On the ultraviolet behaviour of quantum fields over noncommutative manifolds

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Abstract

By exploiting the relation between Fredholm modules and the Segal–Shale–Stinespring version of canonical quantization, and taking as starting point the first-quantized fields described by Connes' axioms for noncommutative spin geometries, a Hamiltonian framework for fermion quantum fields over noncommutative manifolds is introduced. We analyze the ultraviolet behaviour of second-quantized fields over noncommutative three-tori, and discuss what behaviour should be expected on other noncommutative spin manifolds.

1 Introduction

This article considers quantum fields over noncommutative spaces. The fact that compactification of matrix models in M-theory leads to noncommutative tori [1, 2] provides some motivation. But here we address questions of principle, open since Connes characterized the noncommutative manifolds able to sustain matter [3].

First-quantized fermion fields live on noncommutative spin manifolds, in particular NC tori. An odd spin geometry consists of four objects $(\mathcal{A}, \mathcal{H}, J, D)$, where: (1) $\mathcal{A}$ is a unital pre-C^*-algebra; (2) $\mathcal{H}$ is a Hilbert space carrying a representation of $\mathcal{A}$ by bounded operators; (3) J is an antilinear isometry of $\mathcal{H}$ onto itself; (4) D is a selfadjoint operator on $\mathcal{H}$, with compact resolvent. From such a structure, plus some appropriate compatibility conditions formulated as axioms, Connes was able to derive ordinary spin geometry – in which D is the standard Dirac operator $D\!\!\!\!/$ – including all of the Riemannian structure. Leaving out the condition that $\mathcal{A}$ be commutative, we are left with a handle on the vast new realm of noncommutative spin geometries.

Noncommutative geometry is also a language of choice for the formal aspects of quantum field theory. For instance, Wick ordering is intimately related to Connes' Fredholm modules [4–6], reviewed here. The structure of anomalies in gauge field theories can be recast in terms of cyclic cohomology; this was pointed out by Araki [7, 8] and put forward by Mickelsson and Langmann in a splendid series of papers [9]. Very recently, it has been found that the quasi-Hopf algebra structure of Feynman graphs [10] is directly related to Hopf algebras relevant to the general index formula in noncommutative geometry [11].

These facts can be better put into perspective by taking the step proposed in this paper. Indeed, it has long been known that quantum field theory possesses an algebraic core independent of the nature of spacetime [12]. For instance, the description of fermions coupled to external gauge fields is a problem in representation theory of the infinite dimensional orthogonal group. From the latter, with the input of an appropriate single-particle space, it is possible to derive all quantities of interest: current algebra, anomalous transformation terms, Feynman rules [13]. Now, the process is fundamentally unchanged if the "matter field" evolves on a noncommutative space. In a nutshell: we endeavour to apply the canonical quantization machinery to a noncommutative kind of single-particle space.

We couple our proposal here with a description of the simplest imaginable model, generalizing the textbook field quantization with periodic boundary conditions; i.e., we quantize chiral fermions in a 3-dimensional "noncommutative box". (We readily admit to a lingering prejudice in favour of the physical number of dimensions.)

The ultraviolet behaviour of fermion fields depends critically on the dimension of the space. We assume, to minimize infrared troubles, that the latter is compact. The simplest case corresponds to $1 + 1$ field theory, with chiral fermions living on an $\mathbb{R} \times \mathbb{T}^1$ spacetime. Let F be the ordinary Dirac phase operator defining the Wick ordering prescription and let X denote a gauge transformation. Choose the associated Fock space representation of the CAR algebra. Then $[F, X]$ is Hilbert–Schmidt, and so the loop groups of arbitrary Yang–Mills theories are contained in the group of Bogoliubov transformations, and the ordering prescription by itself regularizes the theory. This fact is behind the success of second-quantization methods in the construction of representations of the Virasoro and Kac–Moody algebras [14], and partly behind the development of conformal field theory [15]. In the next odd case, $1 + 3$ field theory, which mainly concerns us, the ultraviolet behaviour, as gauged by the summability of $[F, X]$, is much worse, and extra renormalizations are needed in order to regularize the theory.

A long-standing hope, now amenable to rigorous scrutiny, is that giving up locality, one of the basic tenets of rigorous quantum field theory [16] – and indeed, one of the main selling points by the forefathers [17] – will be rewarded with a better ultraviolet behaviour. After all, noncommutative manifolds – with NC tori with irrational parameters as a case in point – usually are much more disconnected that ordinary ones. We shall see that this hope is not borne out.

The content of the paper is as follows. First, we describe a general framework for fermion fields on noncommutative spaces, in the presence of background fields treated adynamically. For that, we recall in Section 2 Connes' axioms for noncommutative fermionic single-particle spaces. We check the axioms and exhibit the spin structure effecting the neutrino paradigm [18] over the noncommutative 3-torus. In Section 3 we discuss the Fredholm module structure. With the (general) Dirac phase operator F in hand, we proceed to second quantization. The space of spinors on the algebra is an infinite dimensional linear spinor space; we recall in Section 4 the definition of the spin representation for its orthogonal group, whose infinitesimal version yields the quantization prescription for the currents. The construction of the scattering matrix is left for another day, our main purpose here being to show how simple noncommutative quantum field theory really is – and why it belongs in the toolkit of every theorist. We then examine the issue of the ultraviolet behaviour by means of our example. In Section 5 we see by direct computation that, as "measured" by the stick considered in this paper, the ultraviolet behaviour of the theory is the same as for a commutative torus. Finally, in Section 6 we discuss why such behaviour of NC tori should be expected, on general grounds, on any noncommutative manifold. This is related to some of the deeper issues in noncommutative geometry.

The next logical step is to quantize bosonic actions for (noncommutative) gauge fields, perhaps in the presence of external currents. Then it would be time to tackle the full-blown renormalization theory for nonlinear field configurations.

2 First quantization on noncommutative tori

Our method of work in this section is the following: each time that we introduce basic data or axioms, we illustrate/comment on the commutative case and verify them for the noncommutative 3-torus. We rely heavily on our Ref. [19]. We begin, then, by making explicit the objects of a spin geometry $(\mathcal{A}, \mathcal{H}, D, J)$ for 3-tori.

Let θ be a real skewsymmetric $n \times n$ matrix with entries θ_{jk}. The C^*-algebra determined by n unitary generators, with the relations

$$u_k u_j = e^{2\pi i \theta_{jk}} u_j u_k,$$

is called the n-torus algebra A_θ. We focus on the $n = 3$ case with irrational entries θ_{jk}. It is very convenient – and suggested by consideration of the Weyl algebra – to introduce the unitary elements

$$u^r := \exp\{\pi i (r_1 \theta_{12} r_2 + r_1 \theta_{13} r_3 + r_2 \theta_{23} r_3)\}\, u_1^{r_1} u_2^{r_2} u_3^{r_3}$$

for each $r \in \mathbb{Z}^3$; the coefficient is chosen so that $(u^r)^* = u^{-r}$ in all cases. They obey the product rule:

$$u^r\, u^s = \lambda(r, s)\, u^{r+s}, \qquad \lambda(r, s) := \exp\{-\pi i\, r_j \theta_{jk} s_k\}.$$

The noncommutative torus proper $\mathcal{A}_\theta := \mathbb{T}^3_\theta$ is the dense subalgebra of A_θ of "noncommutative Fourier series":

$$\mathbb{T}^3_\theta := \{\, a = a_r\, u^r : \{a_r\} \in \mathcal{S}(\mathbb{Z}^3)\,\},$$

where the coefficients belong to the space $\mathcal{S}(\mathbb{Z}^3)$ of rapidly decreasing sequences, namely, those for which $(1 + |r|^2)^k\, |a_r|^2$ is bounded for all $k = 1, 2, 3, \dots$. In the commutative case $\theta = 0$, and then $\mathbb{T}^3_0 \simeq C^\infty(\mathbb{T}^3)$.

On each torus algebra A_θ there is a faithful tracial state τ, given by $\tau(a_r u^r) := a_0$. If θ is irrational, the tracial state τ on A_θ is unique. Any state on a C^*-algebra A_θ gives rise to a Hilbert space by the well-known Gelfand–Naĭmark–Segal construction. So we introduce the auxiliary Hilbert space $\mathcal{H}_0$ given as the completion of the vector space A_θ in the Hilbert norm

$$\|a\|_2 := \sqrt{\tau(a^* a)}.$$

Since τ is a faithful state, the obvious map $A_\theta \to \mathcal{H}_0$ is injective; we shall denote by $\underline{a}$ the vector in $\mathcal{H}_0$ corresponding to $a \in A_\theta$. The GNS representation of A_θ is just

$$\pi(a) : \underline{b} \mapsto \underline{ab}.$$

We now look for the involution J. The obvious candidate to try is

$$J_0(\underline{a}) := \underline{a^*}.$$

(This is in fact the Tomita involution [20] determined by the cyclic and separating vector $\underline{1}$ for the algebra A_θ.) Notice, however, that $J_0^2 = +1$, whereas we require $J^2 = -1$ in three dimensions (see Axiom 1 below). A simple device allows us to modify the sign: we double the GNS Hilbert space by taking $\mathcal{H} := \mathcal{H}_0 \oplus \mathcal{H}_0$ and define

$$J := \begin{pmatrix} 0 & -J_0 \\ J_0 & 0 \end{pmatrix}.$$

The torus algebra acts on $\mathcal{H}$ by the representation

$$\pi(a) := \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}.$$

When $a \in \mathcal{A}$ and $\xi \in \mathcal{H}$, we shall usually write $a\xi := \pi(a)\xi$. The vectors $\psi_m = \underline{u^m} \oplus 0$ and $\psi'_m = 0 \oplus \underline{u^m}$, for $m \in \mathbb{Z}^3$, form a convenient orthonormal basis of $\mathcal{H}$.

Finally, we produce D. Let us consider the usual Pauli matrices σ_j, and the derivations δ_1, δ_2, δ_3 given by

$$\delta_j(a_r u^r) := 2\pi i\, r_j\, a_r u^r, \qquad (j = 1, 2, 3).$$

We define

$$D := -i(\sigma_1\, \delta_1 + \sigma_2\, \delta_2 + \sigma_3\, \delta_3) = -i \begin{pmatrix} \delta_3 & \delta_1 - i\delta_2 \\ \delta_1 + i\delta_2 & -\delta_3 \end{pmatrix}$$

Then $D^2 = -(\sigma \cdot \delta)^2 = (-\delta_1^2 - \delta_2^2 - \delta_3^2) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. This operator is diagonalized by the orthonormal basis $\{\psi_m, \psi'_m\}$ of $\mathcal{H}$, with eigenvalues $4\pi^2|m|^2$. Using this basis we may express D, its absolute value $|D|$ and the phase operator $F := D|D|^{-1}$ as (matrix) multiplication operators in the index m:

$$D = 2\pi\, m \cdot \sigma, \qquad |D| = 2\pi\, |m|, \qquad F = \frac{m \cdot \sigma}{|m|}.$$

The eigenvalues are then the same as for the ordinary Dirac operator on the ordinary torus (with untwisted boundary conditions). One can even introduce "coherent spin states" as eigenvectors of F: our geometry looks like, and is, a spin one-half system on the NC tori.

Before introducing the further relations and properties that the objects of a spin geometry, in particular for $\mathbb{T}_\theta^3$, must satisfy, we make some precisions of a general nature on the data themselves.

(1) A pre-C^*-algebra $\mathcal{A}$ is a dense involutive subalgebra of a C^*-algebra A that is stable under the holomorphic functional calculus; or, more simply, such that the inverse (in A) of any invertible element of $\mathcal{A}$ lies also in $\mathcal{A}$. This happens, for instance, whenever $\mathcal{A}$ is the smooth domain of a Lie algebra of densely defined derivations of A, since $\delta(a^{-1}) = -a^{-1}\, \delta(a)\, a^{-1}$ for any derivation. The major consequence of stability under the holomorphic functional calculus is that the K-theories of $\mathcal{A}$ and of A are the same [21].

For the algebra A_θ, the common domain of the powers δ_j^k of the commuting derivations $\delta_1, \delta_2, \delta_3$ is precisely the subalgebra $\mathbb{T}_\theta^3$; it is clear then that $\mathbb{T}_\theta^3$ is a pre-C^*-algebra.

(2) That $(D - \lambda)^{-1}$ is compact implies that D has a discrete spectrum of eigenvalues of finite multiplicity. This is assured for the Dirac operator on a compact spin manifold. In most circumstances the finite-dimensional kernel of D is of no consequence, and we have felt free to use the notation D^{-1} when convenient.

In the noncommutative case, we shall also refer to D as the Dirac operator. Connes' axioms are reorganized as follows: three with algebraic flavour, three "analytical" axioms and lastly a "topological" one. (Such labels are a bit deceptive, of course.)

Axiom 1 (Reality)**.** *The antilinear isometry $J\colon \mathcal{H} \to \mathcal{H}$ is such that the representation given by $\pi^0(b) := J\pi(b^*)J^\dagger$ commutes with $\pi(\mathcal{A})$. Moreover the isometry satisfies*

$$J^2 = \pm 1, \qquad JD = \pm DJ,$$

where the signs are precisely given by the following table:

n mod 8	1	3	5	7
$J^2 = \pm 1$	$+$	$-$	$-$	$+$
$JD = \pm DJ$	$-$	$+$	$-$	$+$

This table arises from the structure of real Clifford algebra representations that underlie KR-theory. It is well known that, in the commutative case of Riemannian spin manifolds, one can find conjugation operators J on spinors that satisfy these sign rules.

In the noncommutative case, the antilinear operator J comes from the Tomita involution on a Hilbert space: π^0 is a representation of the opposite algebra $\mathcal{A}^0$, consisting of elements $\{ a^0 : a \in \mathcal{A} \}$ with product $a^0 b^0 = (ba)^0$ – we can write $b^0 = Jb^*J^\dagger$. We have thus required that the representations π and π^0 commute. When $\mathcal{A}$ is commutative, we may also require $J\pi(b^*)J^\dagger = \pi(b)$, whereupon the commutation of representations is automatic.

For 3-tori, the opposite algebra A_θ^0 is just $A_{-\theta}$, and the commuting representation of $A_{-\theta}$ on $\mathcal{H}_0$ is given by right multiplication by elements of A_θ:

$$a^0\, \underline{b} = J_0 a^* J_0^\dagger\, \underline{b} = J_0\, \underline{a^* b^*} = \underline{ba}.$$

From that, verification of the reality axiom is immediate.

Axiom 2 (First-order property)**.** *For all $a, b \in \mathcal{A}$, the following commutation relation moreover holds:*

$$[[D, a], Jb^*J^\dagger] = 0.$$

That can be rewritten as $[[D, \pi(a)], \pi^0(b)] = 0$. In view of this condition, the bimodule over $\mathcal{A}$ given by $C_n(\mathcal{A}, \mathcal{A} \otimes \mathcal{A}^0) := (\mathcal{A} \otimes \mathcal{A}^0) \otimes \mathcal{A}^{\otimes n}$ is represented by operators on $\mathcal{H}$:

$$\pi_D((a \otimes b^0) \otimes a_1 \otimes a_2 \otimes \cdots \otimes a_n) := \pi(a)\pi^0(b)\, [D, \pi(a_1)]\, [D, \pi(a_2)] \cdots [D, \pi(a_n)].$$

The elements of $C_n(\mathcal{A}, \mathcal{A} \otimes \mathcal{A}^0)$ are called Hochschild n-chains with coefficients in $\mathcal{A} \otimes \mathcal{A}^0$.

In the commutative case, we may replace $\mathcal{A} \otimes \mathcal{A}^0$ simply by $\mathcal{A}$, and the axiom expresses that the Dirac operator $D\!\!\!/$ is a first-order differential operator.

The first-order axiom for our NC 3-torus geometry can be readily checked, using that D comes from a derivation of the algebra.

Axiom 3 (Orientability)**.** *There exists a Hochschild cycle $\boldsymbol{c} \in Z_n(\mathcal{A}, \mathcal{A} \otimes \mathcal{A}^0)$ whose representative on $\mathcal{H}$ fulfils*

$$\pi_D(\boldsymbol{c}) = 1.$$

We say that the Hochschild n-chain $\boldsymbol{c}$ is a cycle when its boundary is zero, where the Hochschild boundary operator for $n = 3$ is

$$\begin{aligned} b(m_0 \otimes a_1 \otimes a_2 \otimes a_3) := m_0 a_1 \otimes a_2 \otimes a_3 - m_0 \otimes a_1 a_2 \otimes a_3 \\ + m_0 \otimes a_1 \otimes a_2 a_3 - a_3 m_0 \otimes a_1 \otimes a_2, \end{aligned}$$

for $m_0 \in \mathcal{A} \otimes \mathcal{A}^0$; and similarly for other n. Then $b^2 = 0$, making $C_\bullet(\mathcal{A}, \mathcal{A} \otimes \mathcal{A}^0)$ a chain complex.

The Hochschild cycle $\boldsymbol{c}$ is the algebraic equivalent of a volume form, on a noncommutative manifold. Indeed, in the commutative case, a volume form is a sum of terms $a_0\, da_1 \wedge \cdots \wedge da_n$, which we represent by an antisymmetric sum:

$$\boldsymbol{c}' := \sum_\sigma (-)^\sigma a_0 \otimes a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(n)}$$

in $\mathcal{A}^{\otimes(n+1)} = C_n(\mathcal{A}, \mathcal{A})$. Then $b\boldsymbol{c}' = 0$ by cancellation since $\mathcal{A}$ is commutative. When $\mathcal{A} = C^\infty(M)$, chains are represented by Clifford products: $\pi_{\not{D}}(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = a_0\, \gamma(da_1) \cdots \gamma(da_n)$, with $\gamma(da) := \gamma^j\, \partial_j a$, where the γ^j are essentially the Dirac matrices.

For our 3-torus geometries, consider the Hochschild chain:

$$\boldsymbol{c} := \frac{1}{6(2\pi i)^3}\, \varepsilon^{ijk} u_i^{-1} u_j^{-1} u_k^{-1} \otimes u_k \otimes u_j \otimes u_i.$$

(The first tensor factor can lie in $\mathcal{A}$, since $\mathcal{A} \simeq \mathcal{A} \otimes 1^0 \subset \mathcal{A} \otimes \mathcal{A}^0$.) We check that this $\boldsymbol{c}$ is a Hochschild 3-cycle on any $\mathbb{T}^3_\theta$. In fact,

$$\begin{aligned} 6(2\pi i)^3\, b\boldsymbol{c} = \varepsilon^{ijk}\, (u_i^{-1} u_j^{-1} \otimes u_j \otimes u_i - u_i^{-1} u_j^{-1} u_k^{-1} \otimes u_k u_j \otimes u_i \\ + u_i^{-1} u_j^{-1} u_k^{-1} \otimes u_k \otimes u_j u_i - u_j^{-1} u_k^{-1} \otimes u_k \otimes u_j) \end{aligned}$$

and the first and fourth terms cancel after cyclic permutation of the indices. Therefore

$$6(2\pi i)^3\, b\boldsymbol{c} = \varepsilon^{ijk}\, (u_i^{-1} u_j^{-1} u_k^{-1} \otimes u_k \otimes u_j u_i - u_i^{-1} u_j^{-1} u_k^{-1} \otimes u_k u_j \otimes u_i).$$

The remaining term $\varepsilon^{ijk}\, u_i^{-1} u_j^{-1} u_k^{-1} \otimes u_k \otimes u_j u_i$ vanishes by antisymmetrization, since the commutation relations imply

$$u_i^{-1} u_j^{-1} u_k^{-1} \otimes u_k \otimes u_j u_i = u_j^{-1} u_i^{-1} u_k^{-1} \otimes u_k \otimes u_i u_j.$$

Likewise the second remaining term vanishes by antisymmetrization.

The representative on $\mathcal{H}$ given by the geometry is the identity; in effect, $[D, u_j] = -i\sigma_j\, \delta_j(u_j) = 2\pi\sigma_j\, u_j$, and therefore

$$\pi_D(\boldsymbol{c}) = \frac{(2\pi)^3}{6(2\pi i)^3} \varepsilon^{ijk}\, \sigma_k \sigma_j \sigma_i = \frac{-6i\,(2\pi)^3}{6(2\pi i)^3} = 1.$$

Axiom 4 (Classical dimension)**.** *There is an integer n, the classical dimension of the spin geometry, for which the singular values of $|D|^{-n}$ form a logarithmically divergent series. The coefficient of logarithmic divergence will be denoted by $⨍ ds^n$.*

In our case, since $D^2 = 4\pi^2|m|^2$ on a 2-dimensional eigenspace for each m, we see that

$$\fint ds^n = 2 \lim_{R\to\infty} \frac{1}{3\log R} \sum_{1\leqslant|m|\leqslant R} (2\pi|m|)^{-n} = 2 \lim_{R\to\infty} \frac{1}{3\log R} \int_1^R \frac{4\pi r^2\,dr}{(2\pi r)^n},$$

which is zero for $n > 3$, diverges for $n < 3$ and is positive finite (equal to $(3\pi^2)^{-1}$) for $n = 3$; so indeed the dimension is 3.

Once we know what the correct dimension for a noncommutative manifold $\mathcal{A}$ is, we write $\fint a\,ds^n$ for the coefficient of logarithmic divergence of $a|D|^{-n}$, that exists for $a \in A$. In the commutative case, denoting by μ the canonical measure, Connes' trace theorem (see Section 6) shows that $\fint a\,ds^n = C_n \int a(x)\,d\mu(x)$, with C_n a normalization factor. In dimension 3, the normalization factor is precisely $1/3\pi^2$ [22].

Note that for 3-tori: $\fint u^r\,ds^3 = 0$ unless $r = 0$. This can be proved, for instance, by using the zeta-function recipe [23] for the computation of the noncommutative integral: $\fint u^r\,ds^3 = \operatorname{Res}_{s=1} \operatorname{Tr}(u^r|D|^{-3s}) = 0$ since, for any $r \neq 0$ and $s > 1$, $u^r|D|^{-3s}$ is a traceclass operator with an off-diagonal matrix.

Axiom 5 (Regularity)**.** *For any $a \in \mathcal{A}$, the operator $[D,a]$ is bounded on $\mathcal{H}$, and both a and $[D,a]$ belong to the domain of smoothness $\bigcap_{k=1}^\infty \operatorname{Dom}(\delta^k)$ of the derivation δ on $L(\mathcal{H})$ given by $\delta(T) := [|D|,T]$.*

The regularity axiom has far-reaching implications. As shown by Cipriani *et al* [24], it implies, in particular, that $\fint$ is a trace on the algebra $\mathcal{A}$; i.e., $\fint ab\,ds^n = \fint ba\,ds^n$ for all $a,b \in \mathcal{A}$. This finite trace on $\mathcal{A}$ extends to a finite normal trace on the von Neumann algebra $\mathcal{A}''$ generated by $\mathcal{A}$; therefore $\mathcal{A}''$ can only have components of types I_n and II_1 [20].

In the commutative case, where $[\not{D},a] = \gamma(da)$, this axiom amounts to saying that a has derivatives of all orders, i.e., that $\mathcal{A} \subseteq C^\infty(M)$. This is proved with the pseudodifferential calculus. Consequently, all multiplication operators in $\bigcap_{k=1}^\infty \operatorname{Dom}(\delta^k)$ are multiplications by smooth functions.

Verification of the regularity axiom for our noncommutative torus is straightforward.

Axiom 6 (Finiteness)**.** *Denote by $\langle\cdot \mid \cdot\rangle$ the inner product on $\mathcal{H}$. The space of smooth vectors $\mathcal{H}_\infty := \bigcap_{k=1}^\infty \operatorname{Dom}(D^k)$ is a finite projective left $\mathcal{A}$-module with a Hermitian structure $(\cdot \mid \cdot)$ defined by*

$$\fint (\xi \mid \eta)\,ds^n = C_n\,\langle\xi \mid \eta\rangle.$$

The axiom assumes the trace property for the noncommutative integral, as we see from the following manipulation:

$$\begin{aligned} \fint a\,(\xi \mid \eta)\,ds^n &= \fint (\xi \mid a\eta)\,ds^n = C_n\,\langle\xi \mid a\eta\rangle \\ &= C_n\,\langle a^*\xi \mid \eta\rangle = \fint (a^*\xi \mid \eta)\,ds^n = \fint (\xi \mid \eta)\,a\,ds^n. \end{aligned}$$

In the commutative case, Connes's trace theorem shows that $(\xi \mid \eta)$ is just the hermitian product of spinors given by the metric on the spinor bundle. For our 3-torus, plainly $\mathcal{H}_\infty = \mathbb{T}^3_\theta \oplus \mathbb{T}^3_\theta$ is a projective (indeed, free) left module over $\mathbb{T}^3_\theta$, and the hermitian structure is also manifest.

Axiom 7 (P-duality)**.** *The Fredholm index of the operator D yields a nondegenerate intersection form on the K-theory of the algebra $\mathcal{A} \otimes \mathcal{A}^0$.*

We shall not discuss it here for the NC 3-torus, except to say that the K-theory groups of the 3-tori are $K_j(\mathbb{T}^3_\theta) \simeq \mathbb{Z}^4$ for $j = 0, 1$, and all θ.

If we add an "Axiom 0", establishing that $\mathcal{A}$ is the commutative algebra $C^\infty(M)$ of smooth functions on a compact manifold M, then M is spin, and there is a distinguished representation of the geometry for which π is unitarily equivalent to the representation of $\mathcal{A}$ by multiplication operators on the canonical spinor space, and D to the canonical Dirac operator $\not{D}$ [3]. Also, $C^\infty(M)$ is Morita equivalent to the Clifford algebra over M [25].

Of course, Axiom 7 is then redundant. It is to be hoped that the same conclusions may be obtained by just stipulating commutativity of the algebra; but we know no proof of that yet.

At any rate, it transpires that the previous axioms constitute an appropriate description of noncommutative spin manifolds. To be sure, much work remains to be done: we do not have classification results.

In general, the fermions will be coupled to a given "external" Yang–Mills configuration, that may be time-dependent, but whose dynamics is not involved in the problem. For the commutative geometry $(C^\infty(M), L^2(M,S), \not{D}, J)$, we may have a nonabelian gauge theory, formulated on a Hermitian G-vector bundle E over M. The Dirac operator then acts on the Hilbert space $L^2(M, S \otimes E)$. Gauge transformations are elements of the group $C^\infty(\operatorname{Aut} E)$ [26]. Pointwise multiplication gives the representation of $C^\infty(\operatorname{Aut} E)$ on the Hilbert space. When E is trivial, $C^\infty(\operatorname{Aut} E) \simeq \operatorname{Map}(M, G)$. Infinitesimal gauge transformations are accordingly defined. Gauge potentials, in the commutative case, are E-valued 1-forms on M, represented on spinor space as Clifford multiplication operators. In the noncommutative case, vector bundles are translated into finitely generated projective (right) modules over the algebra $\mathcal{A}$. The vector bundles over noncommutative n-tori have been all constructed [27] and partially classified up to Morita equivalence [28], and the corresponding gauge transformations are easily determined. Gauge potentials can also be translated to the noncommutative case [19]. In what follows, we leave aside all geometrical complications extraneous to the analytical problem at hand.

3 A Fredholm module interlude

A *cycle* is a complex graded associative algebra $\Omega^\bullet = \bigoplus_{k=0}^\infty \Omega^k$, endowed with a differential $d\colon \Omega^\bullet \to \Omega^\bullet$, i.e., a linear map of degree +1 such that $d^2 = 0$ and

$$d(\omega_k \omega_l) = (d\omega_k)\, \omega_l + (-)^k \omega_k\, d\omega_l$$

when ω_k, ω_l are homogeneous elements of respective degrees k, l; and with an integral $\int$, namely, a linear map $\int : \Omega^\bullet \to \mathbb{C}$ such that

$$\int \omega_k \omega_l = (-)^{kl} \int \omega_l \omega_k \quad \text{and} \quad \int d\omega = 0 \text{ for any } \omega \in \Omega^\bullet.$$

We refer to the last property as closedness of the integral. A cycle over an algebra $\mathcal{A}$ is a cycle $(\Omega^\bullet, d, \int)$ together with a homomorphism from $\mathcal{A}$ to Ω^0. The simplest examples are afforded by de Rham complexes.

A truly interesting class of examples comes from *Fredholm modules* over a given algebra $\mathcal{A}$. An odd Fredholm module over $\mathcal{A}$ is given by an involutive representation π of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ and a symmetry (selfadjoint unitary operator) F such that $[F, \pi(a)]$ is a compact operator for all $a \in \mathcal{A}$. Let $\mathcal{H}^{\pm}$ denote the eigenspaces for the ± 1 eigenvalues of F. Then we may write any operator T as

$$T = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

where $\alpha\colon \mathcal{H}^+ \to \mathcal{H}^+$, $\beta\colon \mathcal{H}^- \to \mathcal{H}^+$ and so on. For a given F, T is thus decomposed into "linear" and "antilinear" parts:

$$T = T_+ + T_- := \tfrac{1}{2}(T + FTF) + \tfrac{1}{2}(T - FTF) = \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} + \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}.$$

To define an integral, let us postulate a summability condition on the algebra: for all $a \in \mathcal{A}$ and for some chosen nonnegative integer n, we assume that a_- belongs to the Schatten class $\mathcal{L}^{n+1}(\mathcal{H})$. The graded differential algebra structure is introduced as follows: define $\Omega^k(\mathcal{A})$ as the space spanned by forms $a_0\, \mathrm{d}a_1 \dots \mathrm{d}a_k$ with $a_0, a_1, \dots, a_k \in A$, where $\mathrm{d}a := [F, a]$. The algebra multiplication is the operator product. Given an operator T on $\mathcal{H}$, we introduce its *conditional* trace:

$$\operatorname{Tr}_C T := \operatorname{Tr} T_+.$$

Note that $\operatorname{Tr}_C(AB) = \operatorname{Tr}_C(BA)$ when both sides make sense, and that $\operatorname{Tr}_C T := \operatorname{Tr} T$, if $T \in \mathcal{L}^1$, by cyclicity of the trace. Assuming that n is odd, $(\omega_n)_+ \in \mathcal{L}^1$ [4,6]. Therefore, it makes sense to define the integral by

$$\int \omega_n := \operatorname{Tr}_C \omega_n = \tfrac{1}{2} \operatorname{Tr}\ F\, \mathrm{d}\omega_n.$$

We shall then say that (the cycle associated to) the Fredholm module has dimension n. The *Chern character* of that cycle is defined to be the $(n+1)$-linear functional on $\mathcal{A}$ given by

$$\tau(a_0, a_1, \dots, a_n) := \operatorname{Tr}_C(a_0\, \mathrm{d}a_1\, \mathrm{d}a_2 \cdots \mathrm{d}a_n).$$

Now $b\tau = 0$, since

$$\begin{aligned} &\operatorname{Tr}_C((a_0 a_1\, \mathrm{d}a_2 \cdots \mathrm{d}a_{n+1}) + \sum_{i=1}^{n} (-)^i \operatorname{Tr}_C\big(a_0\, \mathrm{d}a_1 \cdots (\mathrm{d}a_i\, a_{i+1} + a_i\, \mathrm{d}a_{i+1}) \cdots \mathrm{d}a_{n+1}\big) \\ &\qquad + (-)^{n+1} \operatorname{Tr}_C(a_{n+1} a_0\, \mathrm{d}a_1 \cdots \mathrm{d}a_n) \\ &= (-)^n \operatorname{Tr}_C((a_0\, \mathrm{d}a_1 \cdots \mathrm{d}a_n) a_{n+1}) + (-)^{n+1} \operatorname{Tr}_C(a_{n+1} a_0\, \mathrm{d}a_1 \cdots \mathrm{d}a_n) = 0, \end{aligned}$$

by telescoping; the last equality is just the trace property $\int a\omega = \int \omega a$ for $a \in \Omega^0$, $\omega \in \Omega^n$. Thus τ is an n-cocycle. Moreover, τ is *cyclic*:

$$\begin{aligned} \tau(a_0, a_1, \dots, a_n) &= (-)^{n-1} \operatorname{Tr}_C(\mathrm{d}a_2 \cdots \mathrm{d}a_n\, a_0\, \mathrm{d}a_1) \\ &= (-)^n \operatorname{Tr}_C(\mathrm{d}a_2 \cdots \mathrm{d}a_n\, \mathrm{d}a_0\, a_1) \\ &= (-)^n \operatorname{Tr}_C(a_1\, \mathrm{d}a_2 \cdots \mathrm{d}a_n\, \mathrm{d}a_0) = (-)^n \tau(a_1, \dots, a_n, a_0), \end{aligned}$$

where we have used that $\mathrm{d}a_0\, a_1 + a_0\, \mathrm{d}a_1 = \mathrm{d}(a_0 a_1)$ and the closedness of Tr_C.

Given a Dirac operator D on a spin geometry of dimension n (e.g., an n-torus), there is a God-given Fredholm module coming from the phase operator $D/|D|$. The minimal integer for which the character exists, for this Fredholm module structure, we call the "quantum dimension" of the spin space. Note that (non)commutativity of $\mathcal{A}$ does not play any role in the foregoing.

4 Second quantization

We now review the algebraic machinery of canonical quantization, and investigate its general properties of application. It is important to realize that the basic ingredient of the construction is just a real Hilbert space. So suppose an infinite-dimensional real vector space V and a symmetric bilinear form d are given, the metric space (V, d) being complete. The first object in quantization is the field algebra over the space (V, d), which is just the complexified Clifford algebra $\mathfrak{A}(V) := \mathrm{C}\ell(V, d) \otimes \mathbb{C}$, complete in the (inductive limit) C^*-norm [29]. The *fermion field* is a linear map $B\colon V \to \mathfrak{A}(V)$ satisfying $[B(v), B(v')]_+ = 2\, d(v, v')$ for all $v, v' \in V$. Any two C^*-algebras generated by two sets of operators obeying the same rules are isomorphic [7].

The orthogonal group $\mathrm{O}(V)$ is $\{\, g \in \mathrm{GL}_{\mathbb{R}}(V) : d(gu, gv) = d(u, v) \text{ for all } u, v \in V \,\}$. A *complex structure* K is an orthogonal operator on V satisfying $K^2 = -1$. Now, introducing the rule $(\alpha + i\beta)v := \alpha v + \beta K v$ for α, β real, the hermitian form

$$\langle u \mid v \rangle_K := d(u, v) + i d(Ku, v)$$

makes (V, d, K) a complex Hilbert space. Once a particular complex structure K has been selected, one can decompose elements of $\mathrm{O}(V)$ as $g = p_g + q_g$ where p_g, q_g are its linear and antilinear parts: $p_g := \frac{1}{2}(g - KgK)$, $q_g := \frac{1}{2}(g + KgK)$. The "restricted orthogonal group" $\mathrm{O}_K(V)$ is the subgroup of $\mathrm{O}(V)$ consisting of those g for which q_g is Hilbert–Schmidt.

One can construct a faithful irreducible representation π_K of $\mathfrak{A}(V)$ by the GNS construction with respect to the "Fock state" ω_K determined by $\omega_K(B(u)B(v)) := \langle u \mid v \rangle_K$; this is the standard representation on the fermion Fock space $\mathcal{F}_K(V)$, with vacuum Ω, in which the creation and annihilation operators are defined as real-linear operators:

$$a_K^\dagger(v) := \pi_K B(P_K v), \qquad a_K(v) := \pi_K B(P_{-K} v),$$

where $P_K := \frac{1}{2}(1 - iK)$.

For g orthogonal, the map $w \mapsto B(gw)$ extends to a $*$-automorphism of the CAR algebra $\mathfrak{A}(V)$. We then ask when these two quantizations are unitarily equivalent, i.e., whether this $*$-automorphism is unitarily implementable on $\mathcal{F}_K(V)$. For a given $g \in \mathrm{O}(V)$, we seek a unitary operator $\mu(g)$ on $\mathcal{F}_K(V)$ so that

$$\mu(g)B(v) = B(gv)\mu(g), \quad \text{for all} \quad v \in V.$$

The complex structure K is transformed to gKg^{-1}; the creation and annihiliation operators undergo a *Bogoliubov transformation*:

$$\begin{aligned} a_{gKg^{-1}}^\dagger(gv) &= a_K(q_g v) + a_K^\dagger(p_g v), \\ a_{gKg^{-1}}(gv) &= a_K(p_g v) + a_K^\dagger(q_g v). \end{aligned}$$

Were $\mu(g)$ to exist, we would then have

$$\begin{aligned} \mu(g)a_K^\dagger(v) &= a_{gKg^{-1}}^\dagger(gv)\mu(g), \\ \mu(g)a_K(v) &= a_{gKg^{-1}}(gv)\mu(g). \end{aligned}$$

The out-vacuum $\mu(g)\Omega$ is annihilated by $a_{gKg^{-1}}(gv)$, for all $v \in V$. From there the Shale–Stinespring criterion [30] for implementability is easily established: the operator $\mu(g)$ exists if and only if g belongs to the restricted orthogonal group. Naturally, the map $g \mapsto \mu(g)$ is only a projective representation of $\mathrm{O}_K(V)$. The explicit construction of μ was performed in our Ref. [13], on which we mostly rely for this section.

The spin representation allows us to quantize all elements of the Lie algebra $\mathfrak{o}_K(V)$ of the group $\mathrm{O}_K(V)$. Define the *infinitesimal spin representation* $\dot\mu(X)$ of $X \in \mathfrak{o}_K(V)$ by:

$$\dot\mu(X)\,\Psi := \frac{d}{dt}\bigg|_{t=0} e^{i\theta_X(t)}\mu(\exp tX)\,\Psi$$

for $\Psi \in \mathcal{F}_K(V)$, where $\theta_X(t)$ is such that $t \mapsto e^{i\theta_X(t)}\mu(\exp tX)$ is a homomorphism. The vacuum expectation value of $\dot\mu(X)$ is $\langle\Omega \mid \dot\mu(X)\Omega\rangle = i\theta'_X(0)$. We set $\theta'_X(0) = 0$ for all $X \in \mathfrak{o}_K(V)$. The quantization rule $X \mapsto \dot\mu(X)$ then is uniquely specified by the condition of vanishing vacuum expectation values.

The fundamental property of the infinitesimal spin representation is the commutation relations:

$$[\dot\mu(X), B(v)] = B(Xv),$$

an operator-valued equation valid on a dense domain in $\mathcal{F}_K(V)$, that justifies the name "currents" for the quantized observables. An easy computation [13] gives

$$[\dot\mu(X), \dot\mu(Y)] - \dot\mu([X,Y]) = \tfrac{1}{4}\operatorname{Tr}(K[K,X][K,Y])$$

when $X, Y \in \mathfrak{o}_K(V)$, for the Schwinger terms.

We reexpress the quantization prescription in the language of creation and annihilation operators. Given orthonormal bases $\{e_j\}$, $\{f_k\}$ of (V, d, K), the quadratic expressions:

$$\begin{aligned}
a^\dagger T a^\dagger &:= \sum_{j,k} a_K^\dagger(f_k)\,\langle f_k \mid T e_j\rangle\, a_K^\dagger(e_j),\\
aTa &:= \sum_{j,k} a_K(e_j)\,\langle T e_j \mid f_k\rangle\, a_K(f_k),\\
a^\dagger C a &:= \sum_{j,k} a_K^\dagger(f_k)\,\langle f_k \mid C e_j\rangle\, a_K(e_j),
\end{aligned}$$

are independent of the orthonormal bases used; T is antilinear and skew, and C is linear, as operators on V. The series $a^\dagger T a^\dagger$, aTa are meaningful in Fock space if and only if T is Hilbert–Schmidt. If C_X is the linear part of X and A_X the antilinear part, we thus get:

$$\dot\mu(X) = \tfrac{1}{2}(a^\dagger A_X a^\dagger + 2a^\dagger C_X a - a A_X a). \tag{1}$$

In most cases, including our neutrino fields over noncommutative tori, V is a complex Hilbert space to start with. The original complex structure contains important physical information; but we have seen that the first step of second quantization is to forego and replace it with a new complex structure adapted to the dynamical problem at hand. If V has this additional structure, then unitary elements of $\mathcal{L}_{\mathbb{C}}(V)$ are obviously orthogonal; and selfadjoint elements of $\mathcal{L}_{\mathbb{C}}(V)$ are of the form iX, with $X \in \mathfrak{o}_K(V)$. In this context, it is plain that if F is a symmetry defining a Fredholm module, then iF becomes a complex structure on the realification of $\mathcal{H}$.

Suppose, moreover, that F is the phase of the Dirac operator on a spin geometry, commutative if one wishes, and let M denote the underlying manifold, with dimension n. Then F defines the very complex structure we naturally use to quantize fermions over M: we can think of the F-eigenspaces $\mathcal{H}_+$ and $\mathcal{H}_-$ as the spaces of positive and negative energy solutions, respectively, of the Dirac equation

$$i\left(\frac{\partial}{\partial t} - D\right)\psi = 0,$$

so the construction of the new Hilbert space with complex structure iF is equivalent to filling up the Dirac sea. In fact, $iD/|D|$ is the *unique* complex structure for which D becomes a positive generator. Then the quantization prescription (1) effects normal ordering; it is equivalent to the one defined in [9] – although our formalism is more general.

The outcome of the previous discussion is that an orthogonal operator O on the single-particle space can be second-quantized (by means of the spin representation) to an operator on the Fock space associated to the "free" evolution iff $[F, O]$ is Hilbert–Schmidt, and an infinitesimally orthogonal operator Z on the single-particle space can be second-quantized (by means of the infinitesimal spin representation) to an operator on the same Fock space iff $[F, Z]$ is Hilbert–Schmidt. In the complex context, the orthogonal operator will be actually in most cases unitary with respect to the original or fiducial complex structure, and the infinitesimally orthogonal one actually skewadjoint.

In the commutative case, if g is a multiplication operator, then $[F, g] \in \mathcal{L}^{n+1}(\mathcal{H})$. The proof relies on pseudodifferential operators: if T is pseudodifferential of order $m < 0$, then it belongs to the Schatten class $\mathcal{L}^p$ for all $p > -n/m$. This can be deduced from the Cesàro asymptotic development of the spectral density of such operators [31]. Now, F and g are of order 0, so $[F, g]$ is of order -1. Therefore $[F, g] \in \mathcal{L}^p(\mathcal{H})$ for all $p > n$, in particular for $p = n + 1$. As hinted at the end of Section 2, this conclusion is not altered when g is replaced by an element of a more complicated projective module over $C^\infty(M)$, representing a gauge theory on M.

The Schatten class of $[F, g]$, thus the "quantum dimension", measures the degree of ultraviolet divergence of the theory. We have seen that, at least for commutative manifolds, the classical and quantum dimensions coincide. For $1 + 1$ quantum field theory, the character is identical to the Schwinger term; the Shale–Stinespring criterion is satisfied for any g, and so normal ordering is sufficient to regularize the theory. In fact, it is even sufficient to regularize the fully interacting gauged Wess–Zumino–Witten model! [32]. This is not so for $1 + 3$ quantum field theory, where the gauge transformations themselves cannot be unitarily implemented in general.

5 Quantum dimension = classical dimension for NC tori

A gauge transformation for the trivial line bundle over $\mathbb{T}^3_\theta$ is just a unitary element X of this algebra. For irrational θ, $\mathbb{T}^3_\theta$ is a highly nonlocal algebra, and one might expect that its quantum dimension would be less than 3, namely, that typically $[F, X] \in \mathcal{L}^p$ for some $p \leqslant 3$. But this is not the case: indeed, the nonlocality of the irrational 3-torus does nothing to improve that particular test of ultraviolet behaviour.

We may write $X = a_r u^r$ with $\{a_r\} \in \mathcal{S}(\mathbb{Z}^3)$; then $X^* = \bar{a}_r u^{-r} = \bar{a}_{-s} u^s$ and $X^*X = \lambda(m, r)\, \bar{a}_r a_{r+m}\, u^m$, so that X is unitary if and only if

$$\textstyle\sum_r |a_r|^2 = 1, \qquad \sum_r \lambda(m, r)\, \bar{a}_r a_{r+m} = 0 \quad \text{for} \quad m \neq 0.$$

The unitarity conditions in particular imply that a *finite* sum $X = a_r u^r$ can be unitary only if it contains just one summand, i.e., X is a multiple of some u^r.

We start from the computation carried out by Mickelsson and Rajeev [33] ten years ago for commutative tori. With respect to the orthonormal basis $\{\, \psi_n, \psi'_n : n \in \mathbb{Z}^3 \,\}$ for $\mathcal{H}$, the matrix entries of the operator $A = [F, X]$ are given by

$$[F, X]\psi_n = \sum_r \lambda(r, n) a_r \left(\frac{(n+r)\cdot\sigma}{|n+r|} - \frac{n\cdot\sigma}{|n|} \right) \psi_{n+r},$$

and similarly for the ψ'_n. To obtain the Schatten class of A, we must determine the finiteness of the p-norm

$$\|A\|_p := \big(\mathrm{Tr}(A^*A)^{p/2}\big)^{1/p},$$

which is in general hard to compute. A simpler alternative is to calculate

$$|||A|||_p := \left(\sum_n \|A\psi_n\|^p + \|A\psi'_n\|^p \right)^{1/p},$$

or its analogue with any other orthonormal basis of $\mathcal{H}$. However, these are not equivalent norms unless $p = 2$, *pace* Ref. [33]. It is known [34] that $\|A\|_p \leqslant |||A|||_p$ if $1 \leqslant p \leqslant 2$, whereas $|||A|||_p \leqslant \|A\|_p$ if $p \geqslant 2$. Thus, in general, for $p > 2$ the divergence of $|||A|||_p$ implies that $A \notin \mathcal{L}^p$, but not conversely.

For the particular case $A = [F, u^r]$ this does not matter, since A^*A is diagonal in the chosen basis. Indeed,

$$\begin{aligned} [F, u^r]^*[F, u^r]\psi_n &= \bar\lambda(r, n+r)\,\lambda(r, n) \left(\frac{(n+r)\cdot\sigma}{|n+r|} - \frac{n\cdot\sigma}{|n|} \right)^2 \psi_n \\ &= 2\left(1 - \frac{(n+r)\cdot n}{|n+r|\,|n|} \right) \psi_n, \end{aligned}$$

since $\lambda(r, n)\,\bar\lambda(r, n+r) = |\lambda(r, n)|^2 = 1$ (using the antisymmetry of θ). Similar formulas obtain for $[F, u^r]^*[F, u^r]\psi'_n$. Thus

$$\begin{aligned} \|[F, u^r]\|_p^p = |||[F, u^r]|||_p^p &= 2^{1+p/2} \sum_n \left(1 - \frac{(n+r)\cdot n}{|n+r|\,|n|} \right)^{p/2} \\ &= 2 \sum_n \left(\frac{|r|^2}{|n|^2} - \frac{(r\cdot n)^2}{|n|^4} + O(|n|^{-3}) \right)^{p/2}, \end{aligned}$$

so that $[F, u^r] \in \mathcal{L}^p$ if and only if $\sum_{n\neq 0} |n|^{-p}$ converges, if and only if $\int_1^\infty \rho^{2-p}\, d\rho$ converges, if and only if $p > 3$.

For the general case $A = [F, X]$, the matrix of A^*A has off-diagonal terms, but one generally finds that $|||[F, X]|||_p^p$ diverges for $p \leqslant 3$, so that $[F, X] \notin \mathcal{L}^3$. But we can show, with the same type of arguments, that $[F, X] \in \mathcal{L}^4$ for any $X = a_r u^r \in \mathbb{T}^3_\theta$. Since

$$\begin{aligned} [F, X]^*[F, X]\psi_n = \sum_{r,s} \bar\lambda(r, s)\, \bar a_r a_s &\left(\frac{(n+s)\cdot\sigma}{|n+s|} - \frac{(n-r+s)\cdot\sigma}{|n-r+s|} \right) \\ \times \quad &\left(\frac{(n+s)\cdot\sigma}{|n+s|} - \frac{n\cdot\sigma}{|n|} \right) \psi_{n-r+s}, \end{aligned}$$

and $\|[F,X]\|_4^4 = \|B\|_2^2 = \sum_n \|B\psi_n\|^2 + \|B\psi_n'\|^2$ with $B = [F,X]^*[F,X]$, and furtheremore since $\|(p\cdot\sigma)(q\cdot\sigma)\psi_n\|^2 + \|(p\cdot\sigma)(q\cdot\sigma)\psi_n'\|^2 = 2|p|^2|q|^2$, we obtain, after replacing n by $n-s$,

$$\begin{aligned}\|[F,X]\|_4^4 &= 2\sum_{n,r,s} |\bar\lambda(r,s)\,\bar a_r a_s|^2 \left|\frac{n}{|n|} - \frac{(n-r)}{|n-r|}\right|^2 \left|\frac{n}{|n|} - \frac{(n-s)}{|n-s|}\right|^2 \\ &= 8\sum_{n,r,s} |\bar a_r a_s|^2 \left(1 - \frac{n\cdot(n-r)}{|n|\,|n-r|}\right)\left(1 - \frac{n\cdot(n-s)}{|n|\,|n-s|}\right) \\ &= 2\sum_{n,r,s} \left(\frac{|r|^2}{|n|^2} - \frac{(r\cdot n)^2}{|n|^4}\right)|a_r|^2 \left(\frac{|s|^2}{|n|^2} - \frac{(s\cdot n)^2}{|n|^4}\right)|a_s|^2 + O(|n|^{-5}),\end{aligned}$$

which converges since $|r|\,a_r$ is a square-summable sequence because $a \in \mathcal{S}(\mathbb{Z}^3)$. Thus the quantum dimension of $\mathbb{T}_\theta^3$ is 3.

Results of this kind are independent of the torus parameters θ_{jk}, so from the dimensional standpoint the ultraviolet behaviour is exactly the same for all 3-tori, commutative or not.

6 The noncommutative Chern character theorem

We have seen that for NC tori, the quantum dimension, as measured by the character given by the phase operator F, equals the quantum dimension. (It should be clear that the calculations for $n=3$ yield analogous results for higher odd n.) What is the underlying reason for this?

One of the deepest results in noncommutative geometry is that the noncommutative integral defined by a generalized Dirac operator D and the character given by its phase operator F have the same values on "volume forms". This is the content of Connes' Hauptsatz [6, p. 308]: if n is odd,

$$\mathrm{Tr}_C\Big(\textstyle\sum_j a_0^j\,[F,a_1^j]\cdots[F,a_n^j]\Big) = \displaystyle⨍ \textstyle\sum_j a_0^j\,[D,a_1^j]\cdots[D,a_n^j]\,ds^n, \tag{2}$$

whenever $\sum_j a_0^j \otimes a_1^j \otimes \cdots \otimes a_n^j$ is a Hochschild n-cycle on the algebra $\mathcal{A}$.

Assume that the classical dimension of a spin geometry is n, and that Hochschild cohomology of $\mathcal{A}$ is the dual of its Hochschild homology. If the cohomological dimension of the character (what we have called the "quantum dimension" of the geometry) were lower, say $(n-2)$ – it must still be an odd integer – then the character τ_n would necessarily [6, p. 294] be of the form $(-2/n)\,S\tau_{n-2}$, where τ_{n-2} is the analogous character in degree $(n-2)$ and the periodicity operator S promotes cyclic $(n-2)$-cocycles to cyclic n-cocycles. However, promoted cyclic cocycles are always Hochschild-cohomologous to zero; if $\boldsymbol{c}$ denotes the cycle whose representative on $\mathcal{H}$ fulfils $\pi_D(\boldsymbol{c}) = 1$ (Axiom 3 in Section 2), this would imply $⨍ ds^n = ⨍ \pi_D(\boldsymbol{c})\,ds^n = (-2/n)\,S\tau_{n-2}(\boldsymbol{c}) = 0$, which is not possible in classical dimension n. In fine, the quantum dimension is not lower than n. On the other hand [35], the summability of $[F,a]$ is no worse than that of $[D,a]\,|D|^{-1}$, which implies the converse inequality.

By direct computation, Langmann found [36], for the usual spin geometry on $\mathbb{R}^n$, that the character determined by the phase operator $F = \not{D}/|\not{D}|$ is given, up to a constant factor, by an ordinary de Rham integral:

$$\mathrm{Tr}_C(a_0\,[F,a_1]\cdots[F,a_n]) = \widetilde{C}_n \int_{\mathbb{R}^n} \mathrm{tr}(a_0\,da_1\cdots da_n)$$

of smooth, compactly supported matrix-valued functions on $\mathbb{R}^n$. (The constant $\widetilde{C}_n$ and the C_n of Section 2 differ only by a factor of modulus 1.) The integral on the right hand side is in fact a *noncommutative* integral, due to the trace theorem of Connes [37]: on a spin manifold M, the following identity holds:

$$⨍ a_0\,[D\!\!\!/, a_1] \cdots [D\!\!\!/, a_n]\, ds^n = \widetilde{C}_n \int_M a_0\, da_1 \cdots da_n. \tag{3}$$

This is proved for compact manifolds by use of the Wodzicki residue [22,23,37]. Our results in [31] extend the validity of (3) to $\mathbb{R}^n$, for compactly supported functions. Therefore, in the commutative case, the integral identity (2) subsumes the formula given by Langmann.

To summarize, the Fredholm character and the integral give equal results when evaluated on a volume form. Commutativity has nothing to do with the matter – except to allow the noncommutative integral to be rewritten as an ordinary integral.

While a proof of (2) is not given in [6], it is a special case of an even more general index theorem proved in [38]. Thus, at the very heart of NCG, there is a barrier to the improvement of ultraviolet behaviour by abandoning locality of the fields. This is perhaps not a bad thing, given that spacetime behaves at long distances as a commutative manifold of fixed dimension. Of course, time is still counted as a c-number here, both before and after quantization. It may still happen that in fully interacting theories, the noncommutativity of space introduces couplings that soften the ultraviolet divergences. At any rate, we expect fermion fields over noncommutative spaces – in particular over Kronecker foliation algebras, that may prove the more pertinent ones in M-theory – to be regularizable by a direct generalization of the methods developed in [9], which go beyond the $1 + 1$ case; to our mind, this is one of the outstanding issues.

7 Conclusion

Quantization, in the Hamiltonian formalism, amounts to substituting q-numbers for the canonical variables. Connes' mathematical theory leads to consider nc-numbers generalizing c-numbers, probing singular geometries (in fact, one can argue that the Standard Model encodes the true, noncommutative geometry of the world [39–41]). We have shown a conceptually consistent way of making nc-numbers into q-numbers. This points to a fusion of quantum field theory and geometry, and promises to widen the present-day scope of both.

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