infty x^\alpha = \zeta(-\alpha) + \frac{1}{\alpha + 1}, \quad \Re \alpha < -1.$$

We have learned a simple proof that Riemann's zeta function is analytic in  $\mathbb{C} \setminus \{1\}$ , with residue at  $s = 1$  equal to 1, and one realizes that the evaluation of the  $\zeta$  function can be done by Cesàro means (it is only because the zeta function is the outcome of a regularization process that it is useful

for renormalization in quantum field theory). The evaluation  $\langle \sum_{n=1}^{\infty} \delta(x-n) - H(x), x^\alpha \rangle$  is slightly more involved. However, we may write [14]:

$$\left\langle \sum_{n=1}^{\infty} \delta(x-n) - H(x), x^\alpha \right\rangle := Z(\alpha) - \text{F. p.} \int_0^1 x^\alpha dx,$$

where F. p. stands for the Hadamard finite part of the integral. Now,

$$\text{F. p.} \int_0^1 x^\alpha dx = \frac{1}{\alpha + 1}, \quad \alpha \neq -1,$$

therefore, if  $\alpha \neq -1$ ,

$$\zeta(-\alpha) = \sum_{n=1}^{\infty} n^\alpha - \text{F. p.} \int_0^{\infty} x^\alpha dx \quad (C),$$

in the sense that

$$\zeta(-\alpha) = \lim_{x \rightarrow \infty} \left( \sum_{n=1}^{\lfloor x \rfloor} n^\alpha - \text{F. p.} \int_0^x t^\alpha dt \right) \quad (C).$$

This formula gives a nice representation for  $\zeta(\alpha)$  when  $\Re \alpha < 1$ . For instance,  $\zeta(0) = -\frac{1}{2}$  simply because the fractional part  $\{x\} = x - \lfloor x \rfloor$  of  $x$  is periodic of mean  $\frac{1}{2}$ . For  $\alpha = -1$ :

$$\zeta(-1) = \lim_{x \rightarrow \infty} \left( \sum_{n=1}^{\lfloor x \rfloor} n - \int_0^x t dt \right) = \lim_{x \rightarrow \infty} \left( \frac{1}{2} \lfloor x \rfloor (\lfloor x \rfloor + 1) - \frac{1}{2} x^2 \right) \quad (C);$$

we find that

$$\frac{(x - \{x\})(x - \{x\} + 1)}{2} - \frac{x^2}{2} = \frac{\{x\}^2 - \{x\}}{2} + \frac{x(1 - 2\{x\})}{2} = -\frac{1}{12} + o(x^{-\infty}) \quad (C)$$

since  $(1 - 2\{x\})$  and  $(\{x\}^2 - \{x\} + 1/6)$  are periodic of mean zero; we get  $\zeta(-1) = -1/12$ .

Also, the logarithm of the “functional determinant” can be obtained by this method:

$$\zeta'(0) = - \lim_{x \rightarrow \infty} \left( \sum_{n=2}^{\lfloor x \rfloor} \log n - \int_0^x \log t dt \right) \quad (C),$$

on using Lemma 2.1. Stirling’s formula gives

$$\begin{aligned} x \log x - x - \log(\lfloor x \rfloor!) &= x \log x - x - (\lfloor x \rfloor + \frac{1}{2}) \log \lfloor x \rfloor + \lfloor x \rfloor - \log \sqrt{2\pi} + O(x^{-1}) \\ &= -x \log \left( 1 - \frac{\{x\}}{x} \right) - \{x\} + (\{x\} - \frac{1}{2}) \log \lfloor x \rfloor - \frac{1}{2} \log(2\pi) + O(x^{-1}) \\ &= -\frac{1}{2} \log(2\pi) + O(x^{-1}) \quad (C) \end{aligned}$$

since  $x \log(1 - x^{-1}\{x\}) + \{x\} = O(x^{-2})$  and  $(\{x\} - \frac{1}{2})$  is periodic of mean zero. From this it follows that  $\zeta'(0) = -\frac{1}{2} \log(2\pi)$ . This business of Riemann’s zeta function is not merely amusing; it will be useful later.

► We make ready for the main equivalence result.

**Theorem 2.4.** *Let  $f \in \mathcal{D}'$ . If  $\alpha > -1$  then*

$$f(x) = O(|x|^\alpha) \quad (C) \quad \text{as } x \rightarrow \pm\infty \quad (2.4)$$

*if and only if*

$$f(\lambda x) = O(\lambda^\alpha) \quad \text{as } \lambda \rightarrow \infty \quad (2.5)$$

*in the topology of  $\mathcal{D}'$ . If  $-j-1 > \alpha > -j-2$  for some  $j \in \mathbb{N}$ , then (2.4) holds if and only if there are constants  $\mu_0, \dots, \mu_j$  such that*

$$f(\lambda x) = \sum_{k=0}^j \frac{(-1)^k \mu_k \delta^{(k)}(x)}{k! \lambda^{k+1}} + O(\lambda^\alpha)$$

*in the topology of  $\mathcal{D}'$  as  $\lambda \rightarrow \infty$ .*

*Proof.* We prove the theorem in the case  $f$  has support bounded on the left. The general case follows by using a decomposition  $f = f_1 + f_2$ , where  $f_1$  has support bounded on the left and  $f_2$  has support bounded on the right. First we have to clarify the meaning of (2.5). It is a weak or distributional relation: we write  $f(x, \lambda) = O(\lambda^\alpha)$  as  $\lambda \rightarrow \infty$  whenever

$$\langle f(x, \lambda), \phi(x) \rangle = O(\lambda^\alpha) \quad \text{as } \lambda \rightarrow \infty,$$

for all  $\phi \in \mathcal{D}$ . Note that this yields

$$\left\langle \frac{\partial f(x, \lambda)}{\partial x}, \phi(x) \right\rangle = -\langle f(x, \lambda), \phi'(x) \rangle = O(\lambda^\alpha).$$

Now, if (2.5) holds, there exists  $N$  such that the primitive of order  $N$  of  $f(\lambda x)$ , with respect to  $x$ , exists and is bounded by  $M\lambda^\alpha$ , say for  $|x| \leq 1$  and  $\lambda \geq \lambda_0$ . There is then a primitive  $f_N$  of order  $N$  of  $f(x)$ , such that

$$|f_N(\lambda x)| \leq M\lambda^{\alpha+N}, \quad |x| \leq 1, \quad \lambda \geq \lambda_0.$$

Taking  $x = 1$  and replacing  $\lambda$  by  $x$  we obtain

$$|f_N(x)| \leq Mx^{\alpha+N}, \quad x \geq \lambda_0,$$

and thus

$$f(x) = O(x^\alpha) \quad (C, N), \quad \text{as } x \rightarrow \infty.$$

Reciprocally, assume  $\alpha > -1$  and  $f(x) = O(x^\alpha) \quad (C, N)$ , as  $x \rightarrow \infty$ . Then, if  $f_N$  is the (locally integrable for  $x$  large) primitive of order  $N$  of  $f$  with support bounded on the left, an obvious estimate gives  $f_N(\lambda x) = O(\lambda^{\alpha+N})$ , as  $\lambda \rightarrow \infty$ , and on differentiating  $N$  times with respect to  $x$  one obtains  $\lambda^N f(\lambda x) = O(\lambda^{\alpha+N})$ , so that (2.5) follows.

The case when  $\alpha$  is nonintegral and less than  $-1$  is more involved, as one has to deal with the polynomial  $p$  in (2.2). Then one shows that the moments

$$\langle f(x), x^k \rangle = \mu_k \quad (C)$$

up to a certain order exist, those being essentially the coefficients of  $p$ . For the gory details, we refer once again to [12].  $\square$

A characterization of the distributions that have a moment asymptotic expansion follows.

**Theorem 2.5.** *Let  $f \in \mathcal{D}'$ . Then the following are equivalent:*

(a)  $f \in \mathcal{K}'$ .

(b)  $f$  satisfies

$$f(x) = o(|x|^{-\infty}) \quad (C) \quad \text{as } x \rightarrow \pm\infty.$$

(c) *There exist constants  $\mu_0, \mu_1, \mu_2, \dots$  such that*

$$f(\lambda x) \sim \frac{\mu_0 \delta(x)}{\lambda} - \frac{\mu_1 \delta'(x)}{\lambda^2} + \frac{\mu_2 \delta''(x)}{2! \lambda^3} - \dots \quad \text{as } \lambda \rightarrow \infty$$

*in the weak sense.*

*Proof.* It is proved in [15] that the elements of  $\mathcal{K}'$  satisfy the moment asymptotic expansion. For the converse, it is enough, as customary, to consider distributions with support bounded on one side. We show that if (b) holds, then  $f \in \mathcal{K}'_\gamma$  for all  $\gamma$ . From the hypothesis it follows that  $f(x) = O(x^{-\gamma-2})$  (C) as  $x \rightarrow \infty$ . Thus, for a certain  $n$ , the  $n$ -th order primitive  $f_n$  of  $f$  with support bounded on one side is locally integrable and satisfies  $f_n(x) = p(x) + O(x^{-\gamma-2+n})$  as  $x \rightarrow \infty$ , where the polynomial  $p$  has degree at most  $n-1$ . We conjure up a compactly supported continuous function  $g$  whose moments of order up to  $n-1$  coincide with those of  $f$ . If  $g_n$  is the primitive of order  $n$  of  $g$  with support bounded on the left, then  $f_n(x) - g_n(x) = O(x^{-\gamma-2+n})$ . If  $\phi \in \mathcal{K}_{\gamma-n}$ , the integral  $\int_{-\infty}^{\infty} (f_n(x) - g_n(x))\phi(x) dx$  converges. Hence  $f = (f_n - g_n)^{(n)} + g \in \mathcal{K}'_\gamma$ . The rest is clear.  $\square$

We get at once a powerful computational method for duality evaluations.

**Corollary 2.6.** *If  $f \in \mathcal{K}'$  and  $\phi \in \mathcal{K}$ , the evaluation  $\langle f(x), \phi(x) \rangle$  is Cesàro summable.*

*Proof.* It is enough to check for  $\phi = 1$ . But, according to the previous Theorem, if  $f \in \mathcal{K}'$  then  $f(x) = o(x^{-\infty})$  (C) as  $x \rightarrow \infty$ . By the proof of Theorem 2.4,  $\langle f(x), 1 \rangle$  is (C)-summable.  $\square$

Fourier transforms are defined by duality and, in general, if  $f \in \mathcal{S}'$ , we cannot make sense of  $\hat{f}(u)$  because the evaluation  $\langle e^{ixu}, f(x) \rangle$  is not defined. However, if  $\phi \in \mathcal{K}$  and  $u \neq 0$ , Corollary 2.6 guarantees that the Cesàro-sense evaluation  $\langle e^{ixu}, \phi(x) \rangle$  (C) is well defined. Thus

$$\hat{\phi}(u) = \langle e^{ixu}, \phi(x) \rangle \quad (C) \quad \text{when } \phi \in \mathcal{K}, u \neq 0.$$

It is clear that  $\widehat{\mathcal{K}} \subset \mathcal{K}'$ ; this follows also from Proposition 4 of [20].

Note also that the moments of  $f \in \mathcal{K}'$  are (C)-summable. The converse is true:

**Theorem 2.7.** *Let  $f \in \mathcal{D}'$ . If all the moments  $\langle f(x), x^n \rangle = \mu_n$  (C) exist for  $n \in \mathbb{N}$ , then  $f \in \mathcal{K}'$ .*

*Proof.* For the easy proof, we refer to [12].  $\square$

It is clearly worthwhile to characterize spaces of distributions in terms of their Cesàro behaviour. Particularly important is the characterization of tempered distributions.

**Theorem 2.8.** *Let  $f \in \mathcal{D}'$ . Then the following statements are equivalent:*

- (a)  $f$  is a tempered distribution.  
 (b) There exists  $\alpha \in \mathbb{R}$  such that

$$f(\lambda x) = O(\lambda^\alpha), \quad \text{as } \lambda \rightarrow \infty$$

in the weak sense.

- (c) There exists  $\alpha \in \mathbb{R}$  and  $k \in \mathbb{N}$  such that

$$f^{(k)}(x) = O(|x|^{\alpha-k}) \quad (C) \quad \text{as } x \rightarrow \infty.$$

*Proof.* Again, it is enough to consider the case when  $f$  has support bounded on one side. It is well known that if  $f \in \mathcal{S}'$  then there is a primitive  $F$  of some order  $N$  of slow growth at infinity; it follows that  $f(x) = O(|x|^\alpha) \quad (C)$ . The rest is clear, in view of the equivalence theorem 2.4 and the fact that distributional order relations can be differentiated at will.  $\square$

We finish by giving several estimates that we shall need later. The first is just a rewording of the properties of the distribution (2.3).

**Lemma 2.9.** *If  $g \in \mathcal{K}$  and if  $\int_{-\infty}^{\infty} g(x) dx$  is defined, then*

$$\sum_{n=-\infty}^{\infty} g(n\varepsilon) = \frac{1}{\varepsilon} \int_{-\infty}^{\infty} g(x) dx + o(\varepsilon^\infty) \quad \text{as } \varepsilon \downarrow 0. \quad \square$$

By the same token:

**Lemma 2.10.** *If  $g \in \mathcal{K}(\mathbb{R}^n)$  and if  $\int_{\mathbb{R}^n} g(x) dx$  is defined, then*

$$\sum_{k \in \mathbb{Z}^n} g(k\varepsilon) = \varepsilon^{-n} \int_{\mathbb{R}^n} g(x) dx + o(\varepsilon^\infty) \quad \text{as } \varepsilon \downarrow 0. \quad \square$$

**Lemma 2.11.** *If  $g \in \mathcal{K}$  and if  $\int_0^\infty g(x) dx$  is defined, then*

$$\sum_{n=1}^{\infty} g(n\varepsilon) = \frac{1}{\varepsilon} \int_0^\infty g(x) dx + \sum_{n=0}^{\infty} \frac{\zeta(-n)g^{(n)}(0)}{n!} \varepsilon^n + o(\varepsilon^\infty) \quad \text{as } \varepsilon \downarrow 0.$$

*Proof.* This follows from the zeta function example.  $\square$

(Results of this type were used to prove some formulas by Ramanujan in [15].)

### 3 Spectral densities

Let  $\mathcal{H}$  be a concrete Hilbert space, the space of square integrable sections of an Euclidean vector bundle over a Riemannian manifold  $M$ , and let  $H$  be an elliptic positive selfadjoint pseudodifferential operator on  $\mathcal{H}$ , with domain  $\mathcal{X}$ . We consider the derivative, in the distributional sense, of the spectral family of projectors  $E_H(\lambda)$  associated to  $H$ :

$$d_H(\lambda) := \frac{dE_H(\lambda)}{d\lambda}.$$

For instance, if  $H$  is defined on a compact manifold, and  $0 < \lambda_1 \leq \lambda_2 \leq \dots$  is the complete set of its eigenvalues, with orthonormal basis of eigenfunctions  $u_j$ , the kernel of the spectral family is given by [25]:

$$E_H(\lambda) := \sum_{\lambda_j \leq \lambda} |u_j\rangle\langle u_j|,$$

and the derivative is

$$d_H(\lambda) := \sum_j |u_j\rangle\langle u_j| \delta(\lambda - \lambda_j).$$

This *spectral density* is a distribution with values in  $\mathcal{L}(\mathcal{X}, \mathcal{H})$ . The defining properties of  $E(\lambda)$ :

$$I = \int_{-\infty}^{\infty} dE(\lambda), \quad H = \int_{-\infty}^{\infty} \lambda dE(\lambda)$$

(in the weak sense) become, in the language of the previous section:

$$I = \langle d_H(\lambda), 1 \rangle, \quad H = \langle d_H(\lambda), \lambda \rangle.$$

The spectral density is used to construct the functional calculus for  $H$ . Indeed, we can define  $\phi(H)$  whenever  $f$  is a distribution such that the evaluation  $\langle d_H(\lambda), f(\lambda) \rangle$  makes sense, by

$$\phi(H) := \langle d_H(\lambda), \phi(\lambda) \rangle,$$

with domain the subspace of the  $x \in \mathcal{H}$  for which the evaluation  $\langle (y | d_H(\lambda)x), \phi(\lambda) \rangle_\lambda$  is defined for all  $y \in \mathcal{H}$ .

Especially, one is able to deal with the “zeta operator”:

$$H^{-s} := \langle d_H(\lambda), \lambda^{-s} \rangle, \tag{3.1}$$

(for  $0 \notin \text{sp } H$ ); the heat operator:

$$e^{-tH} := \langle d_H(\lambda), e^{-t\lambda} \rangle, \quad t > 0; \tag{3.2}$$

and the unitary group of  $H$ , which is just the Fourier transform of the spectral density:

$$U_H(t) := \langle d_H(\lambda), e^{-it\lambda} \rangle. \tag{3.3}$$

The useful symbolic formula

$$d_H(\lambda) = \delta(\lambda - H)$$

recommends itself, and we shall employ it from now on.

We want to study the asymptotic behaviour of  $\delta(\lambda - H)$ . Let  $\mathcal{X}_n$  be the domain of  $H^n$  and let  $\mathcal{X}_\infty := \bigcap_{n=1}^\infty \mathcal{X}_n$ . The density of  $\mathcal{X}_\infty$  has, in view of the theory of Section 2, momentous consequences. The evaluation

$$H^n = \langle \delta(\lambda - H), \lambda^n \rangle$$

holds in the space  $\mathcal{L}(\mathcal{X}_\infty, \mathcal{H})$ . Hence,  $\delta(\lambda - H)$  belongs to the space  $\mathcal{K}'(\mathbb{R}, \mathcal{L}(\mathcal{X}_\infty, \mathcal{H}))$ . Therefore the moment asymptotic expansion holds:

$$\delta(\lambda\sigma - H) \sim \sum_{n=0}^{\infty} \frac{(-1)^n H^n \delta^{(n)}(\lambda)}{n! \sigma^{n+1}} \quad \text{as } \sigma \rightarrow \infty,$$

and  $\delta(\lambda - H)$  vanishes to infinite order at infinity in the Cesàro sense:

$$\delta(\lambda - H) = o(|\lambda|^{-\infty}) \quad (C) \quad \text{as } |\lambda| \rightarrow \infty.$$

Of course, the last formula is trivial when  $H$  is bounded.

The space  $\mathcal{D}(M)$  of test functions is a subspace of  $\mathcal{X}_\infty$ . We can then realize the spectral density by an associated kernel  $d_H(x, y; \lambda)$ , an element of  $\mathcal{D}'(\mathbb{R}, \mathcal{D}'(M \times M))$ . Ellipticity actually implies that  $d_H(x, y; \lambda)$  is smooth in  $(x, y)$ . The expansion

$$d_H(x, y; \lambda\sigma) \sim \sum_{n=0}^{\infty} \frac{(-1)^n (H^n \delta)(x - y) \delta^{(n)}(\lambda\sigma)}{n! \sigma^{n+1}} \quad \text{as } \sigma \rightarrow \infty$$

holds in principle in the space  $\mathcal{D}'(\mathbb{R}, \mathcal{D}'(M \times M))$ . We also get

$$d_H(x, y; \lambda) = o(|\lambda|^{-\infty}) \quad (C) \quad \text{as } |\lambda| \rightarrow \infty \tag{3.4}$$

in the space  $\mathcal{D}'(M \times M)$ . Eq. (3.4) is the mother of all incoherence principles. For instance, passing to the primitive with respect to  $\lambda$ , for an elliptic operator on a compact manifold with eigenfunctions  $\psi_n$ ,  $n \in \mathbb{N}$ , one concludes that

$$\sum_{\lambda_n \leq \lambda} \bar{\psi}_n(x) \psi_n(y) = o(|\lambda|^{-\infty}) \quad (C) \quad \text{as } |\lambda| \rightarrow \infty,$$

for  $x \neq y$ , which is Carleman's incoherence relation [5].

It should be clear that the expansions cannot hold pointwise in both variables  $x$  and  $y$ , since we cannot set  $x = y$  in the distribution  $\delta(x - y)$ . In fact, our interest in this paper lies in the coincidence limit  $d_H(x, x; \lambda)$ , which is *not* distributionally small. However, it is proved in [13] that, away from the diagonal of  $M \times M$ , the expansions are valid in the sense of uniform convergence of all derivatives on compacta. On the other hand, if  $H_1$  and  $H_2$  are two pseudodifferential operators whose difference over an open subset  $U$  of  $M$  is a smoothing operator, and if  $d_1(x, y; \lambda)$  and  $d_2(x, y; \lambda)$  are the corresponding spectral densities, then [13]:

$$d_1(x, y; \sigma\lambda) = d_2(x, y; \sigma\lambda) + o(\sigma^{-\infty}) \quad \text{as } \sigma \rightarrow \infty$$

in  $\mathcal{D}'(U \times U)$ . Also, it can be shown that

$$d_1(x, y; \lambda) = d_2(x, y; \lambda) + o(\lambda^{-\infty}) \quad (C) \quad \text{as } \lambda \rightarrow \infty$$

uniformly on compacta of  $U \times U$ , even at the diagonal.

We exemplify the reported behaviour with the simplest possible examples. Let  $H$  denote first the Laplacian on the real line. Its spectral density is

$$d_H(x, y; \lambda) = \frac{1}{2\pi\sqrt{\lambda}} \cos(\sqrt{\lambda}(x - y)),$$

and therefore it is clear that  $d_H(x, x; \lambda)$  is not distributionally small, but rather

$$d_H(x, x; \lambda) = \frac{1}{2\pi\sqrt{\lambda}} + o(\lambda^{-\infty}) \quad (C) \quad \text{as } \lambda \rightarrow \infty.$$

Let  $H$  denote now the Laplacian on the circle; the eigenvalues are  $\lambda_n = n^2$ ,  $n = 0, 1, 2, \dots$ , with multiplicity 2 from  $n = 1$  on, with normalized eigenfunctions  $\psi_n^\pm(x) = (2\pi)^{-1/2} e^{\pm inx}$ . Therefore

$$d_H(x, y; \lambda) = \frac{1}{2\pi} \left( \delta(\lambda) + 2 \sum_{n=1}^{\infty} \cos n(x - y) \delta(\lambda - n^2) \right).$$

Then

$$\frac{1}{2\pi} \left( \delta(\lambda\sigma) + 2 \sum_{n=1}^{\infty} \cos n(x - y) \delta(\lambda\sigma - n^2) \right) \sim \sum_{j=0}^{\infty} \frac{\delta^{(2j)}(x - y) \delta^{(j)}(\lambda)}{j! \sigma^{j+1}} \quad \text{as } \sigma \rightarrow \infty$$

in  $\mathcal{D}'(\mathbb{R}, \mathcal{D}'(\mathbb{S}^1 \times \mathbb{S}^1))$ , while

$$\frac{1}{2\pi} \left( \delta(\lambda) + 2 \sum_{n=1}^{\infty} \cos n(x - y) \delta(\lambda - n^2) \right) = o(\lambda^{-\infty}) \quad (C) \quad \text{as } \lambda \rightarrow \infty$$

if  $x$  and  $y$  are fixed,  $x \neq y$ .

On the other hand,

$$d_H(x, x; \lambda) = \frac{1}{2\pi} \left( \delta(\lambda) + 2 \sum_{n=1}^{\infty} \delta(\lambda - n^2) \right)$$

does not belong to  $\mathcal{K}'(\mathbb{R}, C^\infty(\mathbb{S}^1))$ . For the first time in this paper, but not the last, we must find out what is the Cesàro behaviour of a given spectral kernel. We shall have recourse to a variety of tricks. For now, applying Lemma 2.11 to  $g(x) := \phi(x^2)$ , for  $\phi$  a Schwartz function, say, we get:

$$\sum_{n=1}^{\infty} \phi(\varepsilon n^2) = \frac{1}{2\sqrt{\varepsilon}} \int_0^{\infty} x^{-1/2} \phi(x) dx - \frac{1}{2} \phi(0) + o(\varepsilon^\infty) \quad \text{as } \varepsilon \downarrow 0.$$

It is then clear that

$$d_H(x, x; \lambda) = \frac{1}{2\pi\sqrt{\lambda}} + o(\lambda^{-\infty}) \quad (C) \quad \text{as } \lambda \rightarrow \infty.$$

and it is also immediately clear that the distributional and Cesàro behaviour of the spectral density and its kernel are *exactly the same* as in the previous example. That the manifold be compact or not and the spectrum be discrete or continuous is immaterial for that purpose. If we seek a boundary

problem for the Laplacian, say on a bounded interval of the line, we still obtain the same kind of behaviour (off the boundary, where a sharp change takes place). Note also the estimate:

$$\sum_{\pm; \lambda_n \leq \lambda} |\psi_n^\pm(x)|^2 \sim \frac{\sqrt{\lambda}}{\pi} (C) \quad \text{as } \lambda \rightarrow \infty.$$

As an aside, we turn before closing this section to the functional calculus formulas and compare (3.2) with (3.3). Obviously  $e^{-t(\cdot)}$  has an extension belonging to  $\mathcal{K}$ , so there is no difficulty in giving a meaning to the heat operator. Also, as we shall see in Section 6, it is comparatively easy to study the asymptotic development of the corresponding Green function as  $t \downarrow 0$ . One of the motivations of the present approach to spectral asymptotics is to define a sense for expansions of Schrödinger propagators and the like, that do not possess a “true” asymptotic expansion.

Such an approach can be based on the following idea. Theorem 2.8 points to a rough duality between  $\mathcal{K}'$  and  $\mathcal{S}'$ . Let  $g \in \mathcal{S}'(\mathbb{R})$  and find  $\alpha$  so that  $g(\lambda x) = O(\lambda^\alpha)$  weakly as  $\lambda \rightarrow \infty$ . For any  $\phi \in \mathcal{S}(\mathbb{R})$ , the function  $\Phi$  defined by

$$\Phi(x) := \langle g(\lambda x), \phi(\lambda) \rangle_\lambda$$

is smooth for  $x \neq 0$  since  $\Phi(x) = |x|^{-1} \langle g(\lambda), \phi(\lambda x^{-1}) \rangle_\lambda$ , and satisfies

$$\Phi^{(n)}(x) = O(|x|^{\alpha-n}) \quad \text{as } |x| \rightarrow \infty.$$

Therefore, if  $f \in \mathcal{K}'$  with  $0 \notin \text{supp } f$ , we can define  $\langle f(x), g(\lambda x) \rangle_x$  as a tempered distribution.

When  $0 \in \text{supp } f$ , we need to independently ascertain smoothness of  $\Phi$  at the origin. It turns out that, for this purpose, it is enough to demand distributional smoothness of  $g$ , i.e., the existence of the *distributional* values  $g^{(n)}(0)$ , in the sense of [31], for  $n = 0, 1, 2, \dots$ . Then  $g(tH)$  admits a *distributional* expansion in  $\mathcal{L}(\mathcal{X}_\infty, \mathcal{H})$  as  $t \downarrow 0$ . This can eventually lead to a proper treatment of some questions in quantum field theory. We say no more here and refer instead to the forthcoming [13]. In Section 6 of this paper, results will be stated for  $g$  belonging to  $\mathcal{S}(\mathbb{R})$ ; for the rest of the paper we shall venture outside safe territory only in examples.

## 4 The Cesàro asymptotic development of $d_H(x, x; \lambda)$

In this section we obtain the asymptotic expansion for the coincidence limits of spectral density kernels. We are fortified with the results of the previous section, implying that the Cesàro behaviour of the spectral density of pseudodifferential operators is a local matter.

Let  $A$  be any pseudodifferential operator of order a positive integer  $d$ , with complete symbol  $\sigma(A)$ , on the Riemannian manifold  $M$ . To simplify the discussion, we consider only operators acting on scalars; the treatment of matrix-valued symbols presents no further difficulty. The noncommutative or Wodzicki *residue* of  $A$  is defined by integrating (the trace of) the partial symbol  $\sigma_{-n}(A)(x, \xi)$  of order  $-n$  over the cosphere bundle  $\{(x, \xi) : |\xi| = 1\}$ :

$$\text{Wres } A := \int_M \int_{\mathbb{S}^{n-1}} \sigma_{-n}(A)(x, \xi) d\xi dx.$$

Here  $dx$  denotes the canonical volume element on  $M$ . If  $M$  is not compact,  $\text{Wres } A$  may not exist, but there always exists the local density of the residue  $\int_{\mathbb{S}^{n-1}} \sigma_{-n}(A)(x, \omega) d\omega$ , that we denote by  $\text{wres } A(x)$ .

We recall that

$$\sigma(AB) - \sigma(A)\sigma(B) \sim \sum_{|\alpha|>0} \frac{(-i)^{|\alpha|}}{\alpha!} \partial_\xi^\alpha \sigma(A) \partial_x^\alpha \sigma(B).$$

The kernel  $k_A$  of  $A$  is by definition

$$k_A(x, y) := (2\pi)^{-n} \langle e^{i(x-y)\cdot\xi}, \sigma(A)(x, \xi) \rangle_\xi.$$

In particular, on the diagonal:

$$k_A(x, x) := (2\pi)^{-n} \langle 1, \sigma(A)(x, \xi) \rangle_\xi. \quad (4.1)$$

In order to figure out the symbol for a spectral density, we start by considering (the selfadjoint extension of) an elliptic operator  $H$  with constant coefficients. In this case  $\sigma(H^n) = \sigma(H)^n$  and we assert that

$$\sigma(\delta(\lambda - H)) = \delta(\lambda - \sigma(H)),$$

justified by the identities:

$$\int \lambda^n \delta(\lambda - \sigma(H)) d\lambda = \sigma(H^n), \quad \lambda = 0, 1, 2, \dots$$

In the general case of nonconstant coefficients, we make the Ansatz that:

$$\sigma(\delta(\lambda - H)) \sim \delta(\lambda - \sigma(H)) - q_1 \delta'(\lambda - \sigma(H)) + q_2 \delta''(\lambda - \sigma(H)) - q_3 \delta'''(\lambda - \sigma(H)) + \dots \quad (4.2)$$

in the Cesàro sense. Computation of  $\int \lambda^n \sigma(\delta(\lambda - H)) d\lambda$  for  $\lambda = 0, 1, 2, \dots$  then gives

$$q_1 = 0; \quad q_2 = \frac{1}{2}(\sigma(H^2) - \sigma(H)^2); \quad q_3 = \frac{1}{6}(\sigma(H^3) - 3\sigma(H^2)\sigma(H) + 2\sigma(H)^3), \quad (4.3)$$

and so on. This development, it turns out, gives ever lower powers of  $\lambda$  in the asymptotic expansion of  $\sigma(\delta(\lambda - H))$ .

We are interested in explicit formulas for the Cesàro asymptotic development of the coincidence limit for the kernel of a positive operator  $H$  as  $\lambda \rightarrow \infty$ . From (4.1) and (4.2) with  $p := \sigma(H)$ , we get

$$d_H(x, x; \lambda) \sim (2\pi)^{-n} \langle 1, \delta(\lambda - p(x, \xi)) + q_2(x, \xi) \delta''(\lambda - p(x, \xi)) - \dots \rangle_\xi \quad (C).$$

In polar coordinates on the cotangent fibres,  $\xi = |\xi|\omega$  with  $|\omega| = 1$ , this becomes

$$(2\pi)^{-n} \int_{|\omega|=1} d\omega \langle |\xi|^{n-1}, \delta(\lambda - p(x, |\xi|\omega)) + q_2(x, |\xi|\omega) \delta''(\lambda - p(x, |\xi|\omega)) - \dots \rangle_{|\xi|}.$$

Hence, if we denote by  $|\xi|(x, \omega; \lambda)$  the positive solution of the equation  $p(x, |\xi|\omega) = \lambda$ , we need to compute:

$$(2\pi)^{-n} \int_{\mathbb{S}^{n-1}} d\omega \frac{|\xi|^{n-1}(x, \omega; \lambda) + \frac{\partial^2}{\partial \lambda^2} (q_2(x, |\xi|(x, \omega; \lambda)\omega) |\xi|^{n-1}(x, \omega; \lambda)) - \dots}{p'(x, |\xi|(x, \omega; \lambda)\omega)}. \quad (4.4)$$

Write:

$$p(x, |\xi|\omega) \sim p_d(x, \omega) |\xi|^d + p_{d-1}(x, \omega) |\xi|^{d-1} + p_{d-2}(x, \omega) |\xi|^{d-2} \dots$$

To solve  $p(x, |\xi|\omega) = \lambda$  amounts to a series reversion.

► In order to see how that is done, let us assume for a short while that  $H$  is a first-order operator with constant coefficients – for instance, the absolute value of the Dirac operator on  $\mathbb{R}^n$ . We then expect

$$|\xi|(x, \omega; \lambda) \sim \frac{1}{p_1(\omega)} \lambda - \frac{p_0(\omega)}{p_1(\omega)} - p_{-1}(\omega) \lambda^{-1} + \dots .$$

Integration over  $|\omega| = 1$  gives

$$d_H(x, x; \lambda) \sim (2\pi)^{-n} (a_0 \lambda^{n-1} + a_1 \lambda^{n-2} + a_2 \lambda^{n-3} + \dots) \quad (C),$$

where, clearly,  $a_0 = \text{wres } H^{-n}$ .

To compute  $a_1, a_2, \dots$  we can as well assume that the development of  $p$  is analytic as  $|\xi| \rightarrow \infty$ . Let  $\psi(z) := z^{n-1}/p'(z)$ , so that

$$a_0 \lambda^{n-1} + a_1(x) \lambda^{n-2} + a_2(x) \lambda^{n-3} + \dots \sim \int_{\mathbb{S}^{n-1}} \psi(|\xi|(x, \omega; \lambda)) d\omega.$$

If  $\Gamma$  is a circle containing  $|\xi|(x, \omega; \lambda)$ , wound once around  $\infty$ , there is the Cauchy integral:

$$\begin{aligned} \psi(|\xi|(x, \omega; \lambda)) &= \psi(p^{-1}(\lambda)) = -\frac{1}{2\pi i} \oint_{\Gamma} \frac{\psi(z) p'(z) dz}{p(z) - \lambda} \\ &= \frac{1}{2\pi i} \oint_{\Gamma^{-1}} \frac{\psi(\zeta^{-1}) p'(\zeta^{-1}) d\zeta}{\zeta^2 (p(\zeta^{-1}) - \lambda)} = \frac{1}{2\pi i} \oint_{\Gamma^{-1}} \frac{d\zeta}{\zeta^{n+1} (p(\zeta^{-1}) - \lambda)}. \end{aligned}$$

Thus  $a_j(x) = \int_{\mathbb{S}^{n-1}} c_j(x, \omega) d\omega$ , where

$$\begin{aligned} c_j(\omega) &= \frac{1}{2\pi i} \oint_{|s|=\varepsilon} s^{n-j-2} \psi(p^{-1}(1/s)) ds \\ &= \frac{1}{(2\pi i)^2} \oint_{|s|=\varepsilon} s^{n-j-2} ds \oint_{\Gamma^{-1}} \frac{d\zeta}{\zeta^{n+1} (p(1/\zeta) - 1/s)} \\ &= \frac{1}{(2\pi i)^2} \oint_{\Gamma^{-1}} \frac{d\zeta}{\zeta^{n+1}} \oint_{|s|=\varepsilon} \frac{s^{n-j-1} ds}{s p(1/\zeta) - 1} \\ &= \frac{1}{2\pi i} \oint_{\Gamma^{-1}} \frac{d\zeta}{\zeta^{n+1} p(1/\zeta)^{n-j}}, \end{aligned}$$

which is the coefficient of  $\zeta^n$  in the expansion of  $p(1/\zeta)^{j-n}$ . Integrating over  $|\omega| = 1$  yields thus

$$a_j = \text{wres } H^{j-n},$$

so, finally:

$$d_H(x, x; \lambda) \sim \frac{1}{(2\pi)^n} (\text{wres } H^{-n} \lambda^{n-1} + \text{wres } H^{-n+1} \lambda^{n-2} + \dots) \quad (C),$$

where the densities of Wodzicki residues are constant for a constant-coefficient operator. It is amusing that we have arrived at a version of the classical Lagrange–Bürmann expansion [29], with Wodzicki residues in the place of ordinary residues.

Notice that  $a_n = 0$ . This is a very simple “vanishing theorem”: see, for instance, [3].

► Returning to the general case, if  $H$  is a positive pseudodifferential operator of order  $d$ , then  $A := H^{1/d}$  is a positive pseudodifferential operator of first order. Setting  $\mu = \lambda^{1/d}$ , we get

$$\delta(\lambda - H) = \delta(\mu^d - A^d) = \frac{\delta(\mu - A)}{d \mu^{d-1}} = \frac{\delta(\lambda^{1/d} - H^{1/d})}{d \lambda^{(d-1)/d}}.$$

and so

$$d_H(x, x; \lambda) \sim \frac{1}{d(2\pi)^n} (a_0(x)\lambda^{(n-d)/d} + a_1(x)\lambda^{(n-d-1)/d} + a_2(x)\lambda^{(n-d-2)/d} + \dots) \quad (C). \quad (4.5)$$

Clearly,  $a_0 = \text{wres } H^{-n/d}$ . Now, the order of  $q_2$  is at most  $2d - 1$ , therefore its higher order contribution to this development is in principle to  $a_1$ ; the order of  $q_3$  is at most  $3d - 2$ , so it contributes to  $a_2$  at the earliest, and so on.

Formula (4.5), obtained through fairly elementary manipulations, is the main result of this section. To illustrate its power, we show how to reap from it a rich harvest of classical results (with a little extra effort).

**Corollary 4.1** (Connes' trace theorem). *For positive elliptic pseudodifferential operators of order  $-n$  on a compact  $n$ -dimensional manifold, the Dixmier trace and the Wodzicki residue are proportional:*

$$\text{Dtr } H = \frac{1}{n(2\pi)^n} \text{Wres } H.$$

*Proof.* Let  $H$  be of order  $d = -n$  in (4.5). We get

$$d_H(x, x; \lambda) \sim -\frac{1}{n(2\pi)^n} \text{wres } H(x) \lambda^{-2} + \dots \quad (C).$$

Assume the manifold is compact. We then know that  $H$  is a compact operator. Now, heuristically the argument goes as follows:  $N'_H(\lambda) \sim -\lambda^{-2}$ , ergo  $N_H(\lambda) \sim \lambda^{-1}$ , ergo  $\lambda_l(H) \sim l^{-1}$ . A Tauberian argument can be used at this point [37] to ensure that the second asymptotic estimate is valid without the Cesàro condition; and then the result follows. But this is by no means necessary. One can steal a look at Section 6 and, by approaching step functions by elements of  $\mathfrak{S}$ , prove in an elementary way that for any given  $\varepsilon > 0$  there is an  $l(\varepsilon)$  such that

$$\frac{C(1 - \varepsilon)}{l(\varepsilon)} < \lambda_l(H) < \frac{C(1 + \varepsilon)}{l(\varepsilon)},$$

where  $C = n^{-1}(2\pi)^{-n} \text{Wres } H$ . □

On a noncompact spin manifold, consider now the Dirac operator on the space of spinors  $L^2(S)$ . The noncommutative integral of  $|D|^{-n}$  does not exist. However, if  $\int a(x) dx$  is defined, it is computable by a noncommutative integral:

**Theorem 4.2.** *Let  $a$  be an integrable function with respect to the volume form on  $M$ . Then*

$$C_n \int_M a(x) dx = \frac{1}{n(2\pi)^n} \text{Wres}(a|D|^{-n}),$$

where on the right hand side  $A$  is seen as a multiplication operator on  $L^2(S)$ . The constants are  $C_{2k} = (2\pi)^{-k}/k!$  and  $C_{2k+1} = \pi^{-k-1}/(2k+1)!!$

“Proof”. That follows from Theorem 5.3 of [37] if  $a$  is a smooth function with compact support. For  $a$  positive and integrable, use monotone convergence on both sides; the general case follows at once.  $\square$

The former is a small step in the direction of a theory of  $K$ -cycles (or “spectral triples”, as they are nowadays called) over noncompact manifolds.

*Remark.* (Note added later). The statement of the theorem and the second line of the “proof” are false, as stated. The integral identity is indeed valid for smooth functions of compact support. However, it does not extend to positive integrable functions, since the Dixmier trace, and therefore also the Wresidue, is not a normal linear functional, so that the monotone convergence theorem does not apply. This oversight was gleefully pointed out by other authors, e.g., in [39], which studies precise conditions for the validity of the stated formula. In view of such useful spinoffs, we have let the error in the printed version stand; nostra culpa.

**Corollary 4.3** (Weyl’s estimate). *Let  $N_H(\lambda)$  denote the counting function of  $H$ , a Laplacian on a compact manifold or bounded region  $M$  acting on scalar functions. Then*

$$N_H(\lambda) \sim \frac{\Omega_n \text{vol } M}{n(2\pi)^n} \lambda^{n/2},$$

where  $\Omega_n$  is the surface area of the unit ball in  $\mathbb{R}^n$ .

*Proof.* The same type of arguments as in Corollary 4.1 work. Indeed, this estimate is a corollary of it [37].  $\square$

► Next consider Schrödinger operators  $-\Delta + V(x)$ , with symbol  $p(x, \xi) = |\xi|^2 + V(x)$ . We can take a slightly different tack and solve the equation  $p(x, \xi) = \lambda$  by  $|\xi| = \sqrt{(\lambda - V(x))_+}$ .

**Corollary 4.4** (The correspondence principle). *For Schrödinger operators:*

$$N_H(\lambda) \sim \frac{\Omega_n}{n(2\pi)^n} \int (\lambda - V(x))_+^{n/2} dx. \quad \square$$

See [22], for instance, for the reasons for the terminology.

A word of caution is in order here. The development (4.5) cannot be integrated term by term in general. Consider, for instance, the harmonic oscillator hamiltonian  $H = \frac{1}{2}(-d^2/dx^2 + x^2)$  on  $\mathbb{R}$ : according to the theory developed here, its spectral density behaves as  $1/\sqrt{\lambda}$ . If  $\psi_n$ ,  $n \in \mathbb{N}$  denote the normalized wavefunctions, then indeed, like in Fourier series theory,

$$\sum_{n+\frac{1}{2} \leq \lambda} \psi_n^2(x) \sim \frac{\sqrt{\lambda}}{\pi}$$

is true and can be independently checked. But wres  $H^{-1/2}$  is not integrable over the real line, so one cannot conclude that  $N_H(\lambda)$  behaves as  $\sqrt{\lambda}$ . Actually, as we saw in Section 2,  $\sum_{n=0}^{\infty} \delta(\lambda - (n + \frac{1}{2})) = H(\lambda) + o(\lambda^{-\infty})$  (C), so  $N_H(\lambda) = \lambda H(\lambda) + o(\lambda^{-\infty})$  (C). Now, Corollary 4.4 applies, so that

$$N_H(\lambda) \sim \frac{2}{2\pi} \int_{-\sqrt{2\lambda}}^{\sqrt{2\lambda}} \sqrt{2\lambda - x^2} dx = \lambda H(\lambda)$$

precisely as it should be. (See the discussion in [30].)

Consider  $n$ -dimensional Schrödinger operators with (continuous) homogeneous potentials  $V(x) \geq 0$ ,  $V(ax) = t^a V(x)$ . The previous formula gives

$$N_H(\lambda) \propto \lambda^{n/2+n/a} \int_{\mathbb{S}^{n-1}} V(x)^{-n/a} dx.$$

and this means that if the cone  $\{x \in \mathbb{R}^n : V(x) = 0\}$  is too big, in the counting number estimate we are heading for trouble [36]. But the “nonstandard asymptotics” that might then intervene do not detract from the validity of the nonintegrated formula (4.5).

► In the remainder of the section, we focus on the computation of spectral densities for Laplacians. Nothing essential is won or lost by considering general vector bundles, so we work on scalars. The more general Laplacian operator on a Riemannian manifold is (minus) the Laplace–Beltrami operator  $\Delta$  plus potential vector and scalar potential terms, with symbol

$$\begin{aligned} p(x, \xi) &= -g^{ij}(x)(\xi_i \xi_j + (i\Gamma_{ij}^k(x)\xi_k + 2A_i(x)\xi_j) \\ &\quad + (A_i(x)A_j(x) + i(\Gamma_{ij}^k(x)A_k(x) - \partial_i A_j(x)))) + V(x) \\ &=: -g^{ij}(x)\xi_i \xi_j + B^i(x)\xi_i + C(x). \end{aligned}$$

Formula (4.5) would seem to give for this case:

$$d_H(x, x; \lambda) \sim \frac{1}{2(2\pi)^n} (a_0(x)\lambda^{(n-2)/2} + a_1(x)\lambda^{(n-3)/2} + a_2(x)\lambda^{(n-4)/2} + \dots) \quad (C).$$

In fact, it will be seen in a moment that  $a_1 = a_3 = \dots = 0$ . Also, we know already that  $a_0(x) = \text{Wres } \Delta^{-n/2} = \Omega_n$ . Our task is to compute the next coefficients; it is a rather exhausting one, whose results can be inferred from the extensive work already carried out [19] on heat kernel expansions (see Section 6), so we shall limit ourselves to the computation of  $a_2(x)$  to illustrate the relative simplicity of our approach.

Let  $n \geq 3$ . Write  $a$  for  $g^{ij}(x)\omega_i \omega_j$ , then  $b$  for  $B^i(x)\omega_i$  and  $c$  for  $C(x)$ . Our method calls for solving for the positive root of  $a|\xi|^2 + b|\xi| + (c - \lambda) = 0$  and substituting this in  $|\xi|^{n-1}/(2a|\xi| + b)$ . In diminishing powers of  $\lambda$ , we obtain for the latter the development:

$$\frac{1}{2a^{n/2}} \left( \lambda^{(n-2)/2} - \frac{(n-1)b}{2a^{1/2}} \lambda^{(n-3)/2} + \left( \frac{n(n-2)b^2}{8a} - \frac{(n-2)c}{2} \right) \lambda^{(n-4)/2} + \dots \right). \quad (4.6)$$

One sees that odd-numbered terms in this expansion contain odd powers of  $\omega$  and thus give vanishing contributions, after the integration on the cosphere. Also, the contribution of the  $q_2$  term in (4.2) will start at order  $\frac{1}{2}n - 2$  in  $\lambda$ , the contribution of  $q_3$  will start at order  $\frac{1}{2}n - 3$ , and so on: the terms in the asymptotic expansion of the density kernels of Laplacian operators differ by powers of  $\lambda$ , not of  $\sqrt{\lambda}$ , as one would expect on general grounds.

It is now convenient to use geodesic coordinates at each point; this is justified by the nature of the result. In these coordinates,  $\Gamma_{ij}^k(x_0) = 0$  and there is the Taylor expansion

$$g_{ij}(x) \sim \delta_{ij} + \frac{1}{3} R_{iklj}(x_0) (x - x_0)^k (x - x_0)^l + \sum_{|\alpha| \geq 3} \partial^\alpha g(x_0) \frac{(x - x_0)^\alpha}{\alpha!} \quad \text{as } x \rightarrow x_0,$$

where  $R_{iklj}$  denotes the Riemann curvature tensor. Recall that the Ricci tensor is given by  $R_{kj} := \sum_l R_{klj}^l$  and the scalar curvature by  $R := \sum_{kj} g^{kj} R_{kj}$ .

From (4.3) one obtains for  $q_2(x_0, \xi)$ ,

$$\frac{1}{2} \sum_{|\alpha|>0} \frac{i^{-|\alpha|}}{\alpha!} \partial_{\xi}^{\alpha} (-g^{ij}(x_0) \xi_i \xi_j + B^i(x_0) \xi_i) \partial_x^{\alpha} \Big|_{x=x_0} (-g^{ij}(x) \xi_i \xi_j + B^i(x) \xi_i + C(x)). \quad (4.7)$$

Let us take for a moment  $A_i = 0$ . Then in geodesic coordinates  $B^i(x_0) = 0$  and it is not hard to see that the only surviving term in (4.7) is equal to  $\frac{1}{3} R_{kj}(x_0) \xi^k \xi^j$ . Also  $b = 0$  in (4.6). So, in view of (4.4) we are left with two terms at order  $\lambda^{(n-4)/2}$ , to wit:

$$- \int_{\mathbb{S}^{n-1}} d\omega \frac{(n-2)C(x_0)}{2} \lambda^{(n-4)/2}$$

that comes from the third term in (4.6), and the first order contribution of

$$\int_{\mathbb{S}^{n-1}} d\omega \frac{\frac{\partial^2}{\partial \lambda^2} (q_2(x, |\xi|(x, \omega; \lambda) \omega) |\xi|^{n-1}(x, \omega; \lambda))}{p'(x, |\xi|(x, \omega; \lambda) \omega)}.$$

In effect,  $q_2$  contributes here a factor of order  $\lambda$ , so the second derivative in the previous formula gives rise to a term of order  $\lambda^{(n-4)/2}$  also. To finish the computation, we use

$$\int_{\mathbb{S}^{n-1}} d\omega A_{ij} \omega^i \omega^j = \frac{\Omega_n}{n} g^{ij} A_{ij},$$

to get

$$a_2(x_0) = \frac{(n-2)\Omega_n}{2} \left( \frac{1}{6} R(x_0) - C(x_0) \right). \quad (4.8)$$

Notice that for a pure Laplace–Beltrami operator, the contribution to  $a_2$ , when computed in geodesic coordinates, comes exclusively through the  $q_2$  term.

It remains to convince ourselves that vector potentials give no contribution at this stage. On one hand, the  $c$  term in (4.6) would contribute now the extra terms

$$-\frac{(n-2)\Omega_n}{2} (A^j A_j + i \partial_j A^j).$$

On the other, the term in  $b^2$  in the same formula would contribute a term of the form  $\frac{1}{2}(n-2)\Omega_n A^j A_j$ , and in the computation of  $q_2$  there appears now a term  $(2i/n) \partial_j B^j$  that contributes  $\frac{1}{2}(n-2)\Omega_n \partial_j A^j$  and thereby cancels the rest. Therefore (4.8) stands also in that case.

Actually the coefficients of the Cesàro asymptotic expansion of  $d(x, x; \lambda)$  are all (local densities of) Wodzicki residues for  $n$  odd:  $a_{2k}(x) = \text{wres } \Delta^{-n/2+k}(x)$ , for  $k \in \mathbb{N}$ . For  $n$  even it happens that  $a_{2k} = \text{wres } \Delta^{-n/2+k}$  only as long as  $-n/2 + k < 0$  (the Wodzicki residues of nonnegative powers of a differential operator being of course zero); the following coefficients for the parametric expansion are, in our terminology of Section 2 (further explained in the next two sections), not “residues” but “moments”. Note that for  $n = 2$ , the coefficient  $a_2$  is already a “moment” and cannot be computed by a Cesàro development. This strikingly different behaviour of the odd-dimensional and the even-dimensional cases is concealed in the uniformity of the usual heat kernel method, but it reflects

itself in the corresponding zeta functions having an infinite number of poles, corresponding to the residues, in the odd-dimensional case; and a finite number in the even-dimensional case. From [38]:

$$\operatorname{Res}_{s=n/2-k} \zeta_H(s) = \frac{1}{2} \operatorname{Wres} H^{k-n/2},$$

where

$$\zeta_H(s) = \int_M \langle d_H(x, x; \lambda), \lambda^{-s} \rangle_\lambda dx \quad (\Re s \gg 0)$$

is the kernel of the zeta operator (3.1). A direct, “elementary” proof of the essential identity between Wodzicki residues and residues of the poles of the zeta functions is obviously in the cards, but we shall not go further afield here. For a nontrivial use of the noncommutative residue in zeta function theory, have a look at [11].

## 5 Cesàro developments of counting functions

We consider here operators on compact manifolds without boundary and look at the behaviour of the counting function

$$N(\lambda) := \sum_{\lambda_l \leq \lambda} 1.$$

In order to refresh our intuition, we shall follow a deliberately naïve approach and temporarily forget some of what we learned at the end of last section. Envisage first the scalar Laplacian on  $\mathbb{T}^2$  with the flat metric; then the counting function is given by the following table:

$\lambda$	0	1	2	4	5	8	9	10	13	16	17	18	20	25	26	...
$N(\lambda^+)$	1	5	9	13	21	25	29	37	45	49	57	61	69	81	89	...

No doubt,  $N(\lambda) \sim \pi\lambda$  is a reasonable first approximation; but it is also plain that the remainder undergoes wild oscillations. The precise determination of this remainder is a difficult problem, not unlike the problem of determining the next-to-main term in the asymptotic development of prime numbers.

An even simpler and more telling example is provided by the eigenvalues  $\lambda_l$  of the Laplacian on the  $n$ -dimensional sphere. They are given by

$$\lambda_l = l(l+n-1) \quad \text{with respective multiplicities} \quad m_l = \binom{l+n}{n} - \binom{l+n-2}{n}, \quad (5.1)$$

for  $l \in \mathbb{N}$ . For example, if  $n = 2$ , the eigenvalues are  $l(l+1)$  and the multiplicities are  $(2l+1)$ . The leading term is

$$N(\lambda) \sim \frac{2}{n!} \lambda^{n/2} \quad \text{as } \lambda \rightarrow \infty.$$

On the other hand, asymptotically:

$$N(\lambda^+) - N(\lambda^-) \sim \frac{2l^{n-1}}{(n-1)!},$$

and so

$$\lambda^{(1-n)/2} (N(\lambda^+) - N(\lambda^-)) \sim \frac{2}{(n-1)!}.$$

Plainly, we cannot find an asymptotic formula for  $N(\lambda)$  with error term  $o(\lambda^{(n-1)/2})$  and continuous main term. The example is taken from Hörmander's work [24, 25].

The foregoing is a ‘‘Gibbs phenomenon’’ related to the lack of smoothness of the characteristic function. The problem is ‘‘solved’’ if one is prepared to look at the expansions in the Cesàro sense. The feature that higher order terms in the asymptotic expansion of the eigenvalues of the Laplacian were to be understood in an averaged sense was pointed out by Brownell [4] many years ago.

Going back to tori, consider the distribution of nonvanishing eigenvalues  $\{\lambda_l\}_{l=1}^\infty$  of the scalar Laplacian on an  $n$ -dimensional torus  $\mathbb{T}^n$ , with the flat metric. The eigenfunctions  $\{\phi_l\}_{l=1}^\infty$  can be seen as nonzero smooth functions in  $\mathbb{R}^n$  that satisfy

$$\Delta\phi_l + \lambda_l \phi_l = 0$$

and the periodicity conditions

$$\phi_l(x_1 + 2k_1\pi, \dots, x_n + 2k_n\pi) = \phi_l(x_1, \dots, x_n),$$

where the girths of the torus are taken to be  $2\pi$  in all directions.

Those eigenvalues are given by  $\lambda_k = k_1^2 + \dots + k_n^2$  for  $k = (k_1, \dots, k_n) \in \mathbb{Z}^n$ , with corresponding eigenfunctions  $\phi_k(x_1, \dots, x_n) = e^{ik \cdot x}$ . Thus the  $\lambda_l$  are the nonnegative integers  $q_l$  that can be written as a sum of  $n$  squares. The multiplicity of each such value is the number of integral solutions of the Diophantine equation  $q_l = k_1^2 + \dots + k_n^2$ . We wish to compute the terms in the parametric and Cesàro developments of  $N(\lambda)$  next to leading Weyl term (which for this problem actually goes back to Gauss):

$$N(\lambda) \sim \frac{\Omega_n}{n} \lambda^{n/2} \quad \text{as } \lambda \rightarrow \infty.$$

To do so, we start with the derivative  $N'(\lambda)$ ; this is nothing but  $(2\pi)^n d(x, x; \lambda)$ ; however, as advertised, it is more instructive to forget for a while the discussion in Section 4. There holds:

$$N'(\lambda) = \sum_{l=1}^\infty \delta(\lambda - \lambda_l) = \sum_{k \in \mathbb{Z}^n} \delta(\lambda - k_1^2 - \dots - k_n^2).$$

Let  $\phi \in \mathcal{D}(\mathbb{R})$ , let  $\sigma$  be a large real parameter and set  $\varepsilon = 1/\sigma$ , so that  $\varepsilon \downarrow 0$ . Then

$$\begin{aligned} \langle N'(\sigma\lambda), \phi(\lambda) \rangle_\lambda &= \varepsilon \langle N'(x), \phi(\varepsilon\lambda) \rangle_\lambda = \varepsilon \sum_{k \in \mathbb{Z}^n} \phi(\varepsilon|k|^2) \\ &= \varepsilon^{1-n/2} \int_{\mathbb{R}^n} \phi(|x|^2) dx + o(\varepsilon^\infty) \\ &= \frac{1}{2} \Omega_n \varepsilon^{1-n/2} \int_0^\infty r^{(n-2)/2} \phi(r) dr + o(\varepsilon^\infty). \end{aligned}$$

The third equality is just Lemma 2.10.

Hence, weakly:

$$N'(\sigma\lambda) = \frac{1}{2} \Omega_n \sigma^{-1+n/2} \lambda_+^{-1+n/2} + o(\sigma^{-\infty}) \quad \text{as } \sigma \rightarrow \infty,$$

and upon integration

$$N(\sigma\lambda) = \frac{\Omega_n}{n} \lambda_+^{n/2} \sigma^{n/2} + o(\sigma^{-\infty}) \quad \text{as } \sigma \rightarrow \infty.$$

Observe that the constant of integration  $\mu_0$  vanishes, as do all the other moments.

Then Theorem 2.4 yields:

$$N(\lambda) = \frac{\Omega_n}{n} \lambda^{n/2} + o(\lambda^{-\infty}) \quad (C) \quad \text{as } \lambda \rightarrow \infty.$$

Hence the error term, although definitely not small in the ordinary sense, is of rapid decay in the (C) sense.

► We turn to examine some cases of spheres. The derivative of the counting function for  $\mathbb{S}^2$  is  $N'(\lambda) = \sum_{l=0}^{\infty} (2l+1) \delta(\lambda - l(l+1))$ . To deal with this case, we need a heavier gun than Lemmata 2.9–2.11. This is provided by:

**Lemma 5.1.** *Let  $f \in \mathcal{K}'(\mathbb{R}^n)$ , so that it satisfies the moment asymptotic expansion. If  $p$  is an elliptic polynomial and  $\phi \in \mathcal{S}$ , then*

$$\langle f(x), \phi(tp(x)) \rangle \sim \sum_{m=0}^{\infty} \frac{\langle f(x), p(x)^m \rangle \phi^{(m)}(0)}{m!} t^m \quad \text{as } t \rightarrow 0.$$

*Proof.* The proof consists in showing that the Taylor expansion

$$\phi(tp(x)) = \sum_{m=0}^N \frac{\phi^{(m)}(0) p(x)^m}{m!} t^m + O(t^{N+1})$$

holds not only pointwise, but also in the topology of  $\mathcal{K}(\mathbb{R}^n)$ . □

Consider now the distribution

$$f(\lambda) := (2\lambda + 1) \left( \sum_{l=1}^{\infty} \delta(\lambda - l) - H(\lambda) \right),$$

that lies in  $\mathcal{K}'$ . Notice that

$$\begin{aligned} \langle f(\lambda), \phi(t(\lambda^2 + \lambda)) \rangle &= \sum_{l=1}^{\infty} (2l+1) \phi(t(l^2 + l)) - \int_0^{\infty} (2\lambda + 1) \phi(t(\lambda^2 + \lambda)) d\lambda \\ &= \sum_{l=1}^{\infty} (2l+1) \phi(t(l^2 + l)) - \int_0^{\infty} \phi(t\mu) d\mu. \end{aligned}$$

From Lemma 5.1 we conclude that, for  $\phi \in \mathcal{S}$ ,

$$\begin{aligned} \langle N'(\lambda), \phi(t\lambda) \rangle &= \sum_{l=0}^{\infty} (2l+1) \phi(t(l^2 + l)) \\ &\sim \int_0^{\infty} \phi(t\mu) d\mu + \phi(0) + \sum_{j=0}^{\infty} \frac{\langle f(\lambda), (\lambda^2 + \lambda)^j \rangle \phi^{(j)}(0)}{j!} t^j \quad \text{as } t \downarrow 0. \end{aligned}$$

The parametric expansion of  $N'(\lambda)$  is thus

$$N'(\lambda/t) \sim H(\lambda) + \delta(\lambda)t + \sum_{j=0}^{\infty} \frac{(-1)^j \mu_j \delta^{(j)}(\lambda)}{j!} t^{j+1} \quad \text{as } t \downarrow 0,$$

where the “generalized moments”  $\mu_j$  are given by

$$\mu_j = \langle f(\lambda), (\lambda^2 + \lambda)^j \rangle = \sum_{l=1}^{\infty} (2l+1)(l^2+l)^j - \int_0^{\infty} (2\lambda+1)(\lambda^2+\lambda)^j d\lambda \quad (C).$$

It follows that  $N'(\lambda) \sim H(\lambda) + o(\lambda^{-\infty})$  (C) as  $\lambda \rightarrow \infty$ .

In view of our gymnastics with Riemann’s zeta function in Section 2, the computation of the  $\mu_j$  presents no difficulties. We obtain

$$\begin{aligned} \mu_0 &= 2\zeta(-1) + \zeta(0) = -\frac{2}{3}, & \mu_2 &= 2\zeta(-5) + 4\zeta(-3) = \frac{8}{315}, \\ \mu_1 &= 2\zeta(-3) + \zeta(-1) = -\frac{1}{15}, & \mu_3 &= 2\zeta(-7) + 9\zeta(-5) + \zeta(-3) = -\frac{2}{105}, \end{aligned}$$

and so on. On integrating, we get

$$N(\lambda/t) \sim \frac{\lambda}{t} H(\lambda) + \frac{1}{3} H(\lambda) + \frac{1}{15} \delta(\lambda) t + \frac{4}{315} \delta'(\lambda) t^2 + \dots \quad \text{as } t \downarrow 0, \quad (5.2)$$

and  $N(\lambda) \sim \lambda H(\lambda) + \frac{1}{3} H(\lambda) + o(\lambda^{-\infty})$  (C). Note that the  $\lambda^0$ th order term in the Cesàro development for  $N(\lambda)$  comes from the first moment. The curvature of a sphere  $\mathbb{S}^n$  is given by  $R = n(n-1)$ , so the second term in the development is precisely what we had expected.

► We now look at the derivative of the counting function for the Laplace–Beltrami operator on  $\mathbb{S}^3$ . It is slightly simpler to consider the operator  $1 - \Delta$ , for which we find, according to (5.1):  $N'(\lambda) = \sum_{l=0}^{\infty} (l+1)^2 \delta(\lambda - (l+1)^2)$ .

Consider the distribution

$$f(\lambda) := (\lambda+1)^2 \left( \sum_{l=0}^{\infty} \delta(\lambda - l) - H(\lambda+1) \right),$$

lying in  $\mathcal{K}'$ . Then:

$$\langle f(\lambda), \phi(t(\lambda+1)^2) \rangle = \sum_{l=0}^{\infty} (l+1)^2 \phi(t(l+1)^2) - \int_{-1}^{\infty} (\lambda+1)^2 \phi(t(\lambda+1)^2) d\lambda.$$

One sees that the moments all cancel:  $\langle f(\lambda), (\lambda+1)^{2j} \rangle = \zeta(-2j-2) = 0$ , for  $j \in \mathbb{N}$ . Therefore we get simply

$$\langle N'(\lambda), \phi(t\lambda) \rangle \sim \frac{1}{2t^{3/2}} \int_0^{\infty} \phi(u) \sqrt{u} du \quad \text{as } t \downarrow 0,$$

and thus in this case we collect just the Weyl term,

$$N(\lambda) \sim \frac{\lambda^{3/2} H(\lambda)}{3} \quad (C) \quad \text{as } \lambda \rightarrow \infty. \quad (5.3)$$

We may reflect now that the counting number for these Laplacians on  $\mathbb{S}^2$ ,  $\mathbb{S}^3$  behave in the expected way for even and odd dimensional cases, respectively. For a generalized Laplacian which is the square of a Dirac operator, the qualitative picture is the same. In particular, the Chamseddine–Connes expansion corresponds to  $n = 4$ , whereupon the counting functional behaves in much the same way as the one for  $\mathbb{S}^2$ . Therefore, formal application of the Chamseddine–Connes Ansatz to the characteristic function of the spectrum, as done in [6, 26], misses the terms involving  $\delta$  and its derivatives – whose physical meaning, if any, is unclear to us.

## 6 Spectral density and the heat kernel

Now we tackle the issue of the small- $t$  behaviour of the Green functions associated to an elliptic pseudodifferential operator  $H$ . These are the integral kernels of operator-valued functions of  $H$ , of the form

$$G(t, x, y) = \langle d_H(x, y; \lambda), g(t\lambda) \rangle_\lambda$$

where  $g$ , as already advertised, will in this section belong (or can be extended) to the Schwartz space  $\mathcal{S}$  (i.e., we deal with the standard theory as opposed to the framework sketched at the end of Section 3). The basic question is whether  $G(t, x, y)$  has an asymptotic expansion as  $t \downarrow 0$ . In effect, we shall see immediately how to obtain from the (C) asymptotic expansion for the spectral density an *ordinary* asymptotic expansion for Green functions.

The emphasis in recent years has been on Abelian type expansions, the so-called heat kernel techniques [19]. It is common folklore that Cesàro summability implies Abel summability, but not conversely. As we just claimed, one can go from the Cesàro expansion to the heat kernel expansion. The reverse implication does not work quite the same. If we know the coefficients of the heat kernel expansion and we independently know that a Cesàro type expansion for the spectral density exists, we can infer the coefficients of the latter from the former. But it may happen that the formal Abel–Laplace type expansion does not say anything about the “true” asymptotic development.

For instance, if  $f(\lambda) := \sin \lambda e^{\sqrt{\lambda}}$  for  $\lambda > 0$ , then  $\lim_{\lambda \rightarrow \infty} f(\lambda)$  (C) does not exist, since no primitive of  $f$  can have polynomial order in  $\lambda$ . Even so, one can show that  $k(t) = \langle f(\lambda), e^{-t\lambda} \rangle$  has a Laplace expansion  $k(t) \sim a_{-1}t^{-1} + a_0 + a_1t + \dots$  as  $t \downarrow 0$ , that is,  $\lim_{\lambda \rightarrow \infty} f(\lambda) = a_{-1}$  (A). To get an example of a bounded function with this behaviour, one uses the fact that  $f_m(\lambda) = \sin \lambda^{1/m}$  obeys  $\lim_{\lambda \rightarrow \infty} f_m(\lambda) = 0$  (C, N) only for  $N > m$ , together with Baire’s theorem, to construct a bounded function  $f(\lambda) = \sum_{k \geq 1} 2^{-k} f_{m_k}(\lambda)$  that does not have a Cesàro limit as  $\lambda \rightarrow \infty$ , but for which  $f(\lambda) \rightarrow 0$  in the Abel sense.

In order to relate our Cesàro asymptotic expansions with heat kernel developments, we need to examine expansions of distributions  $f(\lambda)$  that may contain nonintegral powers of  $\lambda$ . Suppose that  $\{\alpha_k\}_{k \geq 1}$  is a decreasing sequence of real numbers, not including negative integers, and suppose further that  $f \in \mathcal{S}'$ , supported in  $[0, \infty)$ , has the Cesàro asymptotic expansion

$$f(\lambda) \sim \sum_{k \geq 1} c_k \lambda^{\alpha_k} + \sum_{j \geq 1} b_j \lambda^{-j} \quad (C) \quad \text{as } \lambda \rightarrow \infty.$$

It follows from Theorem 32 of [15], and from Theorem 2.5 above, that  $f$  has the following parametric development:

$$f(\sigma\lambda) \sim \sum_{k \geq 1} c_k (\sigma\lambda_+)^{\alpha_k} + \sum_{j \geq 1} b_j \text{Pf}((\sigma\lambda)^{-j} H(\lambda)) + \sum_{m \geq 0} \frac{(-1)^m \mu_m \delta^{(m)}(\lambda)}{m! \sigma^{m+1}} \quad \text{as } \sigma \rightarrow \infty, \quad (6.1)$$

where the “generalized moments”  $\mu_m$  are given by

$$\mu_m := \left\langle f(x) - \sum_{k \geq 1} c_k x_+^{\alpha_k} - \sum_{j \geq 1} b_j \text{Pf}(x^{-j} H(x)), x^m \right\rangle \quad (6.2)$$

and where  $\text{Pf}$  denotes a “pseudofunction” [14] obtained by taking the Hadamard finite part, that is:  $\langle \text{Pf}(h(x)), g(x) \rangle := \text{F. p.} \int_0^\infty h(x)g(x) dx$  if  $\text{supp } h \subseteq [0, \infty)$ . In particular,

$$\begin{aligned} \langle \text{Pf}(x^{-j} H(x)), g(x) \rangle &= \text{F. p.} \int_0^\infty \frac{g(x)}{x^j} dx \\ &= \int_1^\infty \frac{g(x)}{x^j} dx + \int_0^1 \frac{1}{x^j} \left( g(x) - \sum_{k=0}^{j-1} \frac{g^{(k)}(0)}{k!} x^k \right) dx - \sum_{k=0}^{j-2} \frac{g^{(k)}(0)}{k!(j-k-1)}. \end{aligned} \quad (6.3)$$

Notice that taking the finite part involves dropping a logarithmic term proportional to  $g^{(j-1)}(0)$ . This has the consequence that  $\text{Pf}(x^{-j} H(x))$  fails to be homogeneous of degree  $-j$  by a logarithmic term; indeed,

$$\text{Pf}((\sigma\lambda)^{-j} H(\sigma\lambda)) = \sigma^{-j} \text{Pf}(\lambda^{-j} H(\lambda)) + \frac{(-1)^j \delta^{(j-1)}(\lambda) \log \sigma}{(j-1)! \sigma^j}.$$

Consequently,

$$\begin{aligned} \langle f(\lambda), g(t\lambda) \rangle_\lambda &\sim \sum_{k \geq 1} c_k t^{-\alpha_k - 1} \text{F. p.} \int_0^\infty \lambda^{\alpha_k} g(\lambda) d\lambda \\ &+ \sum_{j \geq 1} b_j t^j \left( \text{F. p.} \int_0^\infty \frac{g(\lambda)}{\lambda^j} d\lambda - \frac{g^{(j-1)}(0)}{(j-1)!} \log t \right) + \sum_{m \geq 0} \frac{\mu_m g^{(m)}(0)}{m!} t^m. \end{aligned} \quad (6.4)$$

The heat kernel development may be recovered by taking  $g(\lambda) = e^{-\lambda}$  for  $\lambda \geq 0$ . In that case,  $d\alpha_k$  is integral,  $\text{F. p.} \int_0^\infty \lambda^{\alpha_k} g(\lambda) d\lambda = \Gamma(\alpha_k + 1)$  and  $g^{(j-1)}(0) = (-1)^{(j-1)}$ . From this it is clear that the heat kernel of a pseudodifferential operator may generally contain logarithmic terms. Indeed, by harking back to (4.5), on using (6.4) we prove:

**Corollary 6.1.** *The general form of the (coincidence limit of) the heat kernel for an elliptic pseudodifferential operator of order  $d$  on a compact manifold  $M$  of dimension  $n$  is given by*

$$K(t, x, x) \sim \sum_{j-n \notin d\mathbb{N}_+} \gamma_{j-n}(x) t^{(j-n)/d} + \sum_{j-n \in d\mathbb{N}_+} \beta_{j-n}(x) t^{(j-n)/d} \log t + \sum_{r=1}^{\infty} r_m(x) t^m$$

as  $t \downarrow 0$ , where

$$\gamma_{j-n}(x) = \frac{\Gamma((n-j)/d)}{d(2\pi)^n} a_j(x),$$

and similarly for the other coefficients.

(See [21, Cor. 4.2.7].)

Now suppose we know *a priori* that  $f(\lambda)$  has a Cesàro asymptotic expansion in falling powers of  $\lambda$ , and that we also know that  $\Phi(t) := \langle f(\lambda), e^{-t\lambda} \rangle_\lambda$  has an asymptotic expansion as  $t \downarrow 0$  without

( $\log t$ )-terms. Then it follows that all  $b_j = 0$  in (6.1), i.e., there are no negative integral exponents in the Cesàro development of  $f$ , and consequently the constants  $\mu_m$  are the moments of  $f$ . Thus (6.4) simplifies to

$$\Phi(t) \sim \sum_{k \geq 1} c_k \Gamma(\alpha_k + 1) t^{-\alpha_k - 1} + \sum_{m \geq 0} \frac{(-1)^m \mu_m}{m!} t^m.$$

This is precisely the case for a (generalized) Laplacian: if  $n$  is *odd*, only half-integer powers of  $\lambda$  appear in the spectral density and logarithmic terms in the heat kernel are thereby ruled out. Notice that the Cesàro development for an odd dimensional Laplacian need not terminate. For *even* dimensions, the term  $k = n/2$  is proportional to  $\text{wres } H^0 \lambda^{-1}$  and later terms are proportional to  $\text{wres } H^r \lambda^{-r-1}$ . However, since  $H^r$  is a differential operator, its local Wodzicki residue vanishes for  $r \in \mathbb{N}$ , and the Cesàro development terminates at the  $\lambda^0$  term. However, as we have seen, at this point the moments (6.2) enter the picture.

► It has become a habit to write the diagonal of the heat kernel for a Laplacian in the form

$$K(t, x, x) \sim (4\pi t)^{-n/2} \sum_{k=0}^{\infty} b_k(x, x) t^{k/2},$$

where  $n$  is the dimension of the manifold and  $b_0(x, x) = 1$ . We see now that  $b_k(x, x) = 0$  for  $k$  odd, whereas

$$b_{2k}(x, x) = \frac{2^k a_{2k}(x)}{\Omega_n (n-2)(n-4) \cdots (n-2k)} \quad \text{for } k > 0.$$

A similar formula holds off-diagonal. As we have noted, these expansions are local in the sense that they do not distinguish between a finite and an infinite region of  $\mathbb{R}^n$ , say. However, the smallness of the terms after the first is not uniform near the boundary, and hence the “partition function”

$$K(t) := \int_M K(t, x, x) dx \sim (4\pi t)^{-n/2} \sum_{k=0}^{\infty} b_k t^{k/2}, \quad (6.5)$$

with  $b_0 = \text{vol}(M)$  for scalars, has an expansion with nontrivial boundary terms in general, starting to contribute in  $b_2$  [33].

As for the examples, the expansion (6.5) for  $\mathbb{S}^2$  was first obtained as the partition function of a diatomic molecule [34] and is well known to physicists. On using  $\text{vol}(\mathbb{S}^2) = 4\pi$ , we read Mulholland’s expansion directly by looking at (5.2):

$$K_{\mathbb{S}^2}(t) \sim \frac{1}{t} + \frac{1}{3} + \frac{1}{15} t + \frac{4}{315} t^2 + \cdots \quad \text{as } t \downarrow 0.$$

As for the  $SU(2)$  group manifold, from (5.3), on using  $\text{vol}(\mathbb{S}^3) = 2\pi^2$  and  $e^{t\Delta} = e^t e^{-t(1-\Delta)}$ , the partition function is seen immediately to be

$$K_{\mathbb{S}^3}(t) \sim \frac{\sqrt{\pi}}{4t^{3/2}} e^t.$$

► We turn at last to the Chamseddine–Connes expansion. The theory of Cesàro and parametric expansions justifies (1.1), in the following way. We work in dimension  $n = 4$  and take  $H = D^2$ , a

generalized Laplacian, acting on a space of sections of a vector bundle  $E$ , over a manifold without boundary. The kernel of its spectral density satisfies

$$d_{D^2}(x, x; \lambda) \sim \frac{\text{rank } E}{16\pi^2} \lambda + \frac{1}{32\pi^4} \text{wres } D^{-2}(x) \quad (C) \quad \text{as } \lambda \rightarrow \infty.$$

Integrating over  $M$  and using the formulas of this section with  $t = \Lambda^{-2}$ , we then get

$$\begin{aligned} \text{Tr } \phi(D^2/\Lambda^2) \sim \frac{1}{16\pi^2} & \left( (\text{rank } E) \Lambda^4 \int_0^\infty \lambda \phi(\lambda) d\lambda + b_2(D^2) \Lambda^2 \int_0^\infty \phi(\lambda) d\lambda \right. \\ & \left. + \sum_{m \geq 0} (-1)^m \phi^{(m)}(0) b_{2m+4}(D^2) \Lambda^{-2m} \right) \quad \text{as } t \downarrow 0. \end{aligned}$$

where  $(-1)^m b_{2m+4}(D^2) = 16\pi^2 \mu_m(D^2)/m!$  are suitably normalized, integrated moment terms of the spectral density of  $D^2$ . Thus, we arrive at (1.1).

We finally take stock of the status of the Chamseddine–Connes development. If  $\phi \in \mathcal{S}$ , then the development becomes a bona fide asymptotic expansion. However, if one wishes to use (for instance) the counting function  $N_{D^2}(\lambda \leq \Lambda^2)$ , which does not lie in  $\mathcal{S}$ , then the present formulas are not directly applicable and one must proceed like in Section 5; moreover the expansion beyond the first piece is only valid in the Cesàro sense. We close by noting that third piece of the Chamseddine–Connes Lagrangian has interesting conformal properties; this is better studied through the corresponding zeta function at the origin [35]. That term is definitely not a Wodzicki residue but a moment; whether this fact has any physical significance is not easy to say.

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