Moyal planes are spectral triples

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Abstract

Axioms for nonunital spectral triples, extending those introduced in the unital case by Connes, are proposed. As a guide, and for the sake of their importance in noncommutative quantum field theory, the spaces $\mathbb{R}^{2N}$ endowed with Moyal products are intensively investigated. Some physical applications, such as the construction of noncommutative Wick monomials and the computation of the Connes–Lott functional action, are given for these noncommutative hyperplanes.

Contents

1 Introduction

Since Seiberg and Witten conclusively confirmed [79] that the endpoints of open strings in a magnetic field background effectively live on a noncommutative space, string theory has given much impetus to noncommutative field theory (NCFT). This noncommutative space turns out to be of the Moyal type, for which there already existed a respectable body of mathematical knowledge, in connection with the phase-space formulation of quantum mechanics [65].

However, NCFT is a problematic realm. Its bane is the trouble with both unitarity and causality [39, 78]. Feynman rules for NCFT can be derived either using the canonical operator formalism for quantized fields, working with the scattering matrix in the Heisenberg picture by means of Yang–Feldman–Källén equations; or from the functional integral formalism. These two approaches clash [3], and there is the distinct possibility that both fail to make sense. The difficulties vanish if we look instead at NCFT in the Euclidean signature. Also, in spite of the tremendous influence on NCFT, direct and indirect, of the work by Connes, it is surprising that NCFT based on the Moyal product as currently practiced does not appeal to the spectral triple formalism.

So we may, and should, raise a basic question: namely, whether the Euclidean version of Moyal noncommutative field theory is compatible with the full strength of Connes' formulation of noncommutative geometry, or not.

The prospective benefits of such an endeavour are mutual. Those interested in applications may win a new toolkit, and Connes' paradigm stands to gain from careful consideration of new examples.

In order to speak of noncommutative spaces endowed with topological, differential and metric structures, Connes has put forward an axiomatic scheme for "noncommutative spin manifolds", which in fact is the end product of a long process of learning how to express the concept of an ordinary spin manifold in algebraic and operatorial terms.

A *compact* noncommutative spin manifold consists of a *spectral triple* $(\mathcal{A}, \mathcal{H}, D)$, subject to the six or seven special conditions laid out in [19] – and reviewed below in due course. Here $\mathcal{A}$ is a unital algebra, represented on a Hilbert space $\mathcal{H}$, together with a distinguished selfadjoint operator, the abstract Dirac operator D, whose resolvent is completely continuous, such that each operator $[D, a]$ for $a \in \mathcal{A}$ is bounded. A spectral triple is even if it possesses a $\mathbb{Z}_2$-grading operator χ commuting with $\mathcal{A}$ and anticommuting with D.

The key result is the reconstruction theorem [19, 20] which recovers the classical geometry of a compact spin manifold M from the noncommutative setup, once the algebra of coordinates is assumed to be isomorphic to the space of smooth functions $C^\infty(M)$. Details of this reconstruction are given in [45, Chapters 10 and 11] and in a different vein in [71].

Thus, for compact noncommutative spaces, the answer to our question is clearly in the affirmative. Indeed the first worked examples of noncommutative differential geometries are the noncommutative tori (NC tori), as introduced already in 1980 [14, 74]. It is a simple observation that the NC torus can be obtained as an ordinary torus endowed with a periodic version of the Moyal product. The NC tori have been thoroughly exploited in NCFT [24, 92].

The restriction to compact noncommutative spaces ("compactness" being a metaphor for the unitality of the coordinate algebra $\mathcal{A}$) is essentially a technical one, and no fundamental obstacle to extending the theory of spectral triples to nonunital algebras was foreseen. However, it is fair to say that so far a complete treatment of the nonunital case has not been written down. (There have been, of course, some noteworthy partial treatments: one can mention [41, 73], which identify some of the outstanding issues.) The time has come to add a new twist to the tale.

In this article we show in detail how to build noncompact noncommutative spin geometries. The indispensable commutative example of noncompact manifolds is considered first. Then the geometry associated to the Moyal product is laid out. One of the difficulties for doing this is to pin down a "natural" compactification or unitization (embedding of the coordinate algebra as an essential ideal in a unital algebra), the main idea being that the chosen Dirac operator must play a role in this choice.

Since the resolvent of D is no longer compact, some adjustments need to be made; for instance, we now ask for $a(D-\lambda)^{-1}$ to be compact for $a \in \mathcal{A}$ and $\lambda \notin \operatorname{sp} D$. Then, thanks to a variation of the famous Cwikel inequality [27, 81] – often used for estimating bound states of Schrödinger operators – we prove that the spectral triple

$$\big((\mathcal{S}(\mathbb{R}^{2N}), \star_\Theta), L^2(\mathbb{R}^{2N}) \otimes \mathbb{C}^{2^N}, -i\partial_\mu \otimes \gamma^\mu\big),$$

where $\mathcal{S}$ denotes the space of Schwartz functions and $\star_\Theta$ a Moyal product, is $2N^+$-summable and has in fact the spectral dimension $2N$. The interplay between all suitable algebras containing $(\mathcal{S}(\mathbb{R}^{2N}), \star_\Theta)$ must be validated by the orientation and finiteness conditions [19, 20]. In so doing, we prove that the classical background of modern-day NCFTs does fit in the framework of the rigorous Connes formalism for geometrical noncommutative spaces.

This accomplished, the construction of noncommutative gauge theories, that we perform by means of the primitive form of the spectral action functional, is straightforward. The issue of understanding the fluctuations of the geometry, in order to develop "noncommutative gravity" [12] has not reached a comparable degree of mathematical maturity, and is not examined yet. As a byproduct of our analysis, and although we do not deal here with NCFT proper, a mathematically satisfactory construction of the Moyal–Wick monomials is also given.

The main results in this paper have been announced and summarized in [38].

The first order of business is to review the Moyal product more carefully with due attention paid to the mathematical details.

2 The theory of distributions and Moyal analysis

In this first paragraph we fix the notations and recall basic definitions. For any finite dimension k, let Θ be a real skewsymmetric $k \times k$ matrix, let $s \cdot t$ denote the usual scalar product on Euclidean $\mathbb{R}^k$ and let $\mathcal{S}(\mathbb{R}^k)$ be the space of complex Schwartz (smooth, rapidly decreasing) functions on $\mathbb{R}^k$. One defines, for $f, h \in \mathcal{S}(\mathbb{R}^k)$, the corresponding Moyal or twisted product:

$$f \star_\Theta h(x) := (2\pi)^{-k} \iint f(x - \tfrac{1}{2}\Theta u)\, h(x+t)\, e^{-iu\cdot t}\, d^k u\, d^k t, \tag{2.1}$$

where $d^k x$ is the ordinary Lebesgue measure on $\mathbb{R}^k$. In Euclidean field theory, the entries of Θ have the dimensions of an area. Because Θ is skewsymmetric, complex conjugation reverses the product: $(f \star_\Theta h)^* = h^* \star_\Theta f^*$.

Assume Θ to be nondegenerate, that is to say, $\sigma(s,t) := s \cdot \Theta t$ to be symplectic. This implies even dimension, $k = 2N$. We note that Θ^{-1} is also skewsymmetric; let $\theta > 0$ be defined by $\theta^{2N} := \det \Theta$. Then formula (2.1) may be rewritten as

$$f \star_\Theta h(x) = (\pi\theta)^{-2N} \iint f(x+s)\, h(x+t)\, e^{-2is\cdot\Theta^{-1}t}\, d^{2N}s\, d^{2N}t. \tag{2.2}$$

The latter form is very familiar from phase-space quantum mechanics [40], where $\mathbb{R}^{2N}$ is parametrized by N conjugate pairs of position and momentum variables, and the entries of Θ have the dimensions of an action; one then selects

$$\Theta = \hbar S := \hbar \begin{pmatrix} 0 & 1_N \\ -1_N & 0 \end{pmatrix}.$$

Indeed, the product $\star$ (or rather, its commutator) was introduced in that context by Moyal [65], using a series development in powers of $\hbar$ whose first nontrivial term gives the Poisson bracket; later, it was rewritten in the above integral form. These are actually oscillatory integrals, of which Moyal's series development,

$$f \star_\hbar g(x) = \sum_{\alpha\in\mathbb{N}^{2N}} \left(\frac{i\hbar}{2}\right)^{|\alpha|} \frac{1}{\alpha!}\, \frac{\partial f}{\partial x^\alpha}(x)\, \frac{\partial g}{\partial (Sx)^\alpha}(x), \tag{2.3}$$

is an asymptotic expansion. The development (2.3) holds – and sometimes becomes exact – under conditions spelled out in [33]. The first integral form (2.1) of the Moyal product was exploited by Rieffel in a remarkable monograph [75], who made it the starting point for a more general deformation theory of C^*-algebras.

Since the problems we are concerned with in this paper are of functional analytic nature, there is little point in using the most general Θ here: we concentrate on the nondegenerate case and adopt the form $\Theta = \theta S$ with θ real. Therefore, the corresponding Moyal products are indexed by the real parameter θ; we denote them by $\star_\theta$ and usually omit explicit reference to N in the notation.

The plan of the rest of this section is roughly as follows. The Schwartz space $\mathcal{S}(\mathbb{R}^{2N})$ endowed with these products is an algebra without unit and its unitization will not be unique. Below, after extending the Moyal product to large classes of distributions, we find and choose unitizations suitable for our construction of a noncompact spectral triple, and show that $(\mathcal{S}(\mathbb{R}^{2N}), \star_\theta)$ is a pre-C^*-algebra. We prove that the left Moyal product by a function $f \in \mathcal{S}(\mathbb{R}^{2N})$ is a regularizing operator on $\mathbb{R}^{2N}$. In connection with that, we examine the matter of Calderón–Vaillancourt-type theorems in Moyal analysis. We inspect as well the relation of our compactifications with NC tori.

2.1 Basic facts of Moyalology

With the choice $\Theta = \theta S$ made, the Moyal product can also be written

$$f \star_\theta g(x) := (\pi\theta)^{-2N} \iint f(y) g(z)\, e^{\frac{2i}{\theta}(x-y)\cdot S(x-z)}\, d^{2N}y\, d^{2N}z. \tag{2.4}$$

Of course, our definitions make sense only under certain hypotheses on f and g. A good chunk of Moyal analysis can be found in [43, 90], from which we extract the following lemma.

Lemma 2.1. [43] *Let $f, g \in \mathcal{S}(\mathbb{R}^{2N})$. Then*

(i) $f \star_\theta g \in \mathcal{S}(\mathbb{R}^{2N})$.

(ii) *$\star_\theta$ is a bilinear associative product on $\mathcal{S}(\mathbb{R}^{2N})$. Moreover, complex conjugation of functions $f \mapsto f^*$ is an involution for $\star_\theta$.*

(iii) *Let $j = 1, 2, \ldots, 2N$. The Leibniz rule is satisfied:*

$$\frac{\partial}{\partial x_j}(f \star_\theta g) = \frac{\partial f}{\partial x_j} \star_\theta g + f \star_\theta \frac{\partial g}{\partial x_j}. \tag{2.5}$$

(iv) *Pointwise multiplication by any coordinate x_j obeys*

$$x_j(f \star_\theta g) = f \star_\theta (x_j g) + \frac{i\theta}{2} \frac{\partial f}{\partial (Sx)_j} \star_\theta g = (x_j f) \star_\theta g - \frac{i\theta}{2} f \star_\theta \frac{\partial g}{\partial (Sx)_j}. \tag{2.6}$$

(v) *The product has the tracial property:*

$$\langle f, g\rangle := \frac{1}{(\pi\theta)^N} \int f \star_\theta g(x)\, d^{2N}x = \frac{1}{(\pi\theta)^N} \int g \star_\theta f(x)\, d^{2N}x = \frac{1}{(\pi\theta)^N} \int f(x)\, g(x)\, d^{2N}x.$$

(vi) *Let $L^\theta_f \equiv L^\theta(f)$ be the left multiplication $g \mapsto f \star_\theta g$. Then $\lim_{\theta\downarrow 0} L^\theta_f\, g(x) = f(x)\, g(x)$, for $x \in \mathbb{R}^{2N}$.*

Property (vi) is a consequence of the distributional identity $\lim_{\varepsilon\downarrow 0} \varepsilon^{-k} e^{ia\cdot b/\varepsilon} = (2\pi)^k \delta(a)\delta(b)$, for $a, b \in \mathbb{R}^k$; convergence takes place in the standard topology [77] of $\mathcal{S}(\mathbb{R}^{2N})$. To simplify notation, we put $\mathcal{S} := \mathcal{S}(\mathbb{R}^{2N})$ and let $\mathcal{S}' := \mathcal{S}'(\mathbb{R}^{2N})$ be the dual space of tempered distributions. In view of (vi), we may denote by L^0_f the pointwise product by f.

Theorem 2.2. [43] *$\mathcal{A}_\theta := (\mathcal{S}, \star_\theta)$ is a nonunital associative, involutive Fréchet algebra with a jointly continuous product and a distinguished faithful trace.*

Introduce the symplectic Fourier transform F by

$$Ff(x) := (2\pi)^{-N} \int f(t) e^{ix\cdot St}\, d^{2N}t. \tag{2.7}$$

It is obviously a symmetry, i.e., an involutive selfadjoint operator. Since $\delta \star_\theta \delta = (\pi\theta)^{-2N}$, the maps $f \mapsto (\pi\theta)^N \delta \star_\theta f$ and $f \mapsto f \star_\theta (\pi\theta)^N \delta$ are unitary, too; they turn out to be

$$[(\pi\theta)^N \delta \star_\theta f](y) = (2/\theta)^N\, Ff(-2y/\theta), \qquad [f \star_\theta (\pi\theta)^N \delta](y) = (2/\theta)^N\, Ff(2y/\theta).$$

This prompts us to consider the unitary dilation operators E_a given by

$$E_a f(x) := a^{N/2} f(a^{1/2}x),$$

and it is immediate from (2.7) that $FE_a = E_{1/a}F$. We also remark that

$$f \star_\theta g = (\theta/2)^{-N/2} E_{2/\theta}(E_{\theta/2} f \star_2 E_{\theta/2} g). \tag{2.8}$$

Nearly all formulas in this paper simplify when $\theta = 2$. Thanks to the scaling relation (2.8), it is often enough, when studying properties of the Moyal product, to work out the case $\theta = 2$.

2.2 The oscillator basis

Definition 2.1. The algebra $\mathcal{A}_\theta$ has a natural basis of eigentransitions f_{mn} of the harmonic oscillator, indexed by $m, n \in \mathbb{N}^N$. As usual, for $m = (m_1, \dots, m_N) \in \mathbb{N}^N$, we write $|m| := m_1 + \cdots + m_N$ and $m! := m_1! \cdots m_N!$. If

$$H_l := \tfrac{1}{2}(x_l^2 + x_{l+N}^2) \quad \text{for } l = 1, \dots, N \quad \text{and} \quad H := H_1 + H_2 + \cdots + H_N,$$

then the f_{mn} diagonalize these harmonic oscillator Hamiltonians:

$$H_l \star_\theta f_{mn} = \theta(m_l + \tfrac{1}{2}) f_{mn}, \qquad f_{mn} \star_\theta H_l = \theta(n_l + \tfrac{1}{2}) f_{mn}. \tag{2.9}$$

They may be defined by

$$f_{mn} := \frac{1}{\sqrt{\theta^{|m|+|n|}\, m!n!}} (a^*)^m \star_\theta f_{00} \star_\theta a^n, \tag{2.10}$$

where f_{00} is the Gaussian function $f_{00}(x) := 2^N e^{-2H/\theta}$, and the annihilation and creation functions respectively are

$$a_l := \frac{1}{\sqrt{2}}(x_l + i x_{l+N}) \quad \text{and} \quad a_l^* := \frac{1}{\sqrt{2}}(x_l - i x_{l+N}). \tag{2.11}$$

One finds that $a^n := a_1^{n_1} \dots a_N^{n_N} = a_1^{\star_\theta n_1} \star_\theta \cdots \star_\theta a_N^{\star_\theta n_N}$.

These Wigner eigentransitions are already found in [46] and also in [6]. (Incidentally, the "first" attributions in [36] are quite mistaken.) The f_{mn} can be expressed with the help of Laguerre functions in the variables H_l: see subsection A.1. The next lemma summarizes their chief properties.

Lemma 2.3. [43] *Let* $m, n, k, l \in \mathbb{N}^N$*. Then* $f_{mn} \star_\theta f_{kl} = \delta_{nk} f_{ml}$ *and* $f_{mn}^* = f_{nm}$*. Thus* f_{nn} *is an orthogonal projector and* f_{mn} *is nilpotent for* $m \neq n$*. Moreover,* $\langle f_{mn}, f_{kl} \rangle = 2^N \, \delta_{mk} \, \delta_{nl}$*. The family* $\{ f_{mn} : m, n \in \mathbb{N}^N \} \subset \mathcal{S} \subset L^2(\mathbb{R}^{2N})$ *is an orthogonal basis.*

It is clear that $e_K := \sum_{|n| \leqslant K} f_{nn}$, for $K \in \mathbb{N}$, defines a (not uniformly bounded) approximate unit $\{e_K\}$ for $\mathcal{A}_\theta$.

As a consequence of Lemma 2.3, the Moyal product has a matricial form.

Proposition 2.4. [43] *Let* $N = 1$*. Then* $\mathcal{A}_\theta$ *has a Fréchet algebra isomorphism with the matrix algebra of rapidly decreasing double sequences* $\mathbf{c} = (c_{mn})$ *such that, for each* $k \in \mathbb{N}$*,*

$$r_k(\mathbf{c}) := \left(\sum_{m,n=0}^{\infty} \theta^{2k} (m + \tfrac{1}{2})^k (n + \tfrac{1}{2})^k |c_{mn}|^2 \right)^{1/2}$$

is finite, topologized by all the seminorms (r_k)*; via the decomposition* $f = \sum_{m,n \in \mathbb{N}^N} c_{mn} f_{mn}$ *of* $\mathcal{S}(\mathbb{R}^2)$ *in the* $\{f_{mn}\}$ *basis.*

For $N > 1$*,* $\mathcal{A}_\theta$ *is isomorphic to the (projective) tensor product of* N *matrix algebras of this kind.*

Definition 2.2. We may as well introduce more Hilbert spaces $\mathcal{G}_{st}$ (for $s, t \in \mathbb{R}$) of those $f \in \mathcal{S}'(\mathbb{R}^2)$ for which the following sum is finite:

$$\|f\|_{st}^2 := \sum_{m,n=0}^{\infty} \theta^{s+t} (m + \tfrac{1}{2})^s (n + \tfrac{1}{2})^t |c_{mn}|^2.$$

We define $\mathcal{G}_{st}$, for s, t now in $\mathbb{R}^N$, as the tensor product of Hilbert spaces $\mathcal{G}_{s_1 t_1} \otimes \cdots \otimes \mathcal{G}_{s_N t_N}$. In other words, the elements $(2\pi)^{-N/2} \theta^{-(N+s+t)/2} (m + \frac{1}{2})^{-s/2} (n + \frac{1}{2})^{-t/2} f_{mn}$ (with an obvious multi-index notation), for $m, n \in \mathbb{N}^N$, are declared to be an orthonormal basis for $\mathcal{G}_{st}$.

If $q \leqslant s$ and $r \leqslant t$ in $\mathbb{R}^N$, then $\mathcal{S} \subset \mathcal{G}_{st} \subseteq \mathcal{G}_{qr} \subset \mathcal{S}'$ with continuous dense inclusions. Moreover, $\mathcal{S} = \bigcap_{s,t \in \mathbb{R}^N} \mathcal{G}_{st}$ topologically (i.e., the projective limit topology of the intersection induces the usual Fréchet space topology on $\mathcal{S}$) and $\mathcal{S}' = \bigcup_{s,t \in \mathbb{R}^N} \mathcal{G}_{st}$ topologically (i.e., the inductive limit topology of the union induces the usual DF topology on $\mathcal{S}'$). In particular, the expansion $f = \sum_{m,n \in \mathbb{N}^N} c_{mn} f_{mn}$ of $f \in \mathcal{S}'$ converges in the strong dual topology.

We shall use the notational convention that if F, G are spaces such that $f \star_\theta g$ is defined whenever $f \in F$ and $g \in G$, then $F \star_\theta G$ is the linear span of the set $\{ f \star_\theta g : f \in F, \, g \in G \}$; in many cases of interest, this set is already a vector space. It is now easy to show that $\mathcal{S} \star_\theta \mathcal{S} = \mathcal{S}$; more precisely, the following result holds.

Proposition 2.5. [43, p. 877] *The algebra* $(\mathcal{S}, \star_\theta)$ *has the (nonunique) factorization property: for all* $h \in \mathcal{S}$ *there exist* $f, g \in \mathcal{S}$ *such that* $h = f \star_\theta g$*.*

2.3 Moyal multiplier algebras

Definition 2.3. The Moyal product can be defined, by duality, on larger sets than $\mathcal{S}$. For $T \in \mathcal{S}'$, write the evaluation on $g \in \mathcal{S}$ as $\langle T, g \rangle \in \mathbb{C}$; then, for $f \in \mathcal{S}$ we may define $T \star_\theta f$ and $f \star_\theta T$ as elements of $\mathcal{S}'$ by $\langle T \star_\theta f, g \rangle := \langle T, f \star_\theta g \rangle$ and $\langle f \star_\theta T, g \rangle := \langle T, g \star_\theta f \rangle$, using the continuity of the star product on $\mathcal{S}$. Also, the involution is extended to $\mathcal{S}'$ by $\langle T^*, g \rangle := \overline{\langle T, g^* \rangle}$.

We shall soon argue [43] that if $T \in \mathcal{S}'$ and $f \in \mathcal{S}$, then $T \star_\theta f,\ f \star_\theta T \in C^\infty(\mathbb{R}^{2N})$.
Consider the left and right multiplier algebras:

$$\mathcal{M}_L^\theta := \{\, T \in \mathcal{S}'(\mathbb{R}^{2N}) : T \star_\theta h \in \mathcal{S}(\mathbb{R}^{2N}) \text{ for all } h \in \mathcal{S}(\mathbb{R}^{2N}) \,\},$$
$$\mathcal{M}_R^\theta := \{\, T \in \mathcal{S}'(\mathbb{R}^{2N}) : h \star_\theta T \in \mathcal{S}(\mathbb{R}^{2N}) \text{ for all } h \in \mathcal{S}(\mathbb{R}^{2N}) \,\},$$

and set $\mathcal{M}^\theta := \mathcal{M}_L^\theta \cap \mathcal{M}_R^\theta$.

It is clear from Lemma 2.1(ii) that the map $h \mapsto T \star_\theta h$, for $T \in \mathcal{M}_L^\theta$, is adjointable, with adjoint given by (left) multiplication by T^*. One can then define the Moyal products $\mathcal{M}_R^\theta \star_\theta \mathcal{S}' = \mathcal{S}'$ and $\mathcal{S}' \star_\theta \mathcal{M}_L^\theta = \mathcal{S}'$ as well.

Theorem 2.6. [90] *$\mathcal{M}^\theta$ is a complete nuclear semireflexive locally convex unital $*$-algebra with hypocontinuous multiplication and continuous involution. Moreover, in view of the previous proposition, $\mathcal{M}^\theta$ is the maximal compactification of $\mathcal{A}_\theta$ defined by duality (see* [45, Sec. 1.3]*).*

This maximal unitization $\mathcal{M}^\theta$ of $\mathcal{A}_\theta$ contains, beyond the constant functions (in particular 1 is the identity), the plane waves. By plane waves we understand all functions of the form $x \mapsto \exp(ik \cdot x)$ for k a $2N$-vector. They are important in physics. Also $\mathcal{M}^\theta$ contains the Dirac δ and all its derivatives, and all monomials $x \mapsto x^\alpha$ for $\alpha \in \mathbb{N}^{2N}$. Clearly $\mathcal{M}^2$ is Fourier invariant, so more generally $F\mathcal{M}^\theta = \mathcal{M}^{4/\theta}$.

When $\theta = 0$, the place of $\mathcal{M}^\theta$ is taken by the space $\mathcal{O}_M$ ("M" for multiplier) of smooth functions of polynomial growth on $\mathbb{R}^{2N}$ in all derivatives.

There is a new way of defining the Moyal product for pairs of distributions lying in the Sobolev-like spaces $\mathcal{G}_{st}$ [43]. If $f = \sum_{m,n} c_{mn} f_{mn} \in \mathcal{G}_{st}$, $g = \sum_{m,n} d_{mn} f_{mn} \in \mathcal{G}_{qr}$ and if $t + q \geqslant 0$, then for $a_{mn} := \sum_k c_{mk} d_{kn}$, the series $h := \sum_{m,n} a_{mn} f_{mn}$ converges in $\mathcal{G}_{sr}$; $f \star_\theta g$ is defined, and $f \star_\theta g = h$. Furthermore, the following useful norm estimates hold:

$$\|f \star_\theta g\|_{st} \leqslant \|f\|_{sq}\, \|g\|_{rt} \quad \text{whenever } q + r \geqslant 0.$$

In particular, $\mathcal{G}_{t,-t}$ is a Banach algebra, for all $t \in \mathbb{R}^N$. This is consistent with the previous definition.
We let $\mathcal{G}_{-\infty,t} := \bigcap_{s\in\mathbb{R}^N} \mathcal{G}_{st}$ (with the projective limit topology) and $\mathcal{G}_{s,+\infty} := \bigcup_{t\in\mathbb{R}^N} \mathcal{G}_{st}$ (with the inductive limit topology). Then $\mathcal{M}_L^\theta = \bigcap_{s\in\mathbb{R}^N} \mathcal{G}_{s,+\infty}$ topologically, and the strong (pre-)dual $(\mathcal{M}_L^\theta)'$ equals $\bigcup_{t\in\mathbb{R}^N} \mathcal{G}_{-\infty,t}$ topologically. Note in passing that $(\mathcal{M}_L^\theta)' \hookrightarrow \mathcal{M}_L^\theta$ with a continuous inclusion.

Yet alternatively, we may work with another algebra of distributions including $(\mathcal{S}, \star_\theta)$, to wit, the multiplier algebra of $\mathcal{G}_{00} = L^2(\mathbb{R}^{2N})$ considered in [56, 90]. We first record the analogue of Lemma 2.1.

Lemma 2.7. [43, 48, 49, 90] *Let $f, g \in L^2(\mathbb{R}^{2N})$. Then*

(i) *For $\theta \neq 0$, $f \star_\theta g$ lies in $L^2(\mathbb{R}^{2N})$. Moreover, $f \star_\theta g$ is uniformly continuous.*

(ii) *$\star_\theta$ is a bilinear associative product on $L^2(\mathbb{R}^{2N})$. The complex conjugation of functions $f \mapsto f^*$ is an involution for $\star_\theta$.*

(iii) *The linear functional* $f \mapsto \int f(x)\,dx$ *on* $\mathcal{S}$ *extends to* $\mathcal{I}_{00}(\mathbb{R}^{2N}) := L^2(\mathbb{R}^{2N}) \star_\theta L^2(\mathbb{R}^{2N})$*, and the product has the tracial property:*

$$\langle f, g\rangle := (\pi\theta)^{-N}\int f\star_\theta g(x)\, d^{2N}x = (\pi\theta)^{-N}\int g\star_\theta f(x)\, d^{2N}x = (\pi\theta)^{-N}\int f(x)\, g(x)\, d^{2N}x.$$

We are not asserting that $h = f \star_\theta g$ *is absolutely integrable. We can nevertheless find* $u \in \mathcal{S}'$ *with* $u^* \star_\theta u = 1$ *and* $|h| \in \mathcal{I}_{00}$ *so that* $h = u \star_\theta |h|$ *and* $|h| = l^* \star_\theta l$ *with* $l \in \mathcal{G}_{00}$*. Writing* $\|h\|_{00,1} := \langle 1, |h|\rangle = \|l\|_{00}^2$*, we obtain a Banach space norm for* $\mathcal{I}_{00}$ *such that* $\|f \star_\theta g\|_{00,1} \leqslant \|f\|_{00}\|g\|_{00}$*.*

(iv) $\lim_{\theta\downarrow 0} L^\theta_f\, g(x) = f(x)\, g(x)$ *almost everywhere on* $\mathbb{R}^{2N}$*.*

In subsection A.1 it is discussed why $\mathcal{I}_{00} \not\subset L^1(\mathbb{R}^{2N})$. Since $f \in \mathcal{I}_{00}$ if and only if the Schrödinger representative $\sigma^\theta(f)$ is trace-class (see the proof of Proposition 2.9 below), one can obtain sufficient conditions for f to belong in $\mathcal{I}_{00}$ from the treatment in [29].

Definition 2.4. Let $A_\theta := \{\, T \in \mathcal{S}' : T \star_\theta g \in L^2(\mathbb{R}^{2N})$ for all $g \in L^2(\mathbb{R}^{2N})\,\}$, provided with the operator norm $\|L^\theta(T)\|_{\text{op}} := \sup\{\, \|T \star_\theta g\|_2/\|g\|_2 : 0 \neq g \in L^2(\mathbb{R}^{2N})\,\}$.

Obviously $\mathcal{A}_\theta = \mathcal{S} \hookrightarrow A_\theta$. But $\mathcal{A}_\theta$ is not dense in A_θ (see below), and we shall denote by A^0_θ its closure in A_θ.

Note that $\mathcal{G}_{00} \subset A_\theta$. This is clear from the following estimate.

Lemma 2.8. [43] *If* $f, g \in L^2(\mathbb{R}^{2N})$*, then* $f \star_\theta g \in L^2(\mathbb{R}^{2N})$ *and* $\|L^\theta_f\|_{\text{op}} \leqslant (2\pi\theta)^{-N/2}\|f\|_2$*.*

Proof. Expand $f = \sum_{m,n} c_{mn}\alpha_{mn}$ and $g = \sum_{m,n} d_{mn}\alpha_{mn}$ with respect to the orthonormal basis $\{\alpha_{nm}\} := (2\pi\theta)^{-N/2}\{f_{nm}\}$ of $L^2(\mathbb{R}^{2N})$. Then

$$\begin{aligned}\|f \star_\theta g\|_2^2 &= (2\pi\theta)^{-2N}\Big\|\sum_{m,l}\Big(\sum_n c_{mn}\, d_{nl}\Big) f_{ml}\Big\|_2^2 = (2\pi\theta)^{-N}\sum_{m,l}\Big|\sum_n c_{mn}\, d_{nl}\Big|^2 \\ &\leqslant (2\pi\theta)^{-N}\sum_{m,j}|c_{mj}|^2\sum_{k,l}|d_{kl}|^2 = (2\pi\theta)^{-N}\|f\|_2^2\,\|g\|_2^2,\end{aligned}$$

on applying the Cauchy–Schwarz inequality. □

The algebra A_θ contains moreover $L^1(\mathbb{R}^{2N})$ and its Fourier transform [57], even the bounded measures and their Fourier transforms; the plane waves; but no nonconstant polynomials, nor derivatives of δ. The algebra A_θ is selfconjugate, and it could have been defined using right Moyal multiplication instead.

Proposition 2.9. [56, 90] $(A_\theta, \|.\|_{\text{op}})$ *is a unital* C^**-algebra of operators on* $L^2(\mathbb{R}^{2N})$*, isomorphic to* $\mathcal{L}(L^2(\mathbb{R}^N))$ *and including* $L^2(\mathbb{R}^{2N})$*. Also,* $(\mathcal{I}_{00})' = A_\theta$*. Moreover, there is a continuous injection of* $*$*-algebras* $\mathcal{A}_\theta \hookrightarrow A_\theta$*, but* $\mathcal{A}_\theta$ *is not dense in* A_θ*, namely* $A^0_\theta \subsetneq A_\theta$*.*

Proof. We prove the non-density result. The left regular representation L^θ of A_θ is a denumerable direct sum of copies of the Schrödinger representation σ^θ on $L^2(\mathbb{R}^N)$ [66]. Indeed, there is a unitary operator, the Wigner transformation W [36, 90], from $L^2(\mathbb{R}^{2N})$ onto $L^2(\mathbb{R}^N) \otimes L^2(\mathbb{R}^N)$, such that

$$W\, L^\theta(f)\, W^{-1} = \sigma^\theta(f) \otimes 1.$$

If $f \in \mathcal{S}$, then $\sigma^\theta(f)$ is a compact (indeed, trace-class) operator on $L^2(\mathbb{R}^N)$, and so A_θ^0 equals $\{ W^{-1}(T \otimes 1)W : T \text{ compact} \}$, while A_θ itself is $\{ W^{-1}(T \otimes 1)W : T \text{ bounded} \}$. Clearly the dual space is $(A_\theta^0)' = \mathcal{I}_{00}$. Notice as well that conjugation by W yields an explicit isomorphism between A_θ and $\mathcal{L}(L^2(\mathbb{R}^N))$. □

Consequently, $\mathcal{A}_\theta$ is a Fréchet algebra whose topology is finer than the $\|\cdot\|_{\mathrm{op}}$-topology. Moreover, it is stable under holomorphic functional calculus in its C^*-completion A_θ^0, as the next proposition shows.

Proposition 2.10. *$\mathcal{A}_\theta$ is a (nonunital) Fréchet pre-C^*-algebra.*

Proof. We adapt the argument for the commutative case in [45, p. 135]. To show that $\mathcal{A}_\theta$ is stable under the holomorphic functional calculus, we need only check that if $f \in \mathcal{A}_\theta$ and $1 + f$ is invertible in A_θ^0 with inverse $1 + g$, then the quasi-inverse g of f must lie in $\mathcal{A}_\theta$. From $f + g + f \star_\theta g = 0$, we obtain $f \star_\theta f + g \star_\theta f + f \star_\theta g \star_\theta f = 0$, and it is enough to show that $f \star_\theta g \star_\theta f \in \mathcal{A}_\theta$, since the previous relation then implies $g \star_\theta f \in \mathcal{A}_\theta$, and then $g = -f - g \star_\theta f \in \mathcal{A}_\theta$ also.

Now, $A_\theta \subset \mathcal{G}_{-r,0}$ for any $r > N$ [90, p. 886]. Since $f \in \mathcal{G}_{s,p+r} \cap \mathcal{G}_{qt}$, for s, t arbitrary and p, q positive, we conclude that $f \star_\theta g \star_\theta f \in \mathcal{G}_{s,p+r} \star_\theta \mathcal{G}_{-r,0} \star_\theta \mathcal{G}_{qt} \subset \mathcal{G}_{st}$; as $\mathcal{S} = \bigcap_{s,t\in\mathbb{R}} \mathcal{G}_{st}$, the proof is complete. □

The Fréchet algebras $\mathcal{A}_\theta$ are automatically good (their sets of quasi-invertible elements are open); and by an old result of Banach [5], the quasi-inversion operation is continuous in a good Fréchet algebra. Note that a good algebra with identity cannot have proper (even one-sided) dense ideals. However, the nonunital $(\mathcal{M}_L^\theta)'$ provides an example of a good Fréchet algebra that harbours $\mathcal{A}_\theta$ as a proper dense left ideal [44].

We noticed already that the extensions $\mathcal{M}^\theta$ and A_θ of $\mathcal{A}_\theta$ are quite different. Clearly $\mathcal{M}^\theta$ is associated with smoothness; however, even though the Sobolev-like spaces $\mathcal{G}_{st}$ grow more regular with increasing s and t [90], $\mathcal{M}^\theta$ includes none of them; in particular, $L^2(\mathbb{R}^{2N}) \not\subset \mathcal{M}^\theta$ for any θ.

Be that as it may, the plane waves belong both to $\mathcal{M}^\theta$ and A_θ. One obtains for the Moyal product of plane waves:

$$\exp(ik\,\cdot) \star_\theta \exp(il\,\cdot) = e^{-\frac{i}{2}\theta\, k\cdot Sl} \, \exp(i(k+l)\cdot), \tag{2.12}$$

or, reinstalling the generic Moyal product:

$$\exp(ik\,\cdot) \star_\Theta \exp(il\,\cdot) = e^{-\frac{i}{2}\, k\cdot\Theta l} \, \exp(i(k+l)\cdot). \tag{2.13}$$

Therefore the plane waves close to an algebra, the *Weyl algebra*. It represents the translation group of $\mathbb{R}^{2N}$:

$$\big(\exp(ik\,\cdot) \star_\theta f \star_\theta \exp(-ik\,\cdot)\big)(x) = f(x + \theta Sk),$$

for $f \in \mathcal{S}$ or $f \in \mathcal{G}_{00}$, say.

2.4 Smooth test function spaces, their duals and the Moyal product

Here there is a fascinating interplay. Recall that a pseudodifferential operator $A \in \Psi\mathrm{DO}$ on $\mathbb{R}^k$ is a linear operator which can be written as

$$A\,h(x) = (2\pi)^{-k} \iint \sigma[A](x,\xi)\, h(y)\, e^{i\xi\cdot(x-y)}\, d^k\xi\, d^k y.$$

Let $\Psi^d := \{ A \in \Psi DO : \sigma[A] \in S^d \}$ be the class of ΨDOs of order d, with

$$S^d := \{ \sigma \in C^\infty(\mathbb{R}^k \times \mathbb{R}^k) : |\partial_x^\alpha \partial_\xi^\beta \sigma(x,\xi)| \leqslant C_{K\alpha\beta}(1+|\xi|^2)^{(d-|\beta|)/2} \text{ for } x \in K \},$$

where K is any compact subset of $\mathbb{R}^k$, $\alpha, \beta \in \mathbb{N}^k$, and $C_{K\alpha\beta}$ is some constant. Also $\Psi^\infty := \bigcup_{d\in\mathbb{R}} \Psi^d$ and $\Psi^{-\infty} := \bigcap_{d\in\mathbb{R}} \Psi^d$. Recall, too, that a ΨDO A is called regularizing or smoothing if $A \in \Psi^{-\infty}$, or equivalently [52,80], if A extends to a continuous linear map from the dual of the space of smooth functions $C^\infty(\mathbb{R}^k)$ to itself.

Lemma 2.11. *If $f \in \mathcal{S}$, then L_f^θ is a regularizing* ΨDO.

Proof. From (2.1), one at once sees that left Moyal multiplication by f is the pseudodifferential operator on $\mathbb{R}^{2N}$ with symbol $f(x - \frac{\theta}{2}S\xi)$. Clearly L_f^θ extends to a continuous linear map from $C^\infty(\mathbb{R}^{2N})' \hookrightarrow \mathcal{S}'$ to $C^\infty(\mathbb{R}^{2N})$. The lemma also follows from the inequality

$$|\partial_x^\alpha \partial_\xi^\beta f(x - \tfrac{\theta}{2}S\xi)| \leqslant C_{K\alpha\beta}(1+|\xi|^2)^{(d-|\beta|)/2},$$

valid for all $\alpha, \beta \in \mathbb{N}^{2N}$, any compact $K \subset \mathbb{R}^{2N}$, and any $d \in \mathbb{R}$, since $f \in \mathcal{S}$. □

Remark 2.12. Unlike for the case of a compact manifold, regularizing ΨDOs are not necessarily compact operators. For instance, for each n, $L^\theta(f_{nn})$ possesses the eigenvalue 1 with infinite multiplicity, so it cannot be compact.

Definition 2.5. For $m \in \mathbb{N}$, $f \in C^m(\mathbb{R}^k)$ – functions with m continuous derivatives – and $\gamma, l \in \mathbb{R}$, let

$$q_{\gamma lm}(f) := \sup\{ (1+|x|^2)^{(-l+\gamma|\alpha|)/2} |\partial^\alpha f(x)| : x \in \mathbb{R}^k,\ |\alpha| \leqslant m \};$$

and then let $\underline{\mathcal{V}}_{\gamma,l}^m$, respectively $\mathcal{V}_{\gamma,l}^m$, be the space of functions in $C^m(\mathbb{R}^k)$ for which

$$(1+|x|^2)^{(-l+\gamma|\alpha|)/2}\, \partial^\alpha f(x)$$

vanishes at infinity for all $|\alpha| \leqslant m$, respectively is finite for all $x \in \mathbb{R}^k$, normed by $q_{\gamma lm}$. Note that $\underline{\mathcal{V}}_{0,l}^m$ is Horváth's space $\mathcal{S}_{-2l}^m$ [53]. We define

$$\mathcal{V}_\gamma := \bigcup_{l\in\mathbb{R}} \bigcap_{m\in\mathbb{N}} \mathcal{V}_{\gamma,l}^m \quad \text{and, more generally,} \quad \mathcal{V}_{\gamma,l} := \bigcap_{m\in\mathbb{N}} \mathcal{V}_{\gamma,l}^m$$

so that $\mathcal{V}_\gamma = \bigcup_{l\in\mathbb{R}} \mathcal{V}_{\gamma,l}$. Particularly interesting cases include the space $\mathcal{K} := \mathcal{V}_1$ of Grossmann–Loupias–Stein functions [47], whose dual $\mathcal{K}'$ is the space of Cesàro-summable distributions [34], the space $\mathcal{O}_C := \mathcal{V}_0$ whose dual $\mathcal{O}_C'$ is the space of convolution multipliers (Fourier transforms of $\mathcal{O}_M$), and the space $\mathcal{O}_T := \mathcal{V}_{-1}$ [43]. Similarly, $\mathcal{K}_r := \mathcal{V}_{1,r}$ and $\mathcal{O}_r := \mathcal{V}_{0,r}$ are defined. We see that

$$\mathcal{S} = \bigcap_{m\in\mathbb{N}} \bigcap_{l\in\mathbb{R}} \underline{\mathcal{V}}_{0,l}^m, \qquad \mathcal{O}_M = \bigcap_{m\in\mathbb{N}} \bigcup_{l\in\mathbb{R}} \underline{\mathcal{V}}_{0,l}^m.$$

Following Schwartz, we denote $\mathcal{B} := \mathcal{O}_0$, the space of smooth functions bounded together with all derivatives.

We shall also need $\dot{\mathcal{B}} := \bigcap_{m\in\mathbb{N}} \underline{\mathcal{V}}_{0,0}^m$, the space of smooth functions vanishing at infinity together with all derivatives, and the weighted test space $\mathcal{D}_{L^2}$, the space of elements of $L^2(\mathbb{R}^{2N})$ all of whose (distributional) derivatives also lie in L^2 [68, 77]; by Sobolev's lemma, these are in fact smooth functions and moreover $\mathcal{D}_{L^2} \subset \dot{\mathcal{B}}$ [77]: actually if $f \in \mathcal{D}_{L^2}$, then the (ordinary) Fourier transform $\mathcal{F}(f)$ satisfies $(1+|\xi|^{2n})\mathcal{F}(f) \in L^2$ for all integer n, and by the Cauchy–Schwarz inequality $\mathcal{F}(f) \in L^1$, thus f tends to zero at infinity. In the notation of [84], $\mathcal{D}_{L^2}$ is $H^{2,\infty}$.

There are continuous inclusions $\mathcal{D} \hookrightarrow \mathcal{V}_\gamma \hookrightarrow \mathcal{V}_{\gamma'} \hookrightarrow \mathcal{O}_M \hookrightarrow \mathcal{D}'$ for $\gamma > \gamma'$; these are all normal spaces of distributions, namely, locally convex spaces which include $\mathcal{S}$ as a dense subspace and are continuously included in $\mathcal{S}'$. Also $\mathcal{D}_{L^2}$ (density of $\mathcal{S}$ in this space follows from density of the Schwartz functions in L^2 and invariance of $\mathcal{S}$ under derivations) and $\mathcal{M}^\theta_L$, $\mathcal{M}^\theta_R$ and $\mathcal{M}^\theta$ [90] are normal space of distributions.

By the way, there are suggestive Tauberian-type theorems for these spaces, establishing when their intersections with their respective dual spaces are included in $\mathcal{S}$. Concretely, we quote the following result from [32].

Proposition 2.13. *If $\mathcal{C}$ is a space of smooth functions on $\mathbb{R}^{2N}$ which is closed under complex conjugation, and if the pointwise product space $\mathcal{K}\mathcal{C}$ lies within $\mathcal{C}$, then $\mathcal{C} \cap \mathcal{C}' \subseteq \mathcal{S}$.*

In particular, $\mathcal{V}_\gamma \cap \mathcal{V}'_\gamma = \mathcal{S}$ for $\gamma \leqslant 1$. Also $\mathcal{O}_M \cap \mathcal{O}'_M = \mathcal{S}$ and $C^\infty \cap (C^\infty)' = \mathcal{D} \subset \mathcal{S}$.

Now, what can be said about the relation of all these spaces with $\mathcal{M}^\theta$? In [43] it is established that $\mathcal{O}'_T$, and a fortiori $\mathcal{O}'_M$, is included in $\mathcal{M}^\theta$, for all θ. Therefore by Fourier analysis $\mathcal{O}_C$ is included in $\mathcal{M}^\theta$ for all θ, and $g \star_\theta f$ is defined as a tempered distribution whenever $f, g \in \mathcal{O}_C$. Growth estimates may be obtained as follows. It is true that $\mathcal{O}_C = \bigcup_{r\in\mathbb{R}} \mathcal{O}_r$ topologically. If $g \in \mathcal{O}_r$ and $f \in \mathcal{O}_s$, the following crucial proposition shows that the $\mathcal{O}_r$ spaces have similar behaviour under pointwise and Moyal products.

Proposition 2.14. *The space $\mathcal{O}_C$ is an associative $*$-algebra under the Moyal product. In fact, the Moyal product is a jointly continuous map from $\mathcal{O}_r \times \mathcal{O}_s$ into $\mathcal{O}_{r+s}$, for all $r, s \in \mathbb{R}$. Moreover, $\mathcal{A}_\theta$ is a two sided essential ideal in $\mathcal{O}_C$.*

Proof. For the reader's convenience, we reproduce part of Theorem 2 of [35]. Let $f \in \mathcal{O}_r$ and $g \in \mathcal{O}_s$. By the Leibniz rule for the Moyal product, $\partial^\alpha(f \star_\theta g) = \sum_{\beta+\gamma=\alpha} \binom{\alpha}{\beta} \partial^\beta f \star_\theta \partial^\gamma g$. Hence we need only show that there are constants C_{rsm} such that

$$(1 + |x|^2)^{-(r+s)/2} |(\partial^\beta f \star_\theta \partial^\gamma g)(x)| \leqslant C_{rsm}\, q_{0rm}(f)\, q_{0sm}(g) \tag{2.14}$$

for all $x \in \mathbb{R}^{2N}$, for large enough $m \geqslant |\beta| + |\gamma|$. If $k \in \mathbb{N}$ (to be determined later), we can write

$$\begin{aligned}
(\partial^\beta f \star_\theta \partial^\gamma g)(x) &= (\pi\theta)^{-2N} \iint \frac{\partial^\beta f(x+y)}{(1+|y|^2)^k} \frac{\partial^\gamma g(x+z)}{(1+|z|^2)^k} (1+|y|^2)^k (1+|z|^2)^k e^{\frac{2i}{\theta} y\cdot Sz}\, d^{2N}y\, d^{2N}z \\
&= (\pi\theta)^{-2N} \iint \frac{\partial^\beta f(x+y)}{(1+|y|^2)^k} \frac{\partial^\gamma g(x+z)}{(1+|z|^2)^k} P_k(\partial_y, \partial_z)\big[e^{\frac{2i}{\theta} y\cdot Sz}\big]\, d^{2N}y\, d^{2N}z \\
&= (\pi\theta)^{-2N} \iint e^{\frac{2i}{\theta} y\cdot Sz} P_k(-\partial_y, -\partial_z)\left[\frac{\partial^\beta f(x+y)}{(1+|y|^2)^k} \frac{\partial^\gamma g(x+z)}{(1+|z|^2)^k}\right] d^{2N}y\, d^{2N}z,
\end{aligned}$$

where P_k is a polynomial of degree $2k$ in both y and z variables. From the elementary estimates $|\partial^\alpha((1+|x|^2)^{-k})| \leqslant c_{\alpha,k}(1+|x|^2)^{-k}$ it follows that

$$\begin{aligned}
|\partial^\beta f \star_\theta \partial^\gamma g|(x) &\leqslant \sum_{k',k''\leqslant 2k} C'_{k'k''} \iint \left| \frac{\partial^{\beta+k'} f(x+y)}{(1+|y|^2)^k} \frac{\partial^{\gamma+k''} g(x+z)}{(1+|z|^2)^k} \right| d^{2N}y\, d^{2N}z \\
&\leqslant C''_{rsm}\, q_{0rm}(f)\, q_{0sm}(g) \iint \frac{(1+|x+y|^2)^{r/2}}{(1+|y|^2)^k} \frac{(1+|x+z|^2)^{s/2}}{(1+|z|^2)^k}\, d^{2N}y\, d^{2N}z \\
&\leqslant C'''_{rsm}\, q_{0rm}(f)\, q_{0sm}(g)\, (1+|x|^2)^{(r+s)/2} \int (1+|y|^2)^{r/2-k}\, d^{2N}y \int (1+|z|^2)^{s/2-k}\, d^{2N}z,
\end{aligned}$$

provided $m \geqslant |\beta| + |\gamma| + 2k$; here the Cauchy inequality $1 + |x + y|^2 \leqslant 2(1 + |x|^2)(1 + |y|^2)$ has been used to extract the x variables. If we now choose $k > N + \max\{r, s\}/2$ (and therefore take $m \geqslant |\beta| + |\gamma| + 2N + \max\{r, s\}$), the integrals will be finite. The joint continuity now follows directly from the estimates (2.14).

That $\mathcal{S}$ is a two-sided ideal in $\mathcal{O}_C$ follows from the inclusion $\mathcal{O}_C \subset \mathcal{M}^\theta$. Essentiality for the ideal $\mathcal{S} = \mathcal{A}_\theta$ is equivalent [45, Prop. 1.8] to $g \star_\theta \mathcal{S} \neq 0$ for any nonzero $g \in \mathcal{O}_s$; but if $g \star_\theta f_{mn} = 0$ for all m, n, then in the expansion $g = \sum_{m,n} c_{mn} f_{mn}$ (as an element of $\mathcal{S}'$, say) all coefficients must vanish, so that $g = 0$. □

Similar results hold for $\mathcal{V}_\gamma$ when $\gamma > 0$. Indeed, the Moyal product $(f, g) \mapsto f \star_\theta g$ is a jointly continuous map from $\mathcal{K}_r \times \mathcal{K}_s$ into $\mathcal{K}_{r+s}$; moreover, $f \star_\theta g - fg \in \mathcal{K}_{r+s-2}$, which is a bonus for semiclassical analysis (while on the contrary the similar statement for $\mathcal{O}_r \times \mathcal{O}_s$ is in general false). For $\gamma < 0$, we lose control of the estimates; indeed, Lassner and Lassner [59] gave an example of two functions in $\mathcal{O}_T$ whose twisted product can be defined but is not a smooth function, but rather a distribution (of noncompact support). Also, in the next subsection we prove by counterexample that $\mathcal{O}_T \not\subset \mathcal{M}_L^\theta$. The integral estimates on the derivatives of $g \star_\theta f$ can be refined to show that in fact $\mathcal{O}_M \star_\theta \mathcal{O}_C = \mathcal{O}_M$. However, since these estimates depend on the order of the derivatives in a complicated way, it is doubtful that the twisted product can be extended to $\mathcal{O}_M$.

The regularizing property of $\star_\theta$ proved at the beginning of the section can be vastly improved, as follows.

Proposition 2.15. [43] *If $T \in \mathcal{S}'$ and $f \in \mathcal{S}$, then $T \star_\theta f$ and $f \star_\theta T$ lie in $\mathcal{O}_T$. Moreover, these bilinear maps of $\mathcal{S}' \times \mathcal{S}$ and $\mathcal{S} \times \mathcal{S}'$ into $\mathcal{O}_T$ are hypocontinuous.*

In fact, $\mathcal{S} \star_\theta \mathcal{S}'$ equals $(\mathcal{M}_L^\theta)'$, so the latter is made of smooth functions. But $(\mathcal{M}_L^\theta)' \cap (\mathcal{M}_L^\theta)'' = (\mathcal{M}_L^\theta)' \cap \mathcal{M}_L^\theta = (\mathcal{M}_L^\theta)' \supsetneq \mathcal{S}$; so $(\mathcal{M}_L^\theta)'$ and $(\mathcal{M}_R^\theta)'$ do not satisfy the conclusion of Proposition 2.13. (Here $''$ of course denotes the strong bidual space, not a bicommutant.) As distributions, the elements of $(\mathcal{M}_L^\theta)'$ and $(\mathcal{M}_R^\theta)'$ belong to $\mathcal{O}'_C$, and a fortiori they are Cesàro summable [34].

Finally, it is important to know when smooth functions give rise to elements of A_θ^0 or A_θ. Sufficient conditions are the following (quite strong) results of the Calderón–Vaillancourt type [36, 54].

Theorem 2.16. *The inclusion $\mathcal{V}_{0,0}^{2N+1} \subset A_\theta$ holds. In particular, $\mathcal{B} \subset A_\theta$. The inclusion $\underline{\mathcal{V}}_{00}^{2N+1} \subset A_\theta^0$ also holds. In particular, $\dot{\mathcal{B}} \subset A_\theta^0$. Moreover, if $b \in \mathcal{V}_{0,0}^{2N+1}$ belongs to A_θ^0, then $b \in \underline{\mathcal{V}}_{0,0}^{2N+1}$.*

We have also proved that the function space $\mathcal{B}$ is a $*$-algebra under the Moyal product $\star_\theta$ for any θ, in which $\mathcal{A}_\theta$ is a two-sided essential ideal. Recall that $\mathcal{D}_{L^2} \subset \dot{\mathcal{B}} \subset \mathcal{M}^\theta$. We shall now show that $\mathcal{D}_{L^2}$ is a $*$-algebra under the Moyal product as well.

Lemma 2.17. *$(\mathcal{D}_{L^2}, \star_\theta)$ is a $*$-algebra with continuous product and involution. Moreover, it is an ideal in $(\mathcal{B}, \star_\theta)$.*

Proof. The closure under the twisted product follows from the Leibniz rule and Lemma 2.8:

$$\|\partial^\alpha(f \star_\theta g)\|_2 \leqslant (2\pi\theta)^{N/2} \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} \|\partial^\beta f\|_2 \, \|\partial^{\alpha-\beta} g\|_2.$$

This also shows that the product is separately continuous, indeed jointly continuous since $\mathcal{D}_{L^2}$ is a Fréchet space. The continuity of the involution $f \mapsto f^*$ is immediate.

The fact that $\mathcal{D}_{L^2}$ is a two sided ideal in $\mathcal{B}$ comes directly from the stability of these spaces under partial derivations and from the inclusion $\mathcal{B} \subset A_\theta$ given by the previous theorem, since then $\|\partial^\alpha f \star_\theta \partial^\beta g\|_2 < \infty$ for all $f \in \mathcal{B}$, $g \in \mathcal{D}_{L^2}$ and all $\alpha, \beta \in \mathbb{N}^{2N}$. □

2.5 The preferred unitization of the Schwartz Moyal algebra

As with Stone–Čech compactifications, the algebras $\mathcal{M}^\theta$ are too vast to be of much practical use (in particular, to define noncommutative vector bundles). A more suitable unitization of $\mathcal{A}_\theta$ is given by the algebra $\widetilde{\mathcal{A}}_\theta := (\mathcal{B}, \star_\theta)$. This algebra possesses an intrinsic characterization as the smooth commutant of *right* Moyal multiplication (see our comments at the end of subsection 4.5). The inclusion of $\mathcal{A}_\theta$ in $\mathcal{B}$ is not dense, but this is not needed. $\widetilde{\mathcal{A}}_\theta$ contains the constant functions and the plane waves, but no nonconstant polynomials and no imaginary-quadratic exponentials, such as $e^{iax_1x_2}$ in the case $N = 1$ (we shall see later the pertinence of this).

Proposition 2.18. *$\widetilde{\mathcal{A}}_\theta$ is a unital Fréchet pre-C^*-algebra.*

Proof. We already know that $\mathcal{B}$ is a unital $*$-algebra with the Moyal product, and that $\star_\theta$ is continuous in the topology of the Fréchet space $\mathcal{B}$ defined by the seminorms q_{00m}, for $m \in \mathbb{N}$. Its elements have all derivatives bounded, and so are uniformly continuous functions on $\mathbb{R}^{2N}$, as are their derivatives: the group of translations $\tau_y f = f(\cdot - y)$, for $y \in \mathbb{R}^{2N}$, acts strongly continuously on $\widetilde{\mathcal{A}}_\theta$ (i.e., $y \mapsto \tau_y f$ is continuous for each f).

This action preserves the seminorms q_{00m}, and it is clear that $\mathcal{B}$ is a subspace of the space of smooth elements for τ, which we provisionally call A_θ^∞. The latter space has its own Fréchet topology, coming from the strongly continuous action. Rieffel [75, Thm. 7.1] proves two important properties in this setting: firstly, based on a density theorem of Dixmier and Malliavin [30], that the inclusion $\mathcal{B} \hookrightarrow A_\theta^\infty$ is continuous and dense. Secondly, using a "Θ-twisting" of C^*-algebras with an $\mathbb{R}^k$-action which generalizes (2.1), whereby the pointwise product can be recovered as $(\mathcal{B}, \star_0) = (\widetilde{\mathcal{A}}_\theta, \star_{-\theta})$, one obtains the reverse inclusion; thus, $\mathcal{B} = A_\theta^\infty$. (Thus, the smooth subalgebra is independent of Θ.)

It is now easy to show that $\widetilde{\mathcal{A}}_\theta$, as a subalgebra of the C^*-algebra A_θ, is stable under the holomorphic functional calculus. Indeed, since $G(\tau_y(f)) = \tau_y(G(f))$ for any function G which is holomorphic in the neighbourhood of $\operatorname{sp} L^\theta(f) = \operatorname{sp} L^\theta(\tau_y(f))$, it is clear that $f \in \widetilde{\mathcal{A}}_\theta$ entails $G(f) \in \widetilde{\mathcal{A}}_\theta$. □

Clearly the C^*-algebra completion of $\widetilde{\mathcal{A}}_\theta$ properly contains A_θ^0; it is not known to us whether it is equal to A_θ. At any rate, $\widetilde{\mathcal{A}}_\theta \equiv \mathcal{B}$ is nonseparable as it stands; there is, however, another topology on $\mathcal{B}$, induced by the topology of $C^\infty(\mathbb{R}^{2N})$ [77, p. 203], under which this space is separable. That latter topology is very natural in the context of commutative and Connes–Landi spaces (see subsections 3.3 and 3.4). To investigate its pertinence in the context of Moyal spaces would take us too far afield.

An advantage of $\widetilde{\mathcal{A}}_\theta$ is that the covering relation of the noncommutative plane to the NC torus is made transparent. To wit, the smooth noncommutative torus algebra $C^\infty(\mathbb{T}_\Theta^{2N})$ can be embedded in $\mathcal{B}$ as periodic functions (with a fixed period parallelogram). In that respect, it is well to recall [76, 87] how far the algebraic structure of $C^\infty(\mathbb{T}^{2N}_\Theta)$ can be obtained from the integral form (2.1) of (a periodic version of) the Moyal product.

Anticipating on the next section, we finally note the main reason for suitability of $\widetilde{\mathcal{A}}_\theta$, namely, that each $[\not{D}, L^\theta(f) \otimes 1_{2^N}]$ lies in $A_\theta \otimes M_{2^N}(\mathbb{C})$, for $f \in \widetilde{\mathcal{A}}_\theta$ and $\not{D}$ the Dirac operator on $\mathbb{R}^{2N}$.

The previous proposition has another useful consequence.

Corollary 2.19. *$(\mathcal{D}_{L^2}, \star_\theta)$ is a (nonunital) Fréchet pre-C^*-algebra, whose C^*-completion is A^0_θ.*

Proof. The argument of the proof of Proposition 2.10 applies, with the following modifications. Firstly, $\mathcal{S} \subset \mathcal{D}_{L^2} \subset A^0_\theta$ with continuous inclusions, so that A^0_θ is indeed the C^*-completion of $(\mathcal{D}_{L^2}, \star_\theta)$. Indeed, for the second inclusion one can notice that if $f \in \mathcal{D}_{L^2}$, then $W\,L^\theta(f)\,W^{-1} = \sigma^\theta(f) \otimes 1$ where $\sigma^\theta(f)$ is a Hilbert–Schmidt operator, hence compact. The same conclusion follows from Theorem 2.16.

Secondly, if $f \in \mathcal{D}_{L^2}$ has a quasi-inverse $g \in A^0_\theta$, then the previous proposition shows that $g \in \widetilde{\mathcal{A}}_\theta$, too. Since $(\mathcal{D}_{L^2}, \star_\theta)$ is an ideal in $\widetilde{\mathcal{A}}_\theta$ by Lemma 2.17, we conclude that $f \star_\theta g \star_\theta f \in \mathcal{D}_{L^2}$, which is enough to establish that $g \in \mathcal{D}_{L^2}$. □

A relevant group of inner automorphisms of the "big algebras" $\mathcal{M}^\theta$ or A_θ is given by the *metaplectic representation*. Real symplectic $2N \times 2N$ matrices act on functions by $Mf(x) := f(M^{-1}x)$. We can consider inhomogeneous symplectic transformations, i.e., affine transformations leaving the symplectic structure invariant. Let (s, M) denote an element of the inhomogeneous symplectic group $\mathrm{ISp}(2N, \mathbb{R})$, i.e., the semidirect product of the group of translations and the symplectic group, with group law

$$(s_1, M_1)(s_2, M_2) = (M_2^{-1}s_1 + s_2, M_1M_2),$$

acting by

$$(s, M)f(x) = f(M^{-1}x - s). \tag{2.15}$$

The equivariance of the twisted product is readily checked:

$$(s, M)f \;\star_\theta\; (s, M)g = (s, M)(f \star_\theta g). \tag{2.16}$$

We concentrate on the homogeneous $(0, M)$ transformations. The symplectic action is realized by the adjoint $\star$-action of unitaries $E(M, \cdot)$, belonging also to the multiplier Moyal algebra $\mathcal{M}^\theta$. They constitute a variant of the metaplectic representation; $E(M, \cdot)$ is a distribution on the space of smooth sections of a nontrivial line bundle over $\mathrm{ISp}(2N, \mathbb{R})$, that works like the exponential kernel of a noncommutative Fourier transform:

$$E(M, \cdot) \;\star_\theta f \star_\theta\; E(M, \cdot)^* = Mf, \tag{2.17}$$

for all $f \in \mathcal{S}$ or $f \in L^2(\mathbb{R}^{2N})$ or even $f \in \mathcal{S}'$. Explicitly, for elements M of $\mathrm{Sp}(2N, \mathbb{R})$ which are "nonexceptional", i.e., $\det(1 + M) \neq 0$, there is the presentation

$$E(M, x) = e^{i\alpha}\,\frac{2^N}{\sqrt{\det(1 + M)}}\exp\!\left(-ix \cdot S\,\frac{1 - M}{\theta(1 + M)}\,x\right). \tag{2.18}$$

Thus, such $E(M,.)$ are imaginary-quadratic exponentials; the quadratic form in the exponent is actually an important symplectic invariant, solving a modified Hamilton–Jacobi equation, introduced by Poincaré [69] and nowadays all but forgotten. The phase prefactor in (2.18) reflects the ambiguity inherent in (2.17), which can be reduced to a sign, so that

$$E(M,\cdot) \star_\theta E(M',\cdot) = \pm E(MM',\cdot).$$

The curious reader can directly check the last two formulas, aided by the method of the stationary phase; a look at [1, 37, 89] will help.

In contradistinction to the Weyl algebra, the $E(M,\cdot)$ do not belong to $\mathcal{B}$, and so they yield *outer* automorphisms of $\widetilde{\mathcal{A}}_\theta$ – and of course of $\mathcal{A}_\theta$.

Note that the values $\exp\bigl(\pm 2i\theta^{-1}(x_1x_{N+1} + \cdots + x_Nx_{2N})\bigr)$ are never reached by E in (2.18). For good reason: these functions do not belong to the multiplier algebras $\mathcal{M}^\theta$ or A_θ, as the following lemma shows.

Lemma 2.20. *Let $h_a(x) := \exp\bigl(ia(x_1x_{N+1} + \cdots + x_Nx_{2N})\bigr)$ for $a \neq 0$. Then $h_a \in \mathcal{M}^\theta$, or $h_a \in A_\theta$, if and only if $|a| \neq 2/\theta$.*

Proof. We show this for $N = 1$, the general case follows immediately. In view of (2.8), it suffices to consider the case $\theta = 2$. We must determine whether $h_a \star_2 f_{mn} \in \mathcal{S}$; because of the multiplication rule (2.6), it is enough to check this for the Gaussian function f_{00}. From (2.4),

$$h_a \star_2 f_{00}(x) = \frac{1}{2\pi^2} \iint \exp\bigl(iay_1y_2 - \tfrac{1}{2}z_1^2 - \tfrac{1}{2}z_2^2 + i(x_1-y_1)(x_2-z_2) - i(x_2-y_2)(x_1-z_1)\bigr)\, d^2y\, d^2z.$$

With $u = (y_1, y_2, z_1, z_2)$, the integral is of the type $\int \exp(-\frac{1}{2}u\cdot Qu - iu\cdot R_x)\, d^4u$, where the quadratic form $u \cdot Qu$, with $\Re Q \geqslant 0$, is degenerate if and only if $\det Q = a^2 - 1 = 0$. Thus if $|a| = 1$, then $h_a \star_2 f_{00} \notin \mathcal{S}$, and also $h_a \star_2 f_{00} \notin L^2$, while if $|a| \neq 1$, an explicit calculation shows that $h_a \star_2 f_{00} \in \mathcal{S}$. □

This shows, by the way, that $\mathcal{O}_T \not\subset \mathcal{M}^\theta$ and that the $\mathcal{M}^\theta$ and A_θ for different θ are all distinct spaces of tempered distributions.

Next we look briefly at the derivations of $\mathcal{A}_\theta$, $\widetilde{\mathcal{A}}_\theta$. Linear functions, not belonging to $\mathcal{B} \equiv \widetilde{\mathcal{A}}_\theta$ either, at the infinitesimal level double as Hamiltonians for the translations; quadratic functions double as Hamiltonians for linear symplectomorphisms. For h affine quadratic

$$[h, f]_{\star_\theta} = i\theta\,\{h, f\},$$

that is, the Moyal and Poisson brackets in this case essentially coincide. (Note that the derivations of $\widetilde{\mathcal{A}}_\theta$ corresponding to quadratic Hamiltonians are unbounded.)

On the other hand, all derivations of $\mathcal{M}^\theta$ are inner, as is easily proved using the Poincaré lemma in the distributional context [31].

An important task is to compute the Hochschild cohomologies of $\mathcal{A}_\theta$ and $\widetilde{\mathcal{A}}_\theta$, as Connes did for the NC torus in [15]; we have already seen that they are not entirely trivial. The "big algebras" $\mathcal{M}^\theta$ and A_θ, on the other hand, risk having uninteresting cohomology.

Rennie has proposed to equip nonunital noncommutative algebras $\mathcal{A}$ like $\mathcal{A}_\theta$ with a "local ideal" $\mathcal{A}_c \subset \mathcal{A}$ [73], which would be a noncommutative generalization of the space $C_c^\infty(M)$ of smooth functions with compact support. A Fréchet algebra $\mathcal{A}$ is local in his sense if it has a dense ideal $\mathcal{A}_c$ with local units; an algebra $\mathcal{A}_c$ has local units when, for any finite subset of elements $\{a_1, \dots, a_k\}$ of $\mathcal{A}_c$, there exists $u \in \mathcal{A}_c$ such that $ua_i = a_i u = a_i$ for $i = 1, \dots, k$.

Certainly the Moyal product $\star_\theta$ is not "local" in the ordinary sense: the formulas (2.3) and (2.4) are two different definitions, as may be noticed in the simple example of a couple f, g with disjoint supports; then (2.3) gives zero outside the supports; while (2.4) does not. The algebras $\mathcal{A}_\theta$ are not known to have bilateral ideals; it is very likely that they are simple, and if so, they would not be local in the sense of [73], either (thus, it is not clear if Rennie's device can carry the full weight of noncommutative spin geometry).

However, one can define a useful weaker notion of locality:

Definition 2.6. A Fréchet algebra $\mathcal{A}$ is *quasilocal* if it has a dense $*$*-subalgebra* $\mathcal{A}_c$ with local units. Here, we choose

$$\mathcal{A}_c := \bigcup_{K \in \mathbb{N}} \mathcal{A}_{c,K}, \quad \text{where} \quad \mathcal{A}_{c,K} := \Big\{ f \in \mathcal{S} : f = \sum_{0 \leqslant |m|,|n| \leqslant K} c_{mn} f_{mn} \Big\}.$$

That is, $\mathcal{A}_c$ is the algebra of finite linear combinations of the $\{ f_{mn} : m, n \in \mathbb{N}^N \}$; it possesses local units, and so $\mathcal{A}_\theta$ is quasilocal.

Rennie further argues that possession of a local ideal in his sense guarantees H-unitality [97] of the original algebra. Certainly our $\mathcal{A}_c$ is algebraically H-unital, as it possesses local units [61]. It would be good to know whether $\mathcal{A}_\theta$ is topologically H-unital.

3 Axioms for noncompact spin geometries

3.1 Generalization of the unital case conditions

To define and construct noncommutative spin manifolds, one starts from an operatorial version of ordinary spin geometry, that can be generalized to noncommutative manifolds. Ideally, one should prove a reconstruction theorem, allowing to recover all of the (topological, smooth, geometrical) concrete structure from the abstract geometry over a suitable commutative algebra; this has been performed to satisfaction for compact manifolds without boundary [18, 19, 45, 71]. However, to rush to that at the present stage would not do. It is better for now to patiently listen to what the possible examples have to say.

As the first part of our task, therefore, we seek a collection of Connes-like axioms for not necessarily compact noncommutative manifolds. Such a list of conditions should be compatible with the previous axiomatic framework, and be fulfilled by noncompact commutative manifolds. We expect it also to encompass other interesting cases. Our main task will then be to prove that noncommutative Moyal-product algebras constitute one of the examples. (To eventually reach this goal, we use heavy machinery wholesale; we do not claim to have the "best" proofs.)

The discussion in this section will be relatively informal; a formal proposal is made in the next section.

We set out by discussing what a *real noncompact spectral triple* might be. As mentioned in the Introduction, the basic data $(\mathcal{A}, \mathcal{H}, D)$ – or $(\mathcal{A}, \mathcal{H}, D, \chi)$ – for a spectral triple consist of an algebra $\mathcal{A}$ represented by bounded operators on a Hilbert space $\mathcal{H}$ and an unbounded selfadjoint operator D

on $\mathcal{H}$, such that each commutator $[D, a]$, for $a \in \mathcal{A}$ (densely defined as an operator on $\mathcal{H}$) extends to a bounded operator; it is understood that $a \operatorname{Dom} D \subseteq \operatorname{Dom} D$.

To get an idea of the difficulties involved in the choice of $\mathcal{A}$, consider the commutative case, say of the manifold $\mathbb{R}^k$. Depending on the fall-off conditions deemed suitable, the smooth nonunital algebras that can represent the manifold are numerous as the stars in the sky. The problem is compounded in the noncommutative case, say when $\mathcal{A}$ is a deformation of an algebra of functions. To be on the safe side, we will take a relatively small algebra at the start of our investigation of the Moyal examples; during its course, a larger candidate will emerge.

Also, when $\mathcal{A}$ is not unital, we need choose a preferred unitization $\widetilde{\mathcal{A}}$. Consideration of the links to K-theory and K-homology makes it prudent to require that $\mathcal{A}, \widetilde{\mathcal{A}}$ be pre-C^*-algebras, whose K-theories then coincide with those of their respective C^*-completions [17].

Denote by $\mathcal{K}(\mathcal{H})$ the compact operators on $\mathcal{H}$, and by $\mathcal{L}^p(\mathcal{H})$ the Schatten ideal in $\mathcal{K}(\mathcal{H})$ defined by a finite norm $\|A\|_p := \operatorname{Tr}(|A|^p)^{1/p}$, for $p \geqslant 1$. For compact or unital spectral triples, it is further required that the operator D have compact resolvent, that is, $(D-\lambda)^{-1}$ must belong to $\mathcal{K}(\mathcal{H})$ for $\lambda \notin \operatorname{sp} D$. Consequently D must have discrete spectrum of finite multiplicity. Since this is clearly not the case for the Dirac operator $\not{D}$ on $\mathbb{R}^k$, in the nonunital case we only demand [18] that $a(D-\lambda)^{-1}$ be compact for $a \in \mathcal{A}$. This condition ensures that the spectral triple $(\mathcal{A}, \mathcal{H}, D)$ corresponds to a well-defined K-homology class [51, Chap. 10], and could be termed 'Axiom 0' for an – in general noncompact – noncommutative geometry.

We turn to the several conditions which spectral triples must satisfy to yield noncommutative spin geometries. To formulate the generalization to the noncompact case, we focus first on commutative geometries. First in line there is a *summability* condition, namely that the operators $a(D^2+\varepsilon^2)^{-1}$ be not merely compact, but belong to the generalized Schatten class called $\mathcal{L}^{k+}$, with k an integer; this is a kind of kth root in the sense of operator products of the Dixmier trace class $\mathcal{L}^{1+}$ [17,45]. More concretely, a compact operator T belongs to $\mathcal{L}^{k+}$ if its singular values satisfy $\mu_m(T) = O(m^{1/k})$ as $m \to \infty$.

In the compact commutative case of a k-dimensional spin manifold, choosing D to be the ordinary Dirac operator $\not{D}$ on a spinor space $\mathcal{H}$, one finds that

$$\operatorname{Tr}^+(a|\not{D}|^{-k}) = C_k \int_M a(x)\, d^k x,$$

where Tr^+ denotes any Dixmier trace, for a universal constant C_k. For the noncompact commutative case, we expect $\operatorname{Tr}^+(\cdot\, |\not{D}|^{-k})$ still to exist for a suitable algebra of integrable functions, and we regard $\operatorname{Tr}^+(\cdot\, |\not{D}|^{-k})$ as a noncommutative integral. These two summability conditions together constitute Axiom 1.

A further necessary condition was *regularity* or *smoothness* of the spectral triple. If $\delta(T) := [|D|, T]$ for an operator T on $\mathcal{H}$, regularity means that each $a \in \mathcal{A}$ and each $[D, a]$ lies in the domain of δ^n for all $n \in \mathbb{N}$. In the commutative case, $|\not{D}|$ is a first-order pseudodifferential operator, and computing $\delta^n(a)$ is onerous; it is somewhat easier to handle the (commuting) operations

$$L(T) := |\not{D}|^{-1}\, [\not{D}^2, T], \qquad R(T) := [\not{D}^2, T]\, |\not{D}|^{-1}, \tag{3.1}$$

and one can show that the smooth domain of δ equals the common smooth domain of L and R [23]. If f is a Schwartz function acting on $L^2(\mathbb{R}^k) \otimes \mathbb{C}^{2^{\lfloor k/2 \rfloor}}$ as an (ordinary) multiplication operator, we can regard it as a pseudodifferential operator with symbol $f(x) \otimes 1_{2^{\lfloor k/2 \rfloor}}$, and one checks that $L^n R^m f$

is a bounded pseudodifferential operator of order (at most) zero. On the subject of regularity, the reader is advised to look at the discussions in [45, Sec. 10.3] and also in [50, 73].

There is no obvious need to modify this axiom in the noncompact case. However, at the technical level, $|D\!\!\!/|^{-1}$ is a somewhat more problematic object than in the compact case, and one must find a substitute for it.

The condition of *finiteness*, in the unital case, is that the smooth domain $\mathcal{H}^\infty$ of D in $\mathcal{H}$ be a finitely generated projective (left) module over the unital algebra $\mathcal{A}$, that is, $\mathcal{H}^\infty \simeq \mathcal{A}^m p$ for some projector $p = p^* = p^2$ in $M_m(\mathcal{A})$ with a suitable m.

In the case of $M = \mathbb{R}^k$, under either the pointwise or the Moyal product, the module of smooth spinors is free since the spinor bundle is trivial. However, when $\mathcal{A}$ is nonunital, to get a projective $\mathcal{A}$-module one should select the projector p in a matrix algebra over the preferred compactification $\widetilde{\mathcal{A}}$. See Rennie [72] for a discussion both of this point and of "pullback modules". Concretely, if $\mathcal{A}_1$ is an ideal of $\widetilde{\mathcal{A}}$ and if $\mathcal{E}$ is a left $\widetilde{\mathcal{A}}$-module, its *pullback* to $\mathcal{A}_1$ is the left $\mathcal{A}_1$-module $\mathcal{E}_1 := \mathcal{A}_1\mathcal{E}$.

The finiteness condition for the nonunital case should then demand that $\mathcal{H}^\infty$ densely contains a pullback of a finite projective $\widetilde{\mathcal{A}}$-module to $\mathcal{A}$; or, better still, that it can be identified with $\mathcal{A}_1^m p$, with $\mathcal{A}$ an ideal in $\mathcal{A}_1$ (thus $\widetilde{\mathcal{A}}$ is also a unitization of $\mathcal{A}_1$), for some m and some projector $p \in M_m(\widetilde{\mathcal{A}})$. Moreover, a hermitian structure should be defined on the module $\mathcal{H}^\infty$ through the noncommutative integral; we shall see the details of this further on.

We bring up next the axioms having an algebraic flavour. The *reality* condition is the existence of an antilinear conjugation operator J on $\mathcal{H}$ such that $a \mapsto Ja^*J^{-1}$ gives a second representation of $\mathcal{A}$ on $\mathcal{H}$ commuting with the original one, and with certain algebraic properties listed in [18, 20] and reviewed later: for the commutative case of spin manifolds, J is just the charge conjugation operator on spinors. There is no need to modify this axiom in the noncompact case.

The *first order* condition is that

$$[[D, a], Jb^*J^{-1}] = 0, \quad \text{for all} \quad a, b \in \mathcal{A}.$$

For the commutative case this is a simple check, since $D = D\!\!\!/$ is a first-order differential operator. There is no need to modify this axiom in the noncompact case.

The *orientability* condition is that the spectral triple $(\mathcal{A}, \mathcal{H}, D)$ carry an algebraic version of a "volume k-form", where k is the integer summability exponent ($k = 2N$ in the nondegenerate Moyal case). Let b_0^{op} denote $b_0 \in \mathcal{A}$ as an element of the opposite algebra $\mathcal{A}^{\mathrm{op}}$, with the product reversed; this algebraic version consists of a Hochschild k-cycle $\mathbf{c}$, that is, a sum of terms of the form $(a_0 \otimes b_0^{\mathrm{op}}) \otimes a_1 \otimes \cdots \otimes a_k$ satisfying $b\,\mathbf{c} = 0$ (cycle property), that we represent by bounded operators

$$\pi_D((a_0 \otimes b_0^{\mathrm{op}}) \otimes a_1 \otimes \cdots \otimes a_k) := a_0\, Jb_0^* J^{-1}\, [D, a_1] \dots [D, a_k], \tag{3.2}$$

and on which we impose $\pi_D(\mathbf{c}) = \chi$ (orientation), where χ is the given $\mathbb{Z}_2$-grading operator on $\mathcal{H}$. We just use $\chi = 1$ if k is odd; and, in the even case for ordinary spinors, one uses $\chi := (-i)^m\, \gamma^1\gamma^2 \dots \gamma^{2m}$.

For the commutative or noncommutative torus $C^\infty(\mathbb{T}^k_\Theta)$, with unitary generators $u_1, \dots, u_k$ satisfying

$$u_k u_j = e^{i\Theta_{jk}}\, u_j u_k, \tag{3.3}$$

the good Hochschild cycle is known [20, 45] to be

$$\mathbf{c} = \frac{(-i)^{\lfloor k/2 \rfloor}}{k!} \sum_\sigma (-1)^\sigma (u_{\sigma(1)} u_{\sigma(2)} \dots u_{\sigma(k)})^{-1} \otimes u_{\sigma(1)} \otimes u_{\sigma(2)} \otimes \cdots \otimes u_{\sigma(k)}, \tag{3.4}$$

where the sum is over all permutations of $1, 2, \dots, k$.

For nonunital algebras, we might expect something similar. However, the fact that the plane waves belong to $\mathcal{B}$ suggests, in the light of the NC torus example, taking the cycle over the unitization $\widetilde{\mathcal{A}}$ rather than $\mathcal{A}$ itself. This has the happy consequence of bypassing the many difficulties of Hochschild cohomology for nonunital algebras.

Poincaré duality for a noncompact orientable manifold M is usually expressed as the isomorphism between the compactly supported de Rham cohomology and the homology of M, mediated by the fundamental class $[M]$. In noncommutative geometry a K-theoretic version is in order. One would expect that some kind of compactly supported K-homology of the initial nonunital algebra $\mathcal{A}$ be isomorphic to its K-theory, through a fundamental K-homology class of $\mathcal{A} \otimes \mathcal{A}^{\mathrm{op}}$ given by the spectral triple itself. We shall actually leave aside the final condition of Poincaré duality in K-theory, since it is not central to the present form of the reconstruction theorem in the compact case [45], and the details of its reformulation in the nonunital noncommutative case are still somewhat clouded.

3.2 Modified conditions for nonunital spectral triples

Definition 3.1. By a *real noncompact spectral triple* of dimension k, we mean the data

$$(\mathcal{A}, \widetilde{\mathcal{A}}, \mathcal{H}, D, J, \chi),$$

where $\mathcal{A}$ is an (a priori nonunital) algebra acting faithfully (via a representation sometimes denoted by π) on the Hilbert space $\mathcal{H}$, $\widetilde{\mathcal{A}}$ is a preferred unitization of $\mathcal{A}$, acting the same Hilbert space, and D is an unbounded selfadjoint operator on $\mathcal{H}$ such that $[D, a]$, for each a in $\widetilde{\mathcal{A}}$, extends to a bounded operator on $\mathcal{H}$.

Furthermore, J and χ are respectively an antiunitary and a selfadjoint operator, such that $\chi = 1$ when k is odd, and otherwise $\chi^2 = 1$, $\chi a = a \chi$ for $a \in \mathcal{A}$, and $D\chi = -\chi D$, satisfying the conditions which follow.

0. *Compactness*:
   The operator $a(D - \lambda)^{-1}$ is compact for $a \in \mathcal{A}$ and $\lambda \notin \operatorname{sp} D$.

1. *Spectral dimension*:
   There is a unique nonnegative integer k, the spectral or "classical" dimension of the geometry, for which $a(D^2 + \varepsilon^2)^{-1/2}$ belongs to the generalized Schatten class $\mathcal{L}^{k+}$ for each $a \in \mathcal{A}$ and moreover $\operatorname{Tr}^+(a(|D| + \varepsilon)^{-k})$ is finite and not identically zero, for any $\varepsilon > 0$. This k is even if and only if the spectral triple is even.

2. *Regularity*:
   The bounded operators a and $[D, a]$, for each $a \in \widetilde{\mathcal{A}}$, lie in the smooth domain of the derivation $\delta : T \mapsto [|D|, T]$.

3. *Finiteness*:
   The algebra $\mathcal{A}$ and its preferred unitization $\widetilde{\mathcal{A}}$ are pre-C^*-algebras. There exists an ideal $\mathcal{A}_1$ of $\widetilde{\mathcal{A}}$, including $\mathcal{A}$, which is also a pre-C^*-algebra with the same C^*-completion as $\mathcal{A}$, such that the space of smooth vectors

$$C^\infty(D) \equiv \mathcal{H}^\infty := \bigcap_{k \in \mathbb{N}} \operatorname{Dom}(D^k)$$

is an $\mathcal{A}_1$-pullback of a finite projective $\widetilde{\mathcal{A}}$-module. Moreover, an $\mathcal{A}_1$-valued hermitian structure $(\cdot \mid \cdot)$ is implicitly defined on $\mathcal{H}^\infty$ with the noncommutative integral, as follows:

$$\operatorname{Tr}^+\big((a\xi \mid \eta)(|D| + \varepsilon)^{-k}\big) = \langle \eta \mid a\xi \rangle, \tag{3.5}$$

where $a \in \widetilde{\mathcal{A}}$ and $\langle \cdot \mid \cdot \rangle$ denotes the standard inner product on $\mathcal{H}$. This is an absolute continuity condition, since $(\cdot \mid \cdot)$ is a kind of Radon–Nikodym derivative with respect to the functional $\operatorname{Tr}^+(\cdot\,(|D| + \varepsilon)^{-k})$.

4. *Reality*:
There is an antiunitary operator J on $\mathcal{H}$, such that $[a, Jb^*J^{-1}] = 0$ for all $a, b \in \widetilde{\mathcal{A}}$ (thus $b \mapsto Jb^*J^{-1}$ is a commuting representation on $\mathcal{H}$ of the opposite algebra $\mathcal{A}^{\mathrm{op}}$). Moreover, $J^2 = \pm 1$ and $JD = \pm DJ$, and also $J\chi = \pm \chi J$ in the even case, where the signs depend only on k mod 8. Here is the table for the even case; see the full table in [45, p. 405].

N mod 4	0	1	2	3
$J^2 = \pm 1$	+	−	−	+
$JD = \pm DJ$	+	+	+	+
$J\chi = \pm \chi J$	+	−	+	−

(3.6)

5. *First order*:
The bounded operators $[D, a]$ also commute with the opposite algebra representation: $[[D, a], Jb^*J^{-1}] = 0$ for all $a, b \in \widetilde{\mathcal{A}}$.

6. *Orientation*:
There is a *Hochschild k-cycle* $\mathbf{c}$ on $\widetilde{\mathcal{A}}$, with values in $\widetilde{\mathcal{A}} \otimes \widetilde{\mathcal{A}}^{\mathrm{op}}$. Such a k-cycle is a finite sum of terms like $(a \otimes b^{\mathrm{op}}) \otimes a_1 \otimes \cdots \otimes a_k$, whose natural representative by operators on $\mathcal{H}$ is given by $\pi_D(\mathbf{c})$ in formula (3.2); the "volume form" $\pi_D(\mathbf{c})$ must solve the equation

$$\pi_D(\mathbf{c}) = \chi \quad \text{(even case)}, \quad \text{or} \quad \pi_D(\mathbf{c}) = 1 \quad \text{(odd case)}. \tag{3.7}$$

Finally, a geometry is called *connected* or irreducible if the only operators commuting with $\mathcal{A}$ and D are the scalars. We are mainly interested in connected noncompact noncommutative geometries.

The discussion in the previous subsection, and this proposal, are very much in the vein of [41]. We may also keep the concept in that article of "star triples", a specialization of the spectral triple to deformations of the algebra of functions on a noncompact manifold, wherein the Dirac operator (is possibly deformed, but) remains an ordinary (pseudo-)differential operator on that original manifold. However, the authors of [41] got carried away in that they confused properties of L^θ_f with properties of the Weyl pseudodifferential operator associated (by the Schrödinger representation) to the "symbol" f. And thus, the dreaded "dimension drop", apparent there, does not actually take place. But before going to the Moyal case, we need to reexamine the commutative case.

3.3 The commutative case

The outcome of the discussion in subsection 3.1 is that the main outstanding issues, in order to obtain noncompact noncommutative spin geometries, are the analytical ones.

Let $\mathcal{A}$ be some appropriate subalgebra of $C^\infty(M)$ and $D\!\!\!/$ be the Dirac operator, with k equal to the ordinary dimension of the spin manifold M. Let $\mathcal{H}$ be the space of square-integrable spinors. Then $[D\!\!\!/, f] = D\!\!\!/(f)$, just as in the unital case, and so the boundedness of $[D, \mathcal{A}]$ is unproblematic. In order to check whether $(\mathcal{A}, \mathcal{H}, D\!\!\!/, \chi)$ is a spectral triple in our sense, one first needs to determine whether products of the form $f(|D\!\!\!/| + \varepsilon)^{-k}$ are compact operators of Dixmier trace class, whose Dixmier trace is (a standard multiple of) $\int f(x)\, d^k x$. This compactness condition is guaranteed in the flat space case (taking $\mathcal{A} = \mathcal{S}(\mathbb{R}^k)$, say) by celebrated estimates in scattering theory [81], that we review in subsection 4.1.

The summability condition is a bit tougher. The Cesàro summability theory of [34] establishes that, for a positive pseudodifferential operator H of order d, acting on spinors, the spectral density asymptotically behaves as

$$d_H(x, x; \lambda') \sim \frac{2^{\lfloor k/2 \rfloor}}{d\,(2\pi)^k} \big(\operatorname{wres} H^{-k/d}\, (\lambda')^{(k-d)/d} + \cdots\big),$$

in the Cesàro sense. Here wres denotes the Wodzicki residue density [45]. (If the operator is not positive, one uses the "four parts" argument.) In our case, $H = a(|D\!\!\!/| + \varepsilon)^{-k}$ is pseudodifferential of order $-k$, so

$$d_H(x, x; \lambda') \sim -\frac{2^{\lfloor k/2 \rfloor}\, \Omega_k\, a(x)}{k\,(2\pi)^k}\, (\lambda'^{-2} + \cdots),$$

as $\lambda' \to \infty$ in the Cesàro sense; here Ω_k is the hyperarea of the unit sphere in $\mathbb{R}^k$. We independently know that H is compact, so on integrating the spectral density over x and over $0 \leqslant \lambda' \leqslant \lambda$, we get

$$N_H(\lambda) \sim \frac{2^{\lfloor k/2 \rfloor}\, \Omega_k \int a(x)\, d^k x}{k\,(2\pi)^k}\, \frac{1}{\lambda} \quad \text{as } \lambda \to \infty.$$

This holds in the ordinary asymptotic sense, and not merely the Cesàro sense, by the "sandwich" argument used in the proof of [34, Cor. 4.1]. So finally,

$$\lambda_m(H) \sim \frac{2^{\lfloor k/2 \rfloor}\, \Omega_k \int a(x)\, d^k x}{k\,(2\pi)^k}\, \frac{1}{m} \quad \text{as } m \to \infty, \tag{3.8}$$

and the Dixmier traceability of $a(|D\!\!\!/| + \varepsilon)^{-k}$, plus the value of its trace, follow at once.

The rest is a long but almost trivial verification. For instance, J is the charge conjugation operator on spinors; the algebra $(\mathcal{B}, \star_0)$ is a suitable compactification; the domain $\mathcal{H}^\infty$ consists of the smooth spinors; and so on. See below the parallel discussion for the Moyal case.

The following theorem sums it up.

Theorem 3.1. *The triple* $(\mathcal{S}(\mathbb{R}^k), L^2(\mathbb{R}^k) \otimes \mathbb{C}^{2^{\lfloor k/2 \rfloor}}, D\!\!\!/)$ *on* $\mathbb{R}^k$ *defines a noncompact commutative geometry of spectral dimension* k.

What about the nonflat case (of a spin manifold such that $D\!\!\!/$ is selfadjoint)? Mainly because the previous Cesàro summability argument is purely local, everything carries over, if we choose for $\mathcal{A}$

the algebra of smooth and compactly supported functions. Of course, in some contexts it may be useful to demand that M also has conic exits.

We want to remark that formula (3.8) for the flat case has been proved by Chakraborty et al in [11], using an ingenious reasoning involving two Laplacians on $\mathbb{R}^k$. Theirs is a kind of "poor man's argument" for ours, because what it is really used is that the spectral density has the same asymptotic behaviour for the two Laplacians. Also, our inference is not confined to flat manifolds, rather it is directly valid on any decent noncompact manifold (without recourse to "lifting" devices).

3.4 On the Connes–Landi spaces example

An interesting family of compact spectral triples was constructed by Connes and Landi [25], by isospectral deformation of a commutative spectral triple wherein the Dirac operator is kept fixed (just as for our Moyal-product example) but the algebra is "twisted". One starts with a smooth boundaryless manifold M carrying a smooth effective action of a torus $\mathbb{T}^k$ of dimension $k \geqslant 2$. The orbits on which $\mathbb{T}^k$ acts freely determine maps $C^\infty(M) \to C^\infty(\mathbb{T}^k)$, and with these maps one can pull back the NC torus structure on $C^\infty(\mathbb{T}^k_\Theta) := (C^\infty(\mathbb{T}^k), *_\Theta)$ to get an algebra $C^\infty(M_\Theta) := (C^\infty(M), *_\Theta)$. This algebra is given in fact by a periodic Moyal product just like (2.1), with the translations replaced by the $\mathbb{T}^k$-action. See [26, 82, 87, 88] for several equivalent formulations of this construction.

Now, as pointed out in [26], there is no need to assume that the manifold M be compact: we only need that the group action on M be periodic. Taking $M = \mathbb{R}^k$, we get a noncompact spectral triple which is not isomorphic to the Moyal product examples considered in this article; one can regard it as intermediate between the commutative case and the full Moyal cases (with nonperiodic action).

Concretely, the sphere $\mathbb{S}^{2N-1} = SO(2N)/SO(2N-1)$ carries an effective action of $\mathbb{T}^N$, namely the rotations by elements of a maximal torus of $SO(2N)$; and this extends to a $\mathbb{T}^N$-action by rotations of $\mathbb{R}^{2N}$ preserving the radial coordinate r. Each $f \in \mathcal{S}$ is a function of coordinates $f(r, \alpha_1, \dots, \alpha_{N-1}, \phi_1, \dots, \phi_N)$ where $\phi = (\phi_1, \dots, \phi_N) \in \mathbb{T}^N$. If the equation (2.1) is interpreted as involving integration over the ϕ_j coordinates only, it defines a new twisted product on $\mathcal{S}$ (for each real skewsymmetric $N \times N$ matrix Θ).

To define a spectral triple over this algebra, we need an operator D which is also $\mathbb{T}^N$-invariant. For instance, one can construct D by extending radially the Dirac operator for (say) the round metric on $\mathbb{S}^{2N-1}$, with its spinor bundle; it will be necessary to lift the torus action to a doubly covering action of $\mathbb{T}^n$ on spinors [26]. It remains to check that $\mathcal{B}$ is still a suitable unitization of $\mathcal{S}$ (note that abstract smoothness of $\mathcal{B}$ is proved like in Section 2 here [87]) in the case of the Connes–Landi twisted $2N$-planes, in order to conclude that these fit into the framework developed in this paper.

4 The Moyal $2N$-plane as a spectral triple

There is a natural star triple associated to the Moyal plane and we shall see that it is part of the data for an even spectral triple fulfilling all required conditions.

Let $\mathcal{A} = (\mathcal{S}(\mathbb{R}^{2N}), \star_\theta)$, with preferred unitization $\widetilde{\mathcal{A}} := (\mathcal{B}(\mathbb{R}^{2N}), \star_\theta)$. The Hilbert space will be $\mathcal{H} := L^2(\mathbb{R}^{2N}) \otimes \mathbb{C}^{2^N}$ of ordinary square-integrable spinors. The representation of $\mathcal{A}$ is given by $\pi^\theta \colon \mathcal{A} \to \mathcal{L}(\mathcal{H}) : f \mapsto L^\theta_f \otimes 1_{2^N}$, where L^θ_f acts on the "reduced" Hilbert space $\mathcal{H}_r := L^2(\mathbb{R}^{2N})$.

In other words, if $a \in \mathcal{A}$ and $\Psi \in \mathcal{H}$, to obtain $\pi^\theta(a)\Psi$ we just left Moyal multiply Ψ by a componentwise.

This operator $\pi^\theta(f)$ is bounded, since it acts diagonally on $\mathcal{H}$ and $\|L^\theta_f\| \leqslant (2\pi\theta)^{-N/2}\|f\|_2$ was proved in Lemma 2.8. Under this action, the elements of $\mathcal{H}$ get the lofty name of *Moyal spinors*.

The selfadjoint Dirac operator is not "deformed": it will be the ordinary Euclidean Dirac operator $D\!\!\!\!/ := -i\,\gamma^\mu\partial_\mu$, where the hermitian Dirac matrices $\gamma^1,\dots,\gamma^{2N}$ satisfying $\{\gamma^\mu,\gamma^\nu\} = +2\,\delta^{\mu\nu}$ irreducibly represent the Clifford algebra $C\ell(\mathbb{R}^{2N})$ associated to $(\mathbb{R}^{2N},\eta)$, with η the standard Euclidean metric.

As a grading operator χ we take the usual chirality associated to the Clifford algebra:

$$\chi := \gamma_{2N+1} := 1_{\mathcal{H}_r} \otimes (-i)^N\gamma^1\gamma^2\dots\gamma^{2N}.$$

The notation γ_{2N+1} is a nod to physicists' γ_5. Thus $\chi^2 = (-1)^N(\gamma^1\dots\gamma^{2N})^2 = (-1)^{2N} = 1$ and $\chi\gamma^\mu = -\gamma^\mu\chi$.

The real structure J is chosen to be the usual charge conjugation operator for spinors on $\mathbb{R}^{2N}$ endowed with an Euclidean metric. Here, we only assume that $J^2 = \pm 1$ according to the "sign table" (3.6) and that

$$J(1_{\mathcal{H}_r} \otimes \gamma^\mu)J^{-1} = -1_{\mathcal{H}_r} \otimes \gamma^\mu$$

which guarantees the other requirements of (3.6). In general, in a given representation, it can be written as

$$J := CK, \tag{4.1}$$

where C denotes a suitable $2^N \times 2^N$ unitary matrix and K means complex conjugation. An explicit form for J in a particular representation can be found in [98] where all γ^μ are hermitian matrices with purely imaginary (respectively real) entries when μ is even (respectively odd).

An important property of J is

$$J\big(L^\theta(f^*) \otimes 1_{2^N}\big)J^{-1} = R^\theta(f) \otimes 1_{2^N}, \tag{4.2}$$

where $R^\theta(f) \equiv R^\theta_f$ is the right Moyal multiplication by f; this follows from the antilinearity of J and the reversal of the twisted product under complex conjugation.

Lemma 2.1(iii) implies that $[D\!\!\!\!/, \pi^\theta(f)] = -iL^\theta(\partial_\mu f) \otimes \gamma^\mu =: \pi^\theta(D\!\!\!\!/(f))$; by Theorem 2.16 this is bounded for $f \in \widetilde{\mathcal{A}}_\theta = \mathcal{B}(\mathbb{R}^{2N})$ – just as in the commutative case.

4.1 The compactness condition

In this subsection and the next, the main tools are techniques developed some time ago for scattering theory problems, as summarized in Simon's booklet [81, Chap. 4]. We adopt the convention that $\mathcal{L}^\infty(\mathcal{H}) := \mathcal{K}(\mathcal{H})$, with $\|A\|_\infty := \|A\|_{\mathrm{op}}$.

Let $g \in L^\infty(\mathbb{R}^{2N})$. We define the operator $g(-i\nabla)$ on $\mathcal{H}_r$ as

$$g(-i\nabla)\psi := \mathcal{F}^{-1}(g\,\mathcal{F}\psi),$$

where $\mathcal{F}$ is the ordinary Fourier transform. More in detail, for ψ in the correct domain,

$$g(-i\nabla)\psi(x) = (2\pi)^{-2N}\iint e^{i\xi\cdot(x-y)}\,g(\xi)\psi(y)\,d^{2N}\xi\,d^{2N}y.$$

The inequality $\|g(-i\nabla)\psi\|_2 = \|\mathcal{F}^{-1}g\mathcal{F}\psi\|_2 \leqslant \|g\|_\infty\|\psi\|_2$ entails that $\|g(-i\nabla)\|_\infty \leqslant \|g\|_\infty$.

Theorem 4.1. *Let $f \in \mathcal{A}$ and $\lambda \notin \operatorname{sp} D\!\!\!/$. Then, if $R_{D\!\!\!/}(\lambda)$ is the resolvent operator of $D\!\!\!/$, then $\pi^\theta(f)\, R_{D\!\!\!/}(\lambda)$ is compact.*

Thanks to the first resolvent equation, $R_{D\!\!\!/}(\lambda) = R_{D\!\!\!/}(\lambda') + (\lambda' - \lambda) R_{D\!\!\!/}(\lambda) R_{D\!\!\!/}(\lambda')$, we may assume that $\lambda = i\mu$ with $\mu \in \mathbb{R}^*$. The theorem will follow from a series of lemmas interesting in themselves.

Lemma 4.2. *If $f \in \mathcal{S}$ and $0 \neq \mu \in \mathbb{R}$, then*

$$\pi^\theta(f) R_{D\!\!\!/}(i\mu) \in \mathcal{K}(\mathcal{H}) \iff \pi^\theta(f) |R_{D\!\!\!/}(i\mu)|^2 \in \mathcal{K}(\mathcal{H}).$$

Proof. We know that $L^\theta(f)^* = L^\theta(f^*)$. The "only if" part is obvious since $R_{D\!\!\!/}(i\mu)$ is a bounded normal operator. Conversely, if $\pi^\theta(f)|R_{D\!\!\!/}(i\mu)|^2$ is compact, then $\pi^\theta(f)|R_{D\!\!\!/}(i\mu)|^2 \pi^\theta(f^*)$ is compact. Since an operator T is compact if and only if TT^* is compact, the proof is complete. □

The usefulness of this lemma stems from the diagonal nature of the action of $\pi^\theta(f)|R_{D\!\!\!/}(i\mu)|^2$ on $\mathcal{H} = \mathcal{H}_r \otimes \mathbb{C}^{2^N}$; so in our arguments it is feasible to replace $\mathcal{H}$ by $\mathcal{H}_r$, $\pi^\theta(f)$ by L^θ_f, and to use the scalar Laplacian $-\Delta := -\sum_{\mu=1}^{2N} \partial_\mu^2$ instead of the square of the Dirac operator $D\!\!\!/^2$.

Lemma 4.3. *When $f, g \in \mathcal{H}_r$, $L^\theta_f\, g(-i\nabla)$ is a Hilbert–Schmidt operator such that, for all real θ,*

$$\|L^\theta_f\, g(-i\nabla)\|_2 = \|L^0_f\, g(-i\nabla)\|_2 = (2\pi)^{-N} \|f\|_2\, \|g\|_2.$$

Proof. To prove that an operator A with integral kernel K_A is Hilbert–Schmidt, it suffices to check that $\int |K_A(x,y)|^2\, dx\, dy$ is finite, and this will be equal to $\|A\|_2^2$ [81, Thm. 2.11]. So we compute $K_{L^\theta(f)\, g(-i\nabla)}$. In view of Lemma 2.11,

$$[L^\theta(f)\, g(-i\nabla)\psi](x) = \frac{1}{(2\pi)^{2N}} \iint f(x - \tfrac{\theta}{2} S\xi)\, g(\xi) \psi(y)\, e^{i\xi\cdot(x-y)}\, d^{2N}\xi\, d^{2N}y.$$

Thus

$$K_{L^\theta(f)\, g(-i\nabla)}(x,y) = \frac{1}{(2\pi)^{2N}} \int f(x - \tfrac{\theta}{2} S\xi)\, g(\xi)\, e^{i\xi\cdot(x-y)}\, d^{2N}\xi,$$

and $\int |K_{L^\theta(f)\, g(-i\nabla)}(x,y)|^2\, dx\, dy$ is given by

$$\begin{aligned} &\frac{1}{(2\pi)^{4N}} \iiiint \bar f(x - \tfrac{\theta}{2} S\xi)\, \bar g(\xi)\, f(x - \tfrac{\theta}{2} S\zeta)\, g(\zeta)\, e^{i(x-y)\cdot(\zeta-\xi)}\, d^{2N}x\, d^{2N}y\, d^{2N}\zeta\, d^{2N}\xi \\ &\quad = \frac{1}{(2\pi)^{2N}} \iint |f(x - \tfrac{\theta}{2} S\xi)|^2\, |g(\xi)|^2\, d^{2N}x\, d^{2N}\xi = (2\pi)^{-2N} \|f\|_2^2\, \|g\|_2^2 < \infty. \end{aligned}$$ □

Remark 4.4. As a consequence, we get

$$\|.\|_2\text{-}\lim_{\theta\to 0} L^\theta_f\, g(-i\nabla) = L^0_f\, g(-i\nabla).$$

Lemma 4.5. *If $f \in \mathcal{H}_r$ and $g \in L^p(\mathbb{R}^{2N})$ with $2 \leqslant p < \infty$, then $L^\theta_f\, g(-i\nabla) \in \mathcal{L}^p(\mathcal{H}_r)$ and*

$$\|L^\theta_f\, g(-i\nabla)\|_p \leqslant (2\pi)^{-N(1/2+1/p)} \theta^{-N(1/2-1/p)}\, \|f\|_2\, \|g\|_p.$$

Proof. The case $p = 2$ (with equality) is just the previous lemma. For $p = \infty$, we estimate $\|L^\theta_f\, g(-i\nabla)\|_\infty \leqslant (2\pi\theta)^{-N/2}\|f\|_2\,\|g\|_\infty$: since $\|L^\theta_f\, g(-i\nabla)\|_\infty \leqslant \|L^\theta_f\|_\infty\,\|g(-i\nabla)\|_\infty$, this follows from Lemma 2.8 and a previous remark.

Now use complex interpolation for $2 < p < \infty$. For that, we first note that we may suppose $g \geqslant 0$: defining the function a with $|a| = 1$ and $g = a|g|$, we see that

$$\begin{aligned}\|L^\theta_f\, g(-i\nabla)\|_2^2 &= \mathrm{Tr}(|L^\theta_f\, g(-i\nabla)|^2) = \mathrm{Tr}(\bar g(-i\nabla)\, L^\theta_{f^*}\, L^\theta_f\, g(-i\nabla)) \\ &= \mathrm{Tr}(|g|(-i\nabla)\, \bar a(-i\nabla)\, L^\theta_{f^*}\, L^\theta_f\, a(-i\nabla)\, |g|(-i\nabla)) \\ &= \mathrm{Tr}(\bar a(-i\nabla)\, |g|(-i\nabla)\, L^\theta_{f^*}\, L^\theta_f\, |g|(-i\nabla)\, a(-i\nabla)) \\ &= \mathrm{Tr}(|L^\theta_f\, |g|(-i\nabla)|^2) = \|L^\theta_f\, |g|(-i\nabla)\|_2^2,\end{aligned}$$

and

$$\begin{aligned}\|L^\theta_f\, g(-i\nabla)\|_\infty &= \|L^\theta_f\, a(-i\nabla)\, |g|(-i\nabla)\|_\infty = \|L^\theta_f\, |g|(-i\nabla)\, a(-i\nabla)\|_\infty \\ &\leqslant \|L^\theta_f\, |g|(-i\nabla)\|_\infty\, \|a(-i\nabla)\|_\infty = \|L^\theta_f\, |g|(-i\nabla)\|_\infty.\end{aligned}$$

Secondly, for any positive, bounded function g with compact support, we define the maps:

$$F_p : z \mapsto L^\theta_f\, g^{zp}(-i\nabla) : S = \{\, z \in \mathbb{C} : 0 \leqslant \Re z \leqslant \tfrac{1}{2} \,\} \to \mathcal{L}(\mathcal{H}_r).$$

For all $y \in \mathbb{R}$, $F_p(iy) = L^\theta_f\, g^{iyp}(-i\nabla) \in \mathcal{L}^\infty(\mathcal{H}_r)$ by Lemma 4.3 since g, being compactly supported, lies in $\mathcal{H}_r$. Moreover, $\|F_p(iy)\|_\infty \leqslant (2\pi\theta)^{-N/2}\|f\|_2$.

Also, by Lemma 4.3, $F_p(\frac{1}{2} + iy) \in \mathcal{L}^2(\mathcal{H}_r)$ and $\|F_p(\frac{1}{2} + iy)\|_2 = (2\pi)^{-N}\|f\|_2\,\|g^{p/2}\|_2$. Then complex interpolation (see [70, Chap. 9] and [81]) yields $F(z) \in \mathcal{L}^{1/\Re z}(\mathcal{H}_r)$, for all z in the strip S. Moreover,

$$\|F_p(z)\|_{1/\Re z} \leqslant \|F(0)\|_\infty^{1-2\Re z}\, \|F(\tfrac{1}{2})\|_2^{2\Re z} = \|f\|_2 (2\pi\theta)^{-\frac{N}{2}(1-2\Re z)} (2\pi)^{-2N\Re z}\, \|g^{p/2}\|_2^{2\Re z},$$

and applying this result at $z = 1/p$, we get for such g:

$$\|L^\theta_f\, g(-i\nabla)\|_p = \|F(1/p)\|_p \leqslant (2\pi)^{-N(1/2+1/p)}\theta^{-N(1/2-1/p)}\|f\|_2\,\|g\|_p.$$

We finish by using the density of compactly supported bounded functions in $L^p(\mathbb{R}^{2N})$. □

Remark 4.6. In the commutative case, if f and g are bounded on $\mathbb{R}^k$, then $\|f(x)\, g(-i\nabla)\|_\infty \leqslant \|f\|_\infty\,\|g\|_\infty$. Complex interpolation [8, 70, 81] leads then to an estimate of the form

$$\|f(x)\, g(-i\nabla)\|_p \leqslant (2\pi)^{-k/p}\, \|f\|_p\, \|g\|_p$$

when $p \geqslant 2$. For $f \in \mathcal{S}$ and for $g(y) := 1/\sqrt{|y|^2 + \mu^2}$, which lies in $L^p(\mathbb{R}^k)$ for all $p > k$ we conclude that $f(x)\, g(-i\nabla)$ is compact and in $\mathcal{L}^p$ for $p > k$. This already strongly pointed to compliance with Axiom 1 (verified above using Cesàro summability considerations), since $\mathcal{L}^{k+}$ is larger than $\mathcal{L}^k$, but smaller than the intersection of the $\mathcal{L}^p$ for $p > k$.

Lemma 4.7. *If $f \in \mathcal{S}$ and $0 \neq \mu \in \mathbb{R}$, then $\pi^\theta(f)\, |R_{\not D}(i\mu)|^2 \in \mathcal{L}^p$ for $p > N$.*

Proof. We see that

$$\pi^\theta(f)\,|R_{D\!\!\!/}(i\mu)|^2 = (L^\theta_f \otimes 1_{2^N})\,(D\!\!\!/ - i\mu)^{-1}(D\!\!\!/ + i\mu)^{-1} = L^\theta_f\,(-\partial^\nu\partial_\nu + \mu^2)^{-1} \otimes 1_{2^N}.$$

So this operator acts diagonally on $\mathcal{H}_r \otimes \mathbb{C}^{2^N}$ and Lemma 4.5 implies that

$$\big\|L^\theta_f\,(-\partial^\nu\partial_\nu + \mu^2)^{-1}\big\|_p \leqslant (2\pi)^{-N(1/2+1/p)}\theta^{-N(1/2-1/p)}\,\|f\|_2\bigg(\int \frac{d^{2N}\xi}{(\xi^\nu\xi_\nu + \mu^2)^p}\bigg)^{1/p},$$

which is finite for $p > N$. □

Proof of Theorem 4.1. By Lemma 4.2, it was enough to prove that $\pi^\theta(f)\,|R_{D\!\!\!/}(i\mu)|^2$ is compact for a nonzero real μ. □

The conclusion is that $(\mathcal{A}, \widetilde{\mathcal{A}}, \mathcal{H}, D\!\!\!/, \chi, J)$ defines a noncompact spectral triple; recall that we proved in Section 2 that both $\mathcal{A}$ and its preferred compactification $\widetilde{\mathcal{A}}$ are pre-C^*-algebras.

4.2 Spectral dimension of the Moyal planes

Theorem 4.8. *The spectral dimension of the Moyal $2N$-plane spectral triple is $2N$.*

We shall first establish existence properties. Thanks to Lemma 4.5 and because $[D\!\!\!/, \pi^\theta(f)] = -iL^\theta(\partial_\mu f) \otimes \gamma^\mu$, we see that $\pi^\theta(f)(D\!\!\!/^2 + \varepsilon^2)^{-l}$ and $[D\!\!\!/, \pi^\theta(f)]\,(D\!\!\!/^2 + \varepsilon^2)^{-l}$ lie in $\mathcal{L}^p(\mathcal{H})$ whenever $p > N/l$ (we always assume $\varepsilon > 0$). In the next lemma, we show that $[|D\!\!\!/|, \pi^\theta(f)]\,(D\!\!\!/^2 + \varepsilon^2)^{-l}$ has the same property of summability; this will become our main technical instrument for the subsection.

Lemma 4.9. *If $f \in \mathcal{S}$ and $\frac{1}{2} \leqslant l \leqslant N$, then $[|D\!\!\!/|, \pi^\theta(f)]\,(D\!\!\!/^2 + \varepsilon^2)^{-l} \in \mathcal{L}^p(\mathcal{H})$ for $p > N/l$.*

Proof. We use the following spectral identity for a positive operator A:

$$A = \frac{1}{\pi}\int_0^\infty \frac{A^2}{A^2 + \mu}\,\frac{d\mu}{\sqrt{\mu}},$$

and another identity for any operators A, B and $\lambda \notin \operatorname{sp} A$:

$$[B, (A - \lambda)^{-1}] = (A - \lambda)^{-1}[A, B](A - \lambda)^{-1}. \tag{4.3}$$

Hence, for any $\rho > 0$,

$$\begin{aligned}
[|D\!\!\!/|, \pi^\theta(f)] = [|D\!\!\!/| + \rho, \pi^\theta(f)] &= \frac{1}{\pi}\int_0^\infty \bigg[\frac{(|D\!\!\!/| + \rho)^2}{(|D\!\!\!/| + \rho)^2 + \mu}, \pi^\theta(f)\bigg]\frac{d\mu}{\sqrt{\mu}} \\
&= \frac{1}{\pi}\int_0^\infty \bigg(1 - \frac{(|D\!\!\!/| + \rho)^2}{(|D\!\!\!/| + \rho)^2 + \mu}\bigg)\big[(|D\!\!\!/| + \rho)^2, \pi^\theta(f)\big]\frac{1}{(|D\!\!\!/| + \rho)^2 + \mu}\,\frac{d\mu}{\sqrt{\mu}} \\
&= \frac{1}{\pi}\int_0^\infty \frac{1}{(|D\!\!\!/| + \rho)^2 + \mu}\big[(|D\!\!\!/| + \rho)^2, \pi^\theta(f)\big]\frac{1}{(|D\!\!\!/| + \rho)^2 + \mu}\sqrt{\mu}\,d\mu \qquad (4.4) \\
&= \frac{1}{\pi}\int_0^\infty \frac{1}{(|D\!\!\!/| + \rho)^2 + \mu}\Big(-\pi^\theta(\partial^\mu\partial_\mu f) - 2i(L^\theta(\partial_\mu f) \otimes \gamma^\mu)D\!\!\!/ + 2\rho\big[|D\!\!\!/|, \pi^\theta(f)\big]\Big) \\
&\qquad \times \frac{1}{(|D\!\!\!/| + \rho)^2 + \mu}\sqrt{\mu}\,d\mu.
\end{aligned}$$

This implies that

$$\big\|[|D\!\!\!/|,\pi^\theta(f)]\,(D\!\!\!/^2+\varepsilon^2)^{-l}\big\|_p \leqslant \frac{1}{\pi}\int_0^\infty \bigg\|\frac{1}{(|D\!\!\!/|+\rho)^2+\mu}\Big(-\pi^\theta(\partial^\mu\partial_\mu f) - 2i(L^\theta(\partial_\mu f)\otimes\gamma^\mu)D\!\!\!/ + 2\rho\big[|D\!\!\!/|,\pi^\theta(f)\big]\Big)\frac{1}{(|D\!\!\!/|+\rho)^2+\mu}\,(D\!\!\!/^2+\varepsilon^2)^{-l}\bigg\|_p\,\sqrt{\mu}\,d\mu.$$

Thus, the proof reduces to show that for any $f \in \mathcal{S}$,

$$\frac{1}{\pi}\int_0^\infty \bigg\|\frac{1}{(|D\!\!\!/|+\rho)^2+\mu}\,\pi^\theta(f)D\!\!\!/\,\frac{1}{(|D\!\!\!/|+\rho)^2+\mu}\,(D\!\!\!/^2+\varepsilon^2)^{-l}\bigg\|_p\,\sqrt{\mu}\,d\mu < \infty. \tag{4.5}$$

Since the Schatten p-norm is a symmetric norm, and since, as in the proof of Theorem 4.1, only the reduced Hilbert space is affected, expression (4.5) is majorized by

$$\frac{1}{\pi}\int_0^\infty \bigg\|\frac{1}{(|D\!\!\!/|+\rho)^2+\mu}\bigg\|^{3/2}\bigg\|\frac{D\!\!\!/}{(D\!\!\!/^2+\varepsilon^2)^{1/2}}\bigg\|\,\bigg\|\pi^\theta(f)\frac{1}{(D\!\!\!/^2+\varepsilon^2)^{l-1/2}}\frac{1}{((|D\!\!\!/|+\rho)^2+\mu)^{1/2}}\bigg\|_p\,\sqrt{\mu}\,d\mu$$
$$\leqslant \frac{1}{\pi}\int_0^\infty \big\|\pi^\theta(f)\,(D\!\!\!/^2+\varepsilon^2)^{-l+1/2}((|D\!\!\!/|+\rho)^2+\mu)^{-1/2}\big\|_p\,\frac{\sqrt{\mu}\,d\mu}{(\mu+\rho^2)^{3/2}}.$$

Thanks to Lemma 4.5, we can estimate the μ-dependence of the last p-norm:

$$\begin{aligned}&\big\|\pi^\theta(f)((|D\!\!\!/|+\rho)^2+\mu)^{-1/2}(D\!\!\!/^2+\varepsilon^2)^{-l+1/2}\big\|_p\\ &\quad\leqslant (2\pi)^{-N(1/2+1/p)}\theta^{-N(1/2-1/p)}\|f\|_2\,\big\|((|\xi|+\rho)^2+\mu)^{-1/2}(|\xi|^2+\varepsilon^2)^{-l+1/2}\big\|_p\\ &\quad\leqslant C(p,\theta)\big\|((|\xi|+\rho)^2+\mu)^{-1/2}\big\|_q\,\big\|(|\xi|^2+\varepsilon^2)^{-l+1/2}\big\|_r;\end{aligned}$$

with $p^{-1} = q^{-1}+r^{-1}$ appropriately chosen, these integrals are finite for $q > 2N$ and $r > 2N/(2l-1)$; for $l = \frac{1}{2}$, take $r = \infty$ and $q = p$. For such values,

$$\begin{aligned}&\big\|\pi^\theta(f)((|D\!\!\!/|+\rho)^2+\mu)^{-1/2}(D\!\!\!/^2+\varepsilon^2)^{-l+1/2}\big\|_p\\ &\leqslant C(p,\theta,N;f)\|(|\xi|^2+\varepsilon^2)^{-l+1/2}\|_r\,\Omega_{2N}^{1/q}\bigg(\int_0^\infty\frac{R^{2N-1}}{((R+\rho)^2+\mu)^{q/2}}\,dR\bigg)^{1/q}\\ &= C(p,\theta,N;f)\|(|\xi|^2+\varepsilon^2)^{-l+1/2}\|_r\,\pi^{N/q}\,\frac{\Gamma^{1/q}(\frac{q}{2}-N)}{\Gamma^{1/q}(\frac{q}{2})}\,\mu^{-1/2+N/q} =: C'(p,q,\theta,N;f)\,\mu^{-1/2+N/q}.\end{aligned}$$

Finally, the integral (4.5) is less than

$$C'(p,q,\theta,N;f)\int_0^\infty\frac{\mu^{N/q}}{(\mu+\rho^2)^{3/2}}\,d\mu,$$

which is finite for $q > 2N$ and $p > N/l$. This concludes the proof. □

Lemma 4.10. *If $f \in \mathcal{S}$, then $\pi^\theta(f)\,(|D\!\!\!/|+\varepsilon)^{-1}\,\pi^\theta(f^*) \in \mathcal{L}^{2N+}(\mathcal{H})$.*

Proof. This is an extension to the Moyal context of the renowned inequality by Cwikel [27, 81, 94]. As remarked before, it is possible to replace $D\!\!\!/^2$ by $-\Delta$, $\pi^\theta(f)$ by L^θ_f and $\mathcal{H}$ by $\mathcal{H}_r$. Consider $g(-i\nabla) := (\sqrt{-\Delta} + \varepsilon)^{-1}$. Since g is positive, it can be decomposed as $g = \sum_{n\in\mathbb{Z}} g_n$ where

$$g_n(x) := \begin{cases} g(x) & \text{if } 2^{n-1} < g(x) \leqslant 2^n, \\ 0 & \text{otherwise.} \end{cases}$$

For each $n \in \mathbb{Z}$, let A_n and B_n be the two operators

$$A_n := \sum_{k\leqslant n} L^\theta_f\, g_k(-i\nabla)\, L^\theta_{f^*}, \quad B_n := \sum_{k>n} L^\theta_f\, g_k(-i\nabla)\, L^\theta_{f^*}.$$

We estimate the uniform norm of the first part:

$$\begin{aligned}\|A_n\|_\infty &\leqslant \|L^\theta_f\|^2 \Big\|\sum_{k\leqslant n} g_k(-i\nabla)\Big\|_\infty \leqslant (2\pi\theta)^{-N}\|f\|_2^2 \Big\|\sum_{k\leqslant n} g_k\Big\|_\infty \\ &\leqslant (2\pi\theta)^{-N}\|f\|_2^2\, 2^n =: 2^n\, c_1(\theta, N; f).\end{aligned}$$

The trace norm of B_n can be computed using Lemma 4.3:

$$\begin{aligned}\|B_n\|_1 &= \Big\|\Big(\sum_{k>n} g_k(-i\nabla)\Big)^{1/2} L^\theta_{f^*}\Big\|_2^2 = \Big\|L^\theta_f\Big(\sum_{k>n} g_k(-i\nabla)\Big)^{1/2}\Big\|_2^2 = (2\pi)^{-2N}\|f\|_2^2 \Big\|\Big(\sum_{k>n} g_k\Big)^{1/2}\Big\|_2^2 \\ &= (2\pi)^{-2N}\|f\|_2^2 \Big\|\sum_{k>n} g_k\Big\|_1 = (2\pi)^{-2N}\|f\|_2^2 \sum_{k>n}\|g_k\|_1 \\ &\leqslant (2\pi)^{-2N}\|f\|_2^2 \sum_{k>n}\|g_k\|_\infty\, \nu\{\operatorname{supp}(g_k)\},\end{aligned}$$

where ν is the Lebesgue measure on $\mathbb{R}^{2N}$. By definition, $\|g_k\|_\infty \leqslant 2^k$ and

$$\begin{aligned}\nu\{\operatorname{supp}(g_k)\} &= \nu\{\, x \in \mathbb{R}^{2N} : 2^{k-1} < g(x) \leqslant 2^k \,\} \leqslant \nu\{\, x \in \mathbb{R}^{2N} : (|x| + \varepsilon)^{-1} \geqslant 2^{k-1} \,\} \\ &\leqslant 2^{2N(1-k)}\, c_2.\end{aligned}$$

Therefore,

$$\begin{aligned}\|B_n\|_1 &\leqslant (2\pi)^{-2N}\|f\|_2^2\, 2^{2N} c_2 \sum_{k>n} 2^{k(1-2N)} \\ &< \pi^{-2N}\, c_2\, \|f\|_2^2\, 2^{n(1-2N)} =: 2^{n(1-2N)}\, c_3(N; f),\end{aligned}$$

where the second inequality follows because $N > \frac{1}{2}$.

We can now estimate the mth singular value μ_m of B_n (arranged in decreasing order with multiplicity): $\|B_n\|_1 = \sum_{k=0}^\infty \mu_k(B_n)$. Note that, for $m = 1, 2, 3, \dots$, $\|B_n\|_1 \geqslant \sum_{k=0}^{m-1}\mu_k(B_n) \geqslant m\,\mu_m(B_n)$. Thus, $\mu_m(B_n) \leqslant \|B_n\|_1\, m^{-1} \leqslant 2^{n(1-2N)}\, c_3\, m^{-1}$. Now Fan's inequality [81, Thm. 1.7] yields

$$\begin{aligned}\mu_m(L^\theta_f\, g(-i\nabla)\, L^\theta_{f^*}) &= \mu_m(A_n + B_n) \leqslant \mu_1(A_n) + \mu_m(B_n) \\ &\leqslant \|A_n\| + \|B_n\|_1\, m^{-1} \leqslant 2^n\, c_1 + 2^{n(1-2N)}\, c_3\, m^{-1}.\end{aligned}$$

Given m, choose $n \in \mathbb{Z}$ so that $2^n \leqslant m^{-1/2N} < 2^{n+1}$. Then

$$\mu_m(L^\theta_f\, g(-i\nabla)\, L^\theta_{f^*}) \leqslant c_1\, m^{-1/2N} + c_3\, m^{-(1-2N)/2N} m^{-1} =: c_4(\theta, N; f)\, m^{-1/2N}.$$

Therefore $L^\theta_f\,(\sqrt{-\Delta} + \varepsilon)^{-1}\, L^\theta_{f^*} \in \mathcal{L}^{2N+}(\mathcal{H}_r)$, and the statement of the lemma follows. □

Corollary 4.11. *If $f, g \in \mathcal{S}$, then $\pi^\theta(f)\,(|\not{D}| + \varepsilon)^{-1}\,\pi^\theta(g) \in \mathcal{L}^{2N+}(\mathcal{H})$.*

Proof. Consider $\pi^\theta(f \pm g^*)\,(|\not{D}| + \varepsilon)^{-1}\,\pi^\theta(f^* \pm g)$ and $\pi^\theta(f \pm ig^*)\,(|\not{D}| + \varepsilon)^{-1}\,\pi^\theta(f^* \mp ig)$. □

Corollary 4.12. *If $h \in \mathcal{S}$, then $\pi^\theta(h)\,(|\not{D}| + \varepsilon)^{-1} \in \mathcal{L}^{2N+}(\mathcal{H})$.*

Proof. Let $h = f \star_\theta g$. Then

$$\pi^\theta(h)\,(|\not{D}| + \varepsilon)^{-1} = \pi^\theta(f)\,(|\not{D}| + \varepsilon)^{-1}\,\pi^\theta(g) + \pi^\theta(f)\,[\pi^\theta(g), (|\not{D}| + \varepsilon)^{-1}],$$

and we obtain from the identity (4.3) that

$$\pi^\theta(h)\,(|\not{D}| + \varepsilon)^{-1} = \pi^\theta(f)\,(|\not{D}| + \varepsilon)^{-1}\,\pi^\theta(g) + \pi^\theta(f)\,(|\not{D}| + \varepsilon)^{-1}\,[|\not{D}|, \pi^\theta(g)]\,(|\not{D}| + \varepsilon)^{-1}.$$

By arguments similar to those of lemmata 4.5 and 4.9, the last term belongs to $\mathcal{L}^p$ for $p > N$, and thus to $\mathcal{L}^{2N+}$. □

Boundedness of $(|\not{D}| + \varepsilon)(\not{D}^2 + \varepsilon^2)^{-1/2}$ follows from elementary Fourier analysis. And so the last corollary means that the spectral triple is "$2N^+$-summable". We have taken care of the first assertion of the theorem. The next lemma is the last property of existence that we need.

Lemma 4.13. *If $f \in \mathcal{S}$, then $\pi^\theta(f)(|\not{D}| + \varepsilon)^{-2N}$ and $\pi^\theta(f)(\not{D}^2 + \varepsilon^2)^{-N}$ are in $\mathcal{L}^{1+}(\mathcal{H})$.*

Proof. It suffices to prove that $\pi^\theta(f)(|\not{D}| + \varepsilon)^{-2N} \in \mathcal{L}^{1+}(\mathcal{H})$. We factorize $f \in \mathcal{S}$ according to Proposition 2.5, with the following notation:

$$\begin{aligned} f &= f_1 \star_\theta f_2 = f_1 \star_\theta f_{21} \star_\theta f_{22} = f_1 \star_\theta f_{21} \star_\theta f_{221} \star_\theta f_{222} \\ &= \cdots = f_1 \star_\theta f_{21} \star_\theta f_{221} \star_\theta \cdots \star_\theta f_{22\cdots 21} \star_\theta f_{22\cdots 22}. \end{aligned}$$

Therefore,

$$\begin{aligned} \pi^\theta(f)\,(|\not{D}| + \varepsilon)^{-2N} &= \pi^\theta(f_1)\,(|\not{D}| + \varepsilon)^{-1}\,\pi^\theta(f_2)\,(|\not{D}| + \varepsilon)^{-2N+1} \\ &\quad + \pi^\theta(f_1)\,(|\not{D}| + \varepsilon)^{-1}\,[|\not{D}|, \pi^\theta(f_2)]\,(|\not{D}| + \varepsilon)^{-2N}. \end{aligned} \tag{4.6}$$

By Lemma 4.5, $\pi^\theta(f_1)(|\not{D}| + \varepsilon)^{-1} \in \mathcal{L}^p(\mathcal{H})$ whenever $p > 2N$; and by Lemma 4.9, the term $[|\not{D}|, \pi^\theta(f_2)](|\not{D}| + \varepsilon)^{-2N}$ lies in $\mathcal{L}^q(\mathcal{H})$ for $q > 1$. Hence, the last term on the right hand side of equation (4.6) lies in $\mathcal{L}^1(\mathcal{H})$. We may write the following equivalence relation:

$$\pi^\theta(f)(|\not{D}| + \varepsilon)^{-2N} \sim \pi^\theta(f_1)(|\not{D}| + \varepsilon)^{-1}\pi^\theta(f_2)(|\not{D}| + \varepsilon)^{-2N+1},$$

where $A \sim B$ for $A, B \in \mathcal{K}(\mathcal{H})$ means that $A - B$ is trace-class. Thus,

$$\begin{aligned}
\pi^\theta(f)(|\not{D}| + \varepsilon)^{-2N} &\sim \pi^\theta(f_1)(|\not{D}| + \varepsilon)^{-1}\pi^\theta(f_2)(|\not{D}| + \varepsilon)^{-2N+1} \\
&= \pi^\theta(f_1)(|\not{D}| + \varepsilon)^{-1}\pi^\theta(f_{21})(|\not{D}| + \varepsilon)^{-1}\pi^\theta(f_{22})(|\not{D}| + \varepsilon)^{-2N+2} \\
&\quad + \pi^\theta(f_1)(|\not{D}| + \varepsilon)^{-1}\pi^\theta(f_{21})(|\not{D}| + \varepsilon)^{-1}\,[|\not{D}|, \pi^\theta(f_{22})]\,(|\not{D}| + \varepsilon)^{-2N+1} \\
&\sim \pi^\theta(f_1)(|\not{D}| + \varepsilon)^{-1}\pi^\theta(f_{21})(|\not{D}| + \varepsilon)^{-1}\pi^\theta(f_{22})(|\not{D}| + \varepsilon)^{-2N+2} \sim \cdots \\
&\sim \pi^\theta(f_1)(|\not{D}| + \varepsilon)^{-1}\pi^\theta(f_{21})(|\not{D}| + \varepsilon)^{-1}\pi^\theta(f_{221})(|\not{D}| + \varepsilon)^{-1} \cdots \pi^\theta(f_{22\cdots 22})(|\not{D}| + \varepsilon)^{-1}.
\end{aligned}$$

The second equivalence relation holds because $\pi^\theta(f_1)(|\not{D}| + \varepsilon)^{-1}\pi^\theta(f_{21})(|\not{D}| + \varepsilon)^{-1} \in \mathcal{L}^p(\mathcal{H})$ for $p > N$ by Lemma 4.5, and $[|\not{D}|, \pi^\theta(f_{22})](|\not{D}| + \varepsilon)^{-2N+1} \in \mathcal{L}^q(\mathcal{H})$ for $q > 2N/(2N-1)$ by Lemma 4.9 again. The other equivalences come from similar arguments. Corollary 4.11, the Hölder inequality (see [45, Prop. 7.16]) and the inclusion $\mathcal{L}^1(\mathcal{H}) \subset \mathcal{L}^{1+}(\mathcal{H})$ finally yield the result. □

Now we go for the computation of the Dixmier trace. Using the regularized trace for a ΨDO:

$$\operatorname{Tr}_\Lambda(A) := (2\pi)^{-2N} \iint_{|\xi| \leqslant \Lambda} \sigma[A](x, \xi)\, d^{2N}\xi\, d^{2N}x,$$

the result can be conjectured because $\lim_{\Lambda\to\infty} \operatorname{Tr}_\Lambda(\cdot)/\log(\Lambda^{2N})$ is heuristically linked with the Dixmier trace, and the following computation:

$$\begin{aligned}
&\lim_{\Lambda\to\infty} \frac{1}{2N \log \Lambda} \operatorname{Tr}_\Lambda\big(\pi^\theta(f)(\not{D}^2 + \varepsilon^2)^{-N}\big) \\
&\quad = \lim_{\Lambda\to\infty} \frac{2^N}{2N(2\pi)^{2N} \log \Lambda} \iint_{|\xi| \leqslant \Lambda} f(x - \tfrac{\theta}{2}S\xi)\,(|\xi|^2 + \varepsilon^2)^{-N}\, d^{2N}\xi\, d^{2N}x \\
&\quad = \frac{2^N\,\Omega_{2N}}{2N\,(2\pi)^{2N}} \int f(x)\, d^{2N}x.
\end{aligned}$$

This is precisely the same result of (3.8), in the commutative case, for $k = 2N$. However, to establish it rigorously in the Moyal context requires a subtler strategy. We shall compute the Dixmier trace of $\pi^\theta(f)\,(\not{D}^2 + \varepsilon^2)^{-N}$ as the residue of the ordinary trace of a related meromorphic family of operators. For this, recent results of Carey and coworkers [10] extending Connes' trace theorem (see [16] and [45, Chap. 7]) come in handy. In turn we are allowed to introduce the explicit symbol formula that will establish measurability [17, 45], too.

In the language of [50], thus, we seek first to verify that $\mathcal{A}_\theta$ has *analytical dimension* equal to $2N$; that is, for $f \in \mathcal{A}_\theta$ the operator $\pi^\theta(f)\,(\not{D}^2 + \varepsilon^2)^{-z/2}$ is trace-class if $\Re z > 2N$.

Lemma 4.14. *If $f \in \mathcal{S}$, then $L^\theta_f\,(\not{D}^2 + \varepsilon^2)^{-z/2}$ is trace-class for $\Re z > 2N$, and*

$$\operatorname{Tr}[L^\theta_f\,(\not{D}^2 + \varepsilon^2)^{-z/2}] = (2\pi)^{-2N} \iint f(x)\,(|\xi|^2 + \varepsilon^2)^{-z/2}\, d^{2N}\xi\, d^{2N}x.$$

Proof. If $a(x, \xi) \in \mathcal{K}_p(\mathbb{R}^{2k})$, for $p < -k$, is the symbol of a pseudodifferential operator A, then the operator is trace-class and moreover

$$\operatorname{Tr} A = (2\pi)^{-k} \iint a(x, \xi)\, d^k x\, d^k\xi.$$

This is easily proved by taking $a \in \mathcal{S}(\mathbb{R}^{2k})$ first and extending the resulting formula by continuity; have a look at [29, 67, 93] as well.

In our case, the symbol formula for a product of ΨDOs yields, for $p > N$,

$$\begin{aligned}\sigma\big[L^\theta_f(-\Delta+\varepsilon^2)^{-p}\big](x,\xi) &= \sum_{\alpha\in\mathbb{N}^N} \frac{(-i)^{|\alpha|}}{\alpha!}\, \partial^\alpha_\xi \sigma[L^\theta_f](x,\xi)\, \partial^\alpha_x \sigma\big[(-\Delta+\varepsilon^2)^{-p}\big](x,\xi) \\ &= \sigma[L^\theta_f](x,\xi)\, \sigma\big[(-\Delta+\varepsilon^2)^{-p}\big](x,\xi) \\ &= f(x-\tfrac{\theta}{2}S\xi)\,(|\xi|^2+\varepsilon^2)^{-p}.\end{aligned}$$

Therefore, for $p > N$,

$$\begin{aligned}\operatorname{Tr}\big(L^\theta_f(-\Delta+\varepsilon^2)^{-p}\big) &= (2\pi)^{-2N} \iint f(x-\tfrac{\theta}{2}S\xi)\,(|\xi|^2+\varepsilon^2)^{-p}\, d^{2N}\xi\, d^{2N}x \\ &= (2\pi)^{-2N} \iint f(x)\,(|\xi|^2+\varepsilon^2)^{-p}\, d^{2N}\xi\, d^{2N}x. \qquad \square\end{aligned}$$

We continue with a technical lemma, in the spirit of [73]. Consider the approximate unit $\{e_K\}_{K\in\mathbb{N}} \subset \mathcal{A}_c$ where $e_K := \sum_{0\leqslant|n|\leqslant K} f_{nn}$. These e_K are projectors with a natural ordering: $e_K \star_\theta e_L = e_L \star_\theta e_K = e_K$ for $K \leqslant L$, and they are local units for $\mathcal{A}_c$.

Lemma 4.15. *Let $f \in \mathcal{A}_{c,K}$. Then*

$$\pi^\theta(f)\,(D\!\!\!/^2+\varepsilon^2)^{-N} - \pi^\theta(f)\,\big(\pi^\theta(e_K)(D\!\!\!/^2+\varepsilon^2)^{-1}\pi^\theta(e_K)\big)^N \in \mathcal{L}^1(\mathcal{H}).$$

Proof. For simplicity we use the notation $e := e_K$ and $e_n := e_{K+n}$. By the boundedness of $\pi^\theta(f)$, we may assume that $f = e \in \mathcal{A}_{c,K}$.

Because $e_n \star_\theta e = e \star_\theta e_n = e$, it is clear that

$$\pi^\theta(e)(D\!\!\!/+\lambda)^{-1}\big(1-\pi^\theta(e_n)\big) = \pi^\theta(e)\,(D\!\!\!/+\lambda)^{-1}\,[D\!\!\!/,\pi^\theta(e_n)]\,(D\!\!\!/+\lambda)^{-1}. \tag{4.7}$$

Also, $\pi^\theta(e)\,[D\!\!\!/,\pi^\theta(e_n)] = [D\!\!\!/,\pi^\theta(e\star_\theta e_n)] - [D\!\!\!/,\pi^\theta(e)]\,\pi^\theta(e_n) = 0$ because $[D\!\!\!/,\pi^\theta(e)]\,\pi^\theta(e_n) = [D\!\!\!/,\pi^\theta(e)]$ for $n = 1$ or bigger – see equation (A.1). We obtain

$$\begin{aligned} A_n &:= \pi^\theta(e)(D\!\!\!/+\lambda)^{-1}[D\!\!\!/,\pi^\theta(e_n)](D\!\!\!/+\lambda)^{-1} \\ &= \pi^\theta(e)(D\!\!\!/+\lambda)^{-1}[D\!\!\!/,\pi^\theta(e_1)](D\!\!\!/+\lambda)^{-1}[D\!\!\!/,\pi^\theta(e_n)](D\!\!\!/+\lambda)^{-1} \\ &= \pi^\theta(e)(D\!\!\!/+\lambda)^{-1}[D\!\!\!/,\pi^\theta(e_1)]\pi^\theta(e_2)(D\!\!\!/+\lambda)^{-1}[D\!\!\!/,\pi^\theta(e_n)](D\!\!\!/+\lambda)^{-1} = \cdots \\ &= \big(\pi^\theta(e)(D\!\!\!/+\lambda)^{-1}\big)\big([D\!\!\!/,\pi^\theta(e_1)](D\!\!\!/+\lambda)^{-1}\big)\big([D\!\!\!/,\pi^\theta(e_2)](D\!\!\!/+\lambda)^{-1}\big)\cdots\big([D\!\!\!/,\pi^\theta(e_n)](D\!\!\!/+\lambda)^{-1}\big).\end{aligned}$$

Taking $n = 2N$ here, A_{2N} appears as a product of $2N+1$ terms in parentheses, each in $\mathcal{L}^{2N+1}(\mathcal{H})$ by Lemma 4.5. Hence, by Hölder's inequality, A_{2N} is trace-class and therefore

$$\pi^\theta(e)(D\!\!\!/+\lambda)^{-1}(1-\pi^\theta(e_{2N})) \in \mathcal{L}^1(\mathcal{H}).$$

Thus,

$$\begin{aligned} &\pi^\theta(e)\,(D\!\!\!/^2+\varepsilon^2)^{-1}\big(1-\pi^\theta(e_{4N})\big) \\ &\quad = \pi^\theta(e)(D\!\!\!/-i\varepsilon)^{-1}\big(1-\pi^\theta(e_{2N})+\pi^\theta(e_{2N})\big)(D\!\!\!/+i\varepsilon)^{-1}\big(1-\pi^\theta(e_{4N})\big) \\ &\quad = \pi^\theta(e)(D\!\!\!/-i\varepsilon)^{-1}\big(1-\pi^\theta(e_{2N})\big)(D\!\!\!/+i\varepsilon)^{-1}\big(1-\pi^\theta(e_{4N})\big) \\ &\qquad + \pi^\theta(e)(D\!\!\!/-i\varepsilon)^{-1}\pi^\theta(e_{2N})(D\!\!\!/+i\varepsilon)^{-1}\big(1-\pi^\theta(e_{4N})\big) \in \mathcal{L}^1(\mathcal{H}). \end{aligned} \tag{4.8}$$

That is to say, $\pi^\theta(e)(D\!\!\!/^2+\varepsilon^2)^{-1} \sim \pi^\theta(e)(D\!\!\!/^2+\varepsilon^2)^{-1}\pi^\theta(e_{4N})$. Shifting this property, we get

$$\begin{aligned}\pi^\theta(e)(D\!\!\!/^2+\varepsilon^2)^{-N} &\sim \pi^\theta(e)(D\!\!\!/^2+\varepsilon^2)^{-1}\pi^\theta(e_{4N})(D\!\!\!/^2+\varepsilon^2)^{-N+1}\\ &\sim \pi^\theta(e)(D\!\!\!/^2+\varepsilon^2)^{-1}\pi^\theta(e_{4N})(D\!\!\!/^2+\varepsilon^2)^{-1}\pi^\theta(e_{8N})(D\!\!\!/^2+\varepsilon^2)^{-N+2} \sim \cdots\\ &\sim \pi^\theta(e)(D\!\!\!/^2+\varepsilon^2)^{-1}\pi^\theta(e_{4N})(D\!\!\!/^2+\varepsilon^2)^{-1}\pi^\theta(e_{8N})\cdots(D\!\!\!/^2+\varepsilon^2)^{-1}\pi^\theta(e_{4N^2}).\end{aligned}$$

By identity (4.3), the last term on the right equals

$$\begin{aligned}&\pi^\theta(e)(D\!\!\!/+i\varepsilon)^{-1}\pi^\theta(e)(D\!\!\!/-i\varepsilon)^{-1}\pi^\theta(e_{4N})(D\!\!\!/^2+\varepsilon^2)^{-1}\pi^\theta(e_{8N})\cdots(D\!\!\!/^2+\varepsilon^2)^{-1}\pi^\theta(e_{4N^2})\\ &+\pi^\theta(e)(D\!\!\!/+i\varepsilon)^{-1}[D\!\!\!/,\pi^\theta(e)](D\!\!\!/^2+\varepsilon^2)^{-1}\pi^\theta(e_{4N})(D\!\!\!/^2+\varepsilon^2)^{-1}\pi^\theta(e_{8N})\cdots(D\!\!\!/^2+\varepsilon^2)^{-1}\pi^\theta(e_{4N^2}).\end{aligned}$$

The last term is trace-class because it is a product of N terms in $\mathcal{L}^p(\mathcal{H})$ for $p > N$ and one term in $\mathcal{L}^q(\mathcal{H})$ for $q > 2N$, by Lemma 4.5. Removing the second $\pi^\theta(e)$ once again, by the ordering property of the local units e_K yields

$$\begin{aligned}&\pi^\theta(e)(D\!\!\!/+i\varepsilon)^{-1}\pi^\theta(e)(D\!\!\!/-i\varepsilon)^{-1}\pi^\theta(e_{4N})(D\!\!\!/^2+\varepsilon^2)^{-1}\pi^\theta(e_{8N})\cdots(D\!\!\!/^2+\varepsilon^2)^{-1}\pi^\theta(e_{4N^2})\\ &=\pi^\theta(e)(D\!\!\!/^2+\varepsilon^2)^{-1}\pi^\theta(e)(D\!\!\!/^2+\varepsilon^2)^{-1}\pi^\theta(e_{8N})\cdots(D\!\!\!/^2+\varepsilon^2)^{-1}\pi^\theta(e_{4N^2})\\ &\quad+\pi^\theta(e)(D\!\!\!/^2+\varepsilon^2)^{-1}[D\!\!\!/,\pi^\theta(e)](D\!\!\!/-i\varepsilon)^{-1}\pi^\theta(e_{4N})(D\!\!\!/^2+\varepsilon^2)^{-1}\pi^\theta(e_{8N})\cdots(D\!\!\!/^2+\varepsilon^2)^{-1}\pi^\theta(e_{4N^2}).\end{aligned}$$

The last term is still trace-class, hence

$$\pi^\theta(e)(D\!\!\!/^2+\varepsilon^2)^{-N} \sim \pi^\theta(e)(D\!\!\!/^2+\varepsilon^2)^{-1}\pi^\theta(e)(D\!\!\!/^2+\varepsilon^2)^{-1}\pi^\theta(e_{8N})\cdots(D\!\!\!/^2+\varepsilon^2)^{-1}\pi^\theta(e_{4N^2}).$$

This algorithm, applied another $(N-1)$ times, yields the result:

$$\pi^\theta(e)(D\!\!\!/^2+\varepsilon^2)^{-N} \sim \big(\pi^\theta(e)(D\!\!\!/^2+\varepsilon^2)^{-1}\pi^\theta(e)\big)^N. \qquad \square$$

We retain the following consequence.

Corollary 4.16. $\operatorname{Tr}^+\big(\pi^\theta(g)\,[\pi^\theta(f),(D\!\!\!/^2+\varepsilon^2)^{-N}]\big)=0$ *for any* $g\in\mathcal{S}$ *and any projector* $f\in\mathcal{A}_c$.

Proof. This follows from Lemma 4.15 applied to $\pi^\theta(f)\,(D\!\!\!/^2+\varepsilon^2)^{-N}$ and its adjoint. □

Now we are finally ready to evaluate the Dixmier traces.

Proposition 4.17. *For* $f\in\mathcal{S}$, *any Dixmier trace* Tr^+ *of* $\pi^\theta(f)\,(D\!\!\!/^2+\varepsilon^2)^{-N}$ *is independent of* ε, *and*

$$\operatorname{Tr}^+\big(\pi^\theta(f)\,(D\!\!\!/^2+\varepsilon^2)^{-N}\big)=\frac{2^N\,\Omega_{2N}}{2N\,(2\pi)^{2N}}\int f(x)\,d^{2N}x=\frac{1}{N!\,(2\pi)^N}\int f(x)\,d^{2N}x.$$

Proof. We will first prove it for $f\in\mathcal{A}_c$. Choose e a unit for f, that is, $e\star_\theta f=f\star_\theta e=f$. By Lemmata 4.13 and 4.15, and because $\mathcal{L}^1(\mathcal{H})$ lies inside the kernel of the Dixmier trace, we obtain

$$\operatorname{Tr}^+(\pi^\theta(f)\,(D\!\!\!/^2+\varepsilon^2)^{-N})=\operatorname{Tr}^+\big(\pi^\theta(f)\,(\pi^\theta(e)(D\!\!\!/^2+\varepsilon^2)^{-1}\pi^\theta(e))^N\big).$$

Lemma 4.15 applied to $f = e$ implies that $\big(\pi^\theta(e)(D\!\!\!/^2 + \varepsilon^2)^{-1}\pi^\theta(e)\big)^N$ is a positive operator in $\mathcal{L}^{1+}(\mathcal{H})$, since it is equal to $\pi^\theta(e)(D\!\!\!/^2 + \varepsilon^2)^{-N}$ plus a term in $\mathcal{L}^1(\mathcal{H})$. Thus, [10, Thm. 5.6] yields (since the limit converges, any Dixmier trace will give the same result):

$$\begin{aligned}\operatorname{Tr}^+\big(\pi^\theta(f)\,(D\!\!\!/^2 + \varepsilon^2)^{-N}\big) &= \lim_{s\downarrow 1}(s-1)\operatorname{Tr}\big[\pi^\theta(f)\,(\pi^\theta(e)(D\!\!\!/^2 + \varepsilon^2)^{-1}\pi^\theta(e))^{Ns}\big] \\ &= \lim_{s\downarrow 1}(s-1)\operatorname{Tr}\big(\pi^\theta(f)\pi^\theta(e)(D\!\!\!/^2 + \varepsilon^2)^{-Ns}\pi^\theta(e) + E_{Ns}\big), \qquad (4.9)\end{aligned}$$

where

$$E_{Ns} := \pi^\theta(f)\,\big(\pi^\theta(e)(D\!\!\!/^2 + \varepsilon^2)^{-1}\pi^\theta(e)\big)^{Ns} - \pi^\theta(f)\pi^\theta(e)(D\!\!\!/^2 + \varepsilon^2)^{-Ns}\pi^\theta(e).$$

Lemma 4.15 again shows that $E_N \in \mathcal{L}^1(\mathcal{H})$.

Now for $s > 1$, the first term $\pi^\theta(f)\,\big(\pi^\theta(e)(D\!\!\!/^2 + \varepsilon^2)^{-1}\pi^\theta(e)\big)^{Ns}$ of E_{Ns} is in $\mathcal{L}^1(\mathcal{H})$. In effect, using Lemma 4.5 and since $\pi^\theta(e)(D\!\!\!/^2+\varepsilon^2)^{-1} \in \mathcal{L}^p(\mathcal{H})$ for $p > N$, we get $\pi^\theta(e)(D\!\!\!/^2+\varepsilon^2)^{-1}\pi^\theta(e) \in \mathcal{L}^{Ns}(\mathcal{H})$. This operator being positive, one concludes that

$$\big(\pi^\theta(e)(D\!\!\!/^2 + \varepsilon^2)^{-1}\pi^\theta(e)\big)^{Ns} \in \mathcal{L}^1(\mathcal{H}).$$

The second term $\pi^\theta(f)\pi^\theta(e)(D\!\!\!/^2 + \varepsilon^2)^{-Ns}\pi^\theta(e)$ lies in $\mathcal{L}^1(\mathcal{H})$ too, because

$$\|\pi^\theta(e)(D\!\!\!/^2 + \varepsilon^2)^{-Ns}\pi^\theta(e)\|_1 = \|(D\!\!\!/^2 + \varepsilon^2)^{-Ns/2}\pi^\theta(e)\|_2^2 = \|\pi^\theta(e)(D\!\!\!/^2 + \varepsilon^2)^{-Ns/2}\|_2^2$$

is finite by Lemma 4.3. So $E_{Ns} \in \mathcal{L}^1(\mathcal{H})$ for $s \geqslant 1$, and (4.9) implies

$$\begin{aligned}\operatorname{Tr}^+\big(\pi^\theta(f)\,(D\!\!\!/^2 + \varepsilon^2)^{-N}\big) &= \lim_{s\downarrow 1}(s-1)\operatorname{Tr}\big(\pi^\theta(f)\pi^\theta(e)(D\!\!\!/^2 + \varepsilon^2)^{-Ns}\pi^\theta(e)\big) \\ &= \lim_{s\downarrow 1}(s-1)\operatorname{Tr}\big(\pi^\theta(f)(D\!\!\!/^2 + \varepsilon^2)^{-Ns}\big).\end{aligned}$$

Applying now Lemma 4.14, we obtain

$$\begin{aligned}\operatorname{Tr}^+\big(\pi^\theta(f)\,(D\!\!\!/^2 + \varepsilon^2)^{-N}\big) &= \lim_{s\downarrow 1}(s-1)\operatorname{Tr}(1_{2^N})\operatorname{Tr}\big(L^\theta_f(-\Delta + \varepsilon^2)^{-Ns}\big) \\ &= 2^N(2\pi)^{-2N}\lim_{s\downarrow 1}(s-1)\iint f(x)\,(|\xi|^2 + \varepsilon^2)^{-Ns}\,d^{2N}\xi\,d^{2N}x \\ &= \frac{1}{N!\,(2\pi)^N}\int f(x)\,d^{2N}x,\end{aligned}$$

where the identity

$$\int (|\xi|^2 + \varepsilon^2)^{-Ns}\,d^{2N}\xi = \pi^N\frac{\Gamma(N(s-1))}{\Gamma(Ns)\,\varepsilon^{2N(s-1)}},$$

and $\Gamma(N\alpha) \sim 1/N\alpha$ as $\alpha \downarrow 0$ have been used. The proposition is proved for $f \in \mathcal{A}_c$.

Finally, take f arbitrary in $\mathcal{S}$, and recall that $\{e_K\}$ is an approximate unit for $\mathcal{A}_\theta$. Since $f = g \star_\theta h$ for some $g, h \in \mathcal{S}$, Corollary 4.16 implies

$$\begin{aligned}&\big|\operatorname{Tr}^+\big((\pi^\theta(f) - \pi^\theta(e_K \star_\theta f \star_\theta e_K))(D\!\!\!/^2 + \varepsilon^2)^{-N}\big)\big| \\ &\quad= \big|\operatorname{Tr}^+\big((\pi^\theta(f) - \pi^\theta(e_K \star_\theta f))\,(D\!\!\!/^2 + \varepsilon^2)^{-N}\big)\big| \\ &\quad= \big|\operatorname{Tr}^+\big((\pi^\theta(g) - \pi^\theta(e_K \star_\theta g))\,\pi^\theta(h)(D\!\!\!/^2 + \varepsilon^2)^{-N}\big)\big| \\ &\quad\leqslant \|\pi^\theta(g) - \pi^\theta(e_K \star_\theta g)\|_\infty\,\operatorname{Tr}^+\big|\pi^\theta(h)\,(D\!\!\!/^2 + \varepsilon^2)^{-N}\big|.\end{aligned}$$

Since $\|\pi^\theta(g) - \pi^\theta(e_K \star_\theta g)\|_\infty \leqslant (2\pi\theta)^{-N/2}\|g - e_K \star_\theta g\|_2$ tends to zero when K increases, the proof is complete because $e_K \star_\theta f \star_\theta e_K$ lies in $\mathcal{A}_c$ and

$$\int [e_K \star_\theta f \star_\theta e_K](x)\, d^{2N}x \to \int f(x)\, d^{2N}x \quad \text{as } K \uparrow \infty. \qquad \square$$

Remark 4.18. Similar arguments to those of this section (or a simple comparison argument) show that for $f \in \mathcal{S}$,

$$\mathrm{Tr}^+\big(\pi^\theta(f)\,(|D\!\!\!/| + \varepsilon)^{-2N}\big) = \mathrm{Tr}^+\big(\pi^\theta(f)\,(D\!\!\!/^2 + \varepsilon^2)^{-N}\big).$$

In conclusion: the analytical and spectral dimension of Moyal planes coincide. Lemma 4.13, Proposition 4.17 and the previous remark have concluded the proof of Theorem 4.8. □

4.3 The regularity condition

Theorem 4.19. *For $f \in \widetilde{\mathcal{A}}_\theta$, the bounded operators $\pi^\theta(f)$ and $[D\!\!\!/, \pi^\theta(f)]$ lie in the smooth domain of the derivation $\delta(T) := [|D\!\!\!/|, T]$.*

The traditional recursive proof [23, 45] does not work in its original form because the useful transformations L and R are undefined in the noncompact case (i.e., $|D\!\!\!/|^{-1}$ is not available). However, an analogue of this proof may exist if instead of (3.1) we define L_λ and R_λ, for real λ, as

$$L_\lambda(\cdot) := (|D\!\!\!/| + i\lambda)^{-1}\,[D\!\!\!/^2, \cdot], \quad R_\lambda(\cdot) := [D\!\!\!/^2, \cdot]\,(|D\!\!\!/| - i\lambda)^{-1}.$$

Here we prefer to prove the theorem by its north face: this approach is still valid for the commutative case, compact or not.

Proof of Theorem 4.19. As before, because $[D\!\!\!/, \pi^\theta(f)] = -iL^\theta(\partial_\mu f) \otimes \gamma^\mu$, it is sufficient to prove that $\pi^\theta(f)$ lies in the smooth domain of δ. For each $n \in \mathbb{N}$ and $\rho > 0$, we may iterate the spectral identity (4.4) n times, to get for $\delta^n(\pi^\theta(f))$:

$$\frac{1}{\pi^n}\int_0^\infty \cdots \int_0^\infty \prod_{i=1}^n \frac{\sqrt{\lambda_i}}{(|D\!\!\!/| + \rho)^2 + \lambda_i}\,(\mathrm{ad}(|D\!\!\!/| + \rho)^2)^n\,(\pi^\theta(f)) \prod_{i=1}^n \frac{1}{(|D\!\!\!/| + \rho)^2 + \lambda_i}\, d\lambda_n \cdots d\lambda_1,$$

with an obvious notation for the n-fold iterated commutators.

Because $[D\!\!\!/^2, \pi^\theta(f)] = D\!\!\!/^2(f) + 2D\!\!\!/(f)\,D\!\!\!/$, with the notation $D\!\!\!/(f) := -iL^\theta(\partial_\mu f) \otimes \gamma^\mu$, we can check that the term with the highest power of $D\!\!\!/$ in the expansion of $(\mathrm{ad}(|D\!\!\!/| + \rho)^2)^n\,(\pi^\theta(f))$ is $2^n D\!\!\!/^n(f)\,D\!\!\!/^n$. For the rest of the proof, we consider only such highest-power terms. As in the proof of Lemma 4.9, all commutators $[|D\!\!\!/|, \pi^\theta(f)]$, which appear due to the artificial presence of ρ, will be treated as a sum of two first order operators. Hence,

$$\begin{aligned}
&\frac{1}{\pi^n}\int_0^\infty \cdots \int_0^\infty \prod_{i=1}^n \frac{\sqrt{\lambda_i}}{(|D\!\!\!/| + \rho)^2 + \lambda_i} 2^n D\!\!\!/^n(f) D\!\!\!/^n \prod_{j=1}^n \frac{1}{(|D\!\!\!/| + \rho)^2 + \lambda_j}\, d\lambda_n \cdots d\lambda_1 \\
&\quad = \frac{1}{\pi^n}\int_0^\infty \cdots \int_0^\infty 2^n D\!\!\!/^n(f)\, D\!\!\!/^n \prod_{i=1}^n \frac{\sqrt{\lambda_i}}{((|D\!\!\!/| + \rho)^2 + \lambda_i)^2}\, d\lambda_n \cdots d\lambda_1 \qquad (4.10)\\
&\qquad + \frac{1}{\pi^n}\int_0^\infty \cdots \int_0^\infty \Big[\prod_{i=1}^n \frac{1}{(|D\!\!\!/| + \rho)^2 + \lambda_i}, 2^n D\!\!\!/^n(f)\Big] D\!\!\!/^n \prod_{i=1}^n \frac{\sqrt{\lambda_i}}{(|D\!\!\!/| + \rho)^2 + \lambda_i}\, d\lambda_n \cdots d\lambda_1.
\end{aligned}$$

Using $\int_0^\infty t(\lambda+t^2)^{-2}\sqrt{\lambda}\,d\lambda = \pi/2$, the first term on the right hand side of (4.10) equals

$$2^n \not{D}^n(f)\frac{\not{D}^n}{(|\not{D}|+\rho)^n}\left(\frac{1}{\pi}\int_0^\infty \frac{|\not{D}|+\rho}{((|\not{D}|+\rho)^2+\lambda)^2}\sqrt{\lambda}\,d\lambda\right)^n = \not{D}^n(f)\frac{\not{D}^n}{(|\not{D}|+\rho)^n},$$

which is a bounded operator.

For the other term, notice that the commutator $\left[\prod_i((|\not{D}|+\rho)^2+\lambda_i)^{-1}, \not{D}^n(f)\right]$ can be rewritten as

$$-\prod_{i=1}^n((|\not{D}|+\rho)^2+\lambda_i)^{-1}\left[\prod_{j=1}^n((|\not{D}|+\rho)^2+\lambda_j), \not{D}^n(f)\right]\prod_{k=1}^n((|\not{D}|+\rho)^2+\lambda_k)^{-1},$$

and the highest-power term of this expression is, up to a constant:

$$\prod_{i=1}^n((|\not{D}|+\rho)^2+\lambda_i)^{-1}\,\not{D}^{n+1}(f)\,\not{D}^{2n-1}\prod_{k=1}^n((|\not{D}|+\rho)^2+\lambda_k)^{-1}.$$

So the proof reduces to showing the finiteness of the following norm:

$$\begin{aligned}
&\left\|\int_0^\infty\cdots\int_0^\infty\prod_{i=1}^n\frac{\sqrt{\lambda_i}}{(|\not{D}|+\rho)^2+\lambda_i}\,\not{D}^{n+1}(f)\,\not{D}^{3n-1}\left(\prod_{j=1}^n\frac{1}{(|\not{D}|+\rho)^2+\lambda_j}\right)^2 d\lambda_n\cdots d\lambda_1\right\| \\
&\leqslant \|\not{D}^{n+1}(f)\|\int_0^\infty\cdots\int_0^\infty\prod_{i=1}^n\left(\left\|\frac{\not{D}^{3-1/n}}{((|\not{D}|+\rho)^2+\lambda_i)^{3/2-1/2n}}\right\|\left\|\frac{1}{(|\not{D}|+\rho)^2+\lambda_i}\right\|^{3/2+1/2n}\sqrt{\lambda_i}\,d\lambda_i\right) \\
&\leqslant \|\not{D}^{n+1}(f)\|\left(\int_0^\infty\frac{\sqrt{\lambda}}{(\rho^2+\lambda)^{3/2+1/2n}}\,d\lambda\right)^n.
\end{aligned}$$

This integral is finite for all $n\in\mathbb{N}$ and so is the norm since $\partial^\alpha f\in\widetilde{\mathcal{A}}_\theta\subset A_\theta$ for $|\alpha|\leqslant n+1$. The proof is complete. ◻

4.4 The finiteness condition

Lemma 4.20. *The smooth vectors for $\not{D}$ are given by*

$$C^\infty(\not{D})\equiv\mathcal{H}^\infty := \bigcap_{k\in\mathbb{N}}\mathrm{Dom}(\not{D}^k)\simeq\mathcal{D}_{L^2}\otimes\mathbb{C}^{2^N}.$$

Proof. Since $\mathcal{D}_{L^2}$ is the common smooth domain of the partial derivatives ∂_μ, for $\mu=1,\dots,2N$, and since $\not{D}=-i\partial_\mu\otimes\gamma^\mu$, the conclusion is clear. ◻

Take $\mathcal{A}_1 := \mathcal{D}_{L^2}$; by Lemma 2.17, this is an ideal in $\widetilde{\mathcal{A}}_\theta$. Then $\mathcal{H}^\infty$ is an $\mathcal{A}_1$-pullback of a free left $\widetilde{\mathcal{A}}_\theta$-module.

On $\mathcal{H}^\infty$ there is a natural $\mathcal{A}_1$-valued hermitian structure, given by

$$(\xi\mid\eta)' := \sum_{j=1}^{2^N}\xi_j\star_\theta\eta_j^*, \quad \text{for all} \quad \xi,\eta\in\mathcal{H}^\infty.$$

Because $\mathcal{D}_{L^2} \subset \mathcal{M}^\theta$, the hermitian pairing $(\pi^\theta(a)\xi \mid \eta)' = a \star_\theta (\xi \mid \eta)'$ is $\mathcal{A}_\theta$-valued whenever $a \in \mathcal{A}_\theta$. Proposition 4.17 and Lemma 2.1(v) now imply

$$\begin{aligned}\operatorname{Tr}^+\big(\pi^\theta((\xi \mid \eta)')\,(D\!\!\!/^2 + \varepsilon^2)^{-N}\big) &= \frac{2^N\,\Omega_{2N}}{2N\,(2\pi)^{2N}} \sum_{j=1}^{2^N} \int (\xi_j \star_\theta \eta_j^*)(x)\,d^{2N}x \\ &= \frac{1}{N!\,(2\pi)^N} \sum_{j=1}^{2^N} \int \eta_j^*(x)\,\xi_j(x)\,d^{2N}x.\end{aligned}$$

Therefore, $(\xi \mid \eta) := N!\,(2\pi)^N(\xi \mid \eta)'$ is the desired hermitian structure satisfying (3.5). Its uniqueness can be checked in the same way as in [45, p. 501]. In summary: the inner product on $\mathcal{H}$ is tightly linked to the natural hermitian structure on $\mathcal{H}^\infty(D)$ by means of the resolvent of D and the noncommutative integral.

Remark 4.21. An obvious integral estimate makes it clear that $\mathcal{O}_r \subset \mathcal{D}_{L^2}$ if and only if $r < -N$. Consider, therefore, $\mathcal{N} := \bigcup_{r<-N} \mathcal{O}_r \subset \mathcal{D}_{L^2}$. It follows from Proposition 2.14 that $\mathcal{N}$ is a $*$-algebra for the twisted product $\star_\theta$, and that it is also an ideal in $\widetilde{\mathcal{A}_\theta} = \mathcal{B}$. This space has already been used with physical motivations in [58].

4.5 The other axioms for the Moyal $2N$-plane

- The signs for the table (3.6) are easily checked in the representation (4.1); indeed, since neither J nor $D\!\!\!/$ depend directly on θ, it suffices to check these signs in the commutative case. The reality property follows at once.

- The first-order property comes directly from (4.2), since

  $$[[D\!\!\!/, \pi^\theta(f)], J\pi^\theta(g)J^{-1}] = [\pi^\theta(D\!\!\!/(f)), J\pi^\theta(g)J^{-1}] = -i\,[L^\theta(\partial_\mu f) \otimes \gamma^\mu, R^\theta(g^*) \otimes 1] = 0,$$

  because left and right twisted multiplications commute.

- The orientation property requires a suitable Hochschild $2N$-cycle over the preferred unitization $\widetilde{\mathcal{A}_\theta} = \mathcal{B}$. As already mentioned, there is a natural embedding of the noncommutative torus $C^\infty(\mathbb{T}^k_\Theta)$ in $\mathcal{B}$ as periodic functions. Indeed, the generators u_j, $j = 1, \dots, 2N$, of the NC torus correspond exactly to the elementary plane waves $u_j(x) := e^{ix_j}$, which are unitary elements for the twisted product and satisfy the same algebraic relations (3.3). Now the very same formula (3.4), rewritten with the Moyal product, yields the desired Hochschild $2N$-cycle; and the volume-form relation (3.7) can be checked with the same calculation [45] as for the NC torus:

  $$\frac{(-i)^N}{(2N)!} \sum_\sigma (-1)^\sigma (u_{\sigma(1)} \star_\theta u_{\sigma(2)} \star_\theta \cdots \star_\theta u_{\sigma(2N)})^{\star_\Theta -1} [D\!\!\!/, u_{\sigma(1)}] \cdots [D\!\!\!/, u_{\sigma(2N)}] = \chi.$$

  Here the π^θ-representation has been understood.

  The plane waves belong not to the Schwartz algebra $\mathcal{S}$, but rather to its unitization $\mathcal{B}$. No finite sum of tensors with entries from the "small" algebra will make up a Hochschild cycle $\mathbf{c}$ satisfying $\pi_{D\!\!\!/}(\mathbf{c}) = \chi$, because of decay at infinity; thus, if one wants to avoid approximation sequences both of operators and volume forms, passage to a compactification containing at least the plane waves is ineluctable.

The commutant of $\mathcal{A}_\theta$ acting by left multiplication consists of right multipliers. Indeed, it has been shown [64] that among those operators on $L^2(\mathbb{R}^{2N})$ which are smooth for the adjoint action of the Heisenberg group, the commutant of $R^\theta(\mathcal{S})$ is exactly $L^\theta(\mathcal{B})$. Right multipliers do not commute with $\not{D}$ unless they are scalars. Therefore the Moyal spin geometries are *connected*. In this respect, left Moyal quantization behaves like a prequantization [85, 86]: see our remark at the end of the conclusions section.

Checking back our arguments and estimates, we find that we have proved something stronger than what we set out to show: most properties hold for $\mathcal{A}_1$, which is determined solely by D; and so, the outcome of the tug-of-war between the operator and the algebra witnessed in the previous pages is the triumph of the operator, which goes a very long way to determine both the algebra and the inner product on the triple's Hilbert space.

Theorem 4.22. *The Moyal planes $(\mathcal{A}, \widetilde{\mathcal{A}}, \mathcal{H}, \not{D}, J, \chi)$ are connected real noncompact spectral triples of spectral dimension $2N$, for which $\mathcal{A}_1$ (as introduced in subsections 3.1, 3.2), is equal to $\mathcal{D}_{L^2}$. Moreover, all postulates for noncompact noncommutative geometries except the first are fulfilled if we replace $\mathcal{A}$ by $\mathcal{A}_1$ throughout.* □

This is the main result of the paper.

5 Moyal–Wick monomials

5.1 An algebraic mould

In this section we put the theory developed in the previous sections to good use in clarifying some fundamentals of quantum noncommutative field theory. In the NCFT literature, models based on spacetime equations like $(\square + m^2)\varphi(x) = g\,\varphi^{\star r}(x)$, where φ is a quantized scalar field and r a suitable integer, are commonplace (we suppress for a while the θ in the notation for the star product). However, this is a formal equation, and in practically all the treatments $\varphi^{\star r}$ is in want of rigorous definition. The Moyal product does not help with the ordering issue in quantum field theory, and therefore that equation should be given in normally ordered form

$$(\square + m^2)\varphi(x) = g\,{:}\varphi^{\star r}(x){:}\ .$$

Thus we need a concept of normally-ordered Moyal products of fields, or Moyal–Wick monomials. Such a definition should work at least for free fields, to serve as basis for a perturbative treatment of interacting ones.

In order to avoid excessively model-dependent casuistics in the definition of what noncommutative Wick monomials should be, it is imperative to employ an algebraic framework. Such a framework fortunately exists [2], and it turns out to mesh very well with Connes' formulation of noncommutative geometry in terms of spectral triples. We contend with Baez, Segal and Zhou as well that it is natural to regard those monomials as quadratic form-valued, rather than operator-valued, distributions; this improves and simplifies the usual definition *à la* Wightman and Gårding [95]. In turn this will be helpful with the explicit rigorous construction of noncommutative Wick monomials in the Moyal algebra context, that we perform next, in which $\mathcal{A}_1 = \mathcal{D}_{L^2}$ is again found to play the main role. In short: the theory of noncompact spectral triples is born in intimate contact with quantum field theory.

According to Segal, the boson quantization of a separable complex Hilbert space $\mathcal{H}$ with inner product $\langle \cdot \mid \cdot \rangle$ (assuming the simplest circumstance in which a suitably unique quantum vacuum can be chosen) consists of a quadruple $(\mathcal{K}, |0\rangle, \beta, \Gamma)$, where $\mathcal{K} \equiv \Gamma(\mathcal{H})$ is another separable Hilbert space; $|0\rangle$ is a distinguished unit vector in $\mathcal{K}$; β is a strongly continuous map from $\mathcal{H}$ to the group of unitary operators on $\mathcal{K}$ satisfying:

$$\beta(v)\beta(v') = \beta(v + v') \exp[-i\Im\langle v \mid v'\rangle]$$

for all $v, v' \in \mathcal{H}$, and such that the span of $\{\, \beta(v)|0\rangle : v \in \mathcal{H} \,\}$ is dense in $\mathcal{K}$; and Γ is a unitary representation on $\mathcal{K}$ of the group of unitaries on $\mathcal{H}$, fulfilling the covariance condition

$$\Gamma(U)\beta(U^{-1}v)\Gamma(U)^{-1} = \beta(v),$$

for which $|0\rangle$ is stationary, and such that the infinitesimal generator $d\Gamma(A)$ of the one-parameter group $\Gamma(\exp(itA))$ is positive selfadjoint on $\mathcal{K}$ whenever A is positive selfadjoint on $\mathcal{H}$. Up to unitary equivalence, this abstract setting uniquely leads to the standard boson (Hopf) algebra on $\mathcal{H}$, with the customary construction of the second-quantized operators.

The most important condition on a compact Connes triple, in regard to our subject, is the finiteness prescription. For the purposes of this paper, where the vector bundle aspect is completely trivial, we could as well identify $\mathcal{A} \equiv \mathcal{H}^\infty(D)$ as vector spaces. In the present nonunital case, in order to have projective modules an auxiliary multiplier algebra $\widetilde{\mathcal{A}}$ of $\mathcal{A}$ is needed; still $\mathcal{H}^\infty(D)$ must be an $\mathcal{A}$-pullback of a finite projective $\widetilde{\mathcal{A}}$-module, and we can keep the identification of $\mathcal{A}$ and $\mathcal{H}^\infty(D)$.

Assume, then, that the Hilbert space $\mathcal{H}$ for Segal's framework has been identified. It is clear that some more structure is required if one is to construct singular operators like the Wick polynomials. The role of a distinguished operator $\underline{D}$ – and of its quantum counterpart $d\Gamma(\underline{D})$ – is precisely to determine domains of regularity for them. For technical reasons, in the context of [2] the operator $\underline{D}$ must be taken strictly positive: $\underline{D} \geqslant \varepsilon$ for some $\varepsilon > 0$; so in particular it is invertible. The operator $\underline{D}$ might not be strictly positive at the outset; but a related operator will do. For instance, for the scalar case, commutative or not, we may use $\underline{D} := (D\!\!\!/^2 + \varepsilon^2)^{1/2}$.

Denote $\mathcal{K}^\infty(\underline{D}) := \bigcap_{k\in\mathbb{N}} \operatorname{Dom}(d\Gamma(\underline{D})^k) \subset \mathcal{K}$. A typical element of $\mathcal{K}^\infty(\underline{D})$ is a symmetrized tensor power of elements of $\mathcal{H}^\infty(\underline{D})$; in fact the algebraic span of such vectors is dense in $\mathcal{K}^\infty(\underline{D})$. The boson field $\varphi(v)$ is just the selfadjoint generator of $\beta(tv)$; the Segal field $\varphi(v) = a(v) + a^\dagger(v)$ (here $a(v)$ and $a^\dagger(v)$ are the usual annihilation and creation operators) is essentially selfadjoint on $\mathcal{K}^\infty(\underline{D})$, and it is easy to see that for $v \in \mathcal{H}^\infty(\underline{D})$, it sends $\mathcal{K}^\infty(\underline{D})$ continuously into itself.

It is advantageous to think of $\varphi(v)$ as a quadratic form; we recall how this comes about. Let L be a dense subspace of $\mathcal{H}$, gifted with a topology stronger than that of $\mathcal{H}$ (in our case, $\mathcal{H}^\infty(\underline{D})$ and $\mathcal{K}^\infty(\underline{D})$ are given the projective Fréchet space topologies associated to the families of norms $\|\underline{D}^n(\cdot)\|$ and $\|d\Gamma(\underline{D})^n(\cdot)\|$, respectively), and let f be a continuous sesquilinear form on L. One could try to introduce a Hilbert space operator $T_f^{\mathcal{H}}$ through $f(u, v) =: \langle u \mid T_f^{\mathcal{H}} v\rangle$, defined on elements v of L for which $f(u, v) \leqslant c_v\|u\|$ for all u; but that condition might only hold for, say, $v = 0$. However, if $L^\sharp$ is the antidual of L, then $\mathcal{H} \hookrightarrow L^\sharp$ with a continuous embedding, since $u \mapsto \langle u \mid v\rangle$ is an antilinear continuous functional on L, and f defines a map $T_f \colon L \to L^\sharp$ by $T_f v(u) := f(u, v)$. The elements of $L^\sharp$ in a concrete representation for $\mathcal{H}$ are distributions; and so quadratic forms are generalized operators. Often, $(\mathcal{H}^\infty(\underline{D}))^\sharp$ is denoted $\mathcal{H}^{-\infty}(\underline{D})$.

We refer to [2, Sec. 7.3] for the following estimate: for all $v \in \mathcal{H}$, $\Phi \in \mathcal{K}^\infty(\underline{D})$ and $m \geqslant 1$,

$$\|a(v)\Phi\| \leqslant C\,\|\underline{D}^{-m}v\|\,\|d\Gamma(\underline{D})^m\Phi\|.$$

From that, and the formula

$$\langle \Psi \mid {:}\varphi(w_1)\cdots\varphi(w_n){:} \mid \Phi\rangle = \sum_{I\subseteq\{1,\dots,n\}} \Big\langle \prod_{i\in I^c} a(w_i)\Psi \,\Big|\, \prod_{j\in I} a(w_j)\Phi \Big\rangle, \tag{5.1}$$

with $\Psi, \Phi \in \mathcal{K}^\infty(\mathcal{H})$ and $I^c = \{1,\dots,n\} \setminus I$, it is immediate that one can define a *Wick map* from monomials in the free algebra over $\mathcal{H}^{-\infty}(\underline{D})$ to quadratic forms on $\mathcal{K}^\infty(\underline{D})$, extending the similar map in the subalgebra generated by $\mathcal{H}$.

To fix ideas in the following, the reader can put $2N = 4$. We work on Euclidean space rather than on Minkowski spacetime, but formal passage to relativistic field theory (where however everything takes place on-shell) is quite simple. The conservative approach is to have the ${:}\varphi^{\star r}(x){:}$ living in the commutative context, that is, in the boson algebra $\mathcal{K} = \Gamma(\mathcal{H})$ over the Hilbert space $\mathcal{H}$ of square-summable functions on momentum space. As already indicated, $\mathcal{K} \simeq \bigoplus_{n=0}^\infty \mathcal{H}^{\vee n}$, where $\mathcal{H}^{\vee n}$ is identified to the space of complex symmetric functions Φ, square-integrable with respect to the standard volume form $d^{2N}p_1 \cdots d^{2N}p_n$ in $\mathbb{R}^{2Nn}$. Precisely, the norm on $\mathcal{H}^{\vee n}$ is taken to be

$$\|\Phi^{(n)}\|^2 := \int \cdots \int n!\,|\Phi^{(n)}(p_1,\dots,p_n)|^2 \prod_{i=1}^n d^{2N}p_i.$$

Then $\mathcal{H}^\infty(\underline{D})$ is none other than the space $\mathcal{D}_{L^2}$! Furthermore, it is possible to take $w_i(x) = \delta(x - x_i)$ in the above (5.1), as the distribution $\delta(\cdot - x_i)$ belongs to $\mathcal{H}^{-\infty}(\underline{D}) = H^{2,-\infty} = \mathcal{D}'_{L^2}$.

An outcome of the previous discussion is that the Wick products

$${:}\varphi(x_1)\cdots\varphi(x_l){:} \;:=\; {:}\varphi(\delta(x - x_1))\cdots\varphi(\delta(x - x_l)){:}$$

used by physicists make perfect sense as continuous sesquilinear forms on the corresponding $\mathcal{K}^\infty(\underline{D})$, and *a fortiori* on the space of Fock vectors with finitely many nonvanishing components, each one belonging to (a symmetrized tensor power of) $\mathcal{D}_{L^2}$. The function from $\mathbb{R}^{2Nl} \times (\mathcal{K}^\infty(\underline{D}))^2$ to $\mathbb{C}$ given by

$$(x_1,\dots,x_n;\Psi,\Phi) \mapsto \langle \Psi \mid {:}\varphi(x_1)\dots\varphi(x_l){:} \mid \Phi\rangle,$$

being continuous (indeed, smooth) in $x_1,\dots,x_l$, can be restricted to the diagonal; and this defines the (ordinary) Wick monomials ${:}\varphi^l(x){:}$ for any l. That is to say,

$$\langle \Psi \mid {:}\varphi^l(x){:} \mid \Phi\rangle = \langle \Psi \mid \langle\, {:}\varphi(x_1)\dots\varphi(x_l){:}\,, \delta(x - x_1)\dots\delta(x - x_l)\rangle_{x_1,\dots,x_l} \mid \Phi\rangle \tag{5.2}$$

is a well-defined expression. Thus, and more important still, we have established that manipulations with Dirac delta functions – such as the ones we are going to use later to define Moyal–Wick monomials – are justifiable. In this respect, the good behaviour of $\mathcal{D}_{L^2}$ under the Moyal product, as under the ordinary one, becomes crucial. Also, for the same reason that the better algebra to represent the Moyal plane is $(\mathcal{D}_{L^2}, \star_\theta)$ rather than $(\mathcal{S}, \star_\theta)$, the use of Schwartz functions and tempered distributions in the classic paper by Wightman and Gårding [95], in which Wick products and Wick monomials were defined as operator-valued distributions, has been revealed as artificial.

5.2 The noncommutative Wick monomials

For ease of reference, we give here the explicit expression of the ordinary commuting Wick products

$$
\begin{aligned}
&\big[{:}\varphi(x_1) \dots \varphi(x_l){:}\ \Phi \big]^{(n)}(p_1, \dots, p_n) \\
&\quad = (2\pi)^{-Nl} \sum_{j=0}^{l} \int \cdots \int \sum_{|X|=l-j} \frac{1}{j!\,(l-j)!} \sum_P \big[P\, e^{i(x_1\eta_1 + \dots + x_j\eta_j - x_{j+1}\eta_{j+1} - \dots - x_l\eta_l)} \big] \\
&\qquad\qquad \times \Phi^{(n-l+2j)}(\eta_1, \dots, \eta_j, p_1, \dots, \widehat{\eta_{j+1}}, \dots, \widehat{\eta_l}, \dots, p_n) \prod_{k=1}^{j} d^{2N}\eta_k, \qquad (5.3)
\end{aligned}
$$

where P runs over all permutations of the momentum variables, and $X = \{\eta_{j+1}, \dots, \eta_l\}$ ranges over all subsets of $l - j$ distinct elements of $\{p_1, \dots, p_n\}$. Consequently, for good measure:

$$
\begin{aligned}
&\big[{:}\varphi^l{:}\,(x)\, \Phi \big]^{(n)}(p_1, \dots, p_n) := \big[{:}\varphi^l(x){:}\ \Phi \big]^{(n)}(p_1, \dots, p_n) \\
&\quad = (2\pi)^{-Nl} \sum_{j=0}^{l} \int \cdots \int \sum_{|X|=l-j} \frac{1}{j!\,(l-j)!} \sum_P \big[P\, e^{ix(\eta_1 + \dots + \eta_j - \eta_{j+1} - \dots - \eta_l)} \big] \\
&\qquad\qquad \times \Phi^{(n-l+2j)}(\eta_1, \dots, \eta_j, p_1, \dots, \widehat{\eta_{j+1}}, \dots, \widehat{\eta_l}, \dots, p_n) \prod_{k=1}^{j} d^{2N}\eta_k.
\end{aligned}
$$

We have used operator rather than sesquilinear-form notation, although ${:}\varphi^l(x){:}\ \Phi$ for $\Phi \in \mathcal{K}^\infty(\underline{D})$ is not in $\mathcal{K}$, instead it is an (actually rather tame) vector-valued distribution. But it is guaranteed that $\langle \mathcal{K}^\infty(\underline{D}) \mid {:}\varphi^l(x){:} \mid \mathcal{K}^\infty(\underline{D}) \rangle$ is finite.

Let us now reinstate the Moyal product associated to a $k \times k$ skewsymmetric matrix Θ; for now, we assume Θ to be nondegenerate. Formula (2.2) can be construed as meaning

$$
\delta(x - s) \star_\Theta \delta(x - t) = (\pi\theta)^{-2N} e^{-2i(s\cdot\Theta^{-1}t)} e^{-2i(x\cdot\Theta^{-1}s + t\cdot\Theta^{-1}x)}.
$$

The left hand side could of course have been written, somewhat more correctly, as $(\delta_s \star_\Theta \delta_t)(x)$. More generally, an easy two-step induction gives

$$
\begin{aligned}
&\delta(x - x_1) \star_\Theta \cdots \star_\Theta \delta(x - x_{2m}) \\
&\quad = (\pi\theta)^{-2Nm}\, e^{2i\sum_{i<j}(-)^{i+j} x_i\cdot\Theta^{-1}x_j}\ e^{-2ix\cdot\Theta^{-1}(x_1 - x_2 + x_3 - \dots - x_{2m})}, \qquad (5.4a) \\
&\delta(x - x_1) \star_\Theta \cdots \star_\Theta \delta(x - x_{2m+1}) \\
&\quad = (\pi\theta)^{-2Nm}\, e^{2i\sum_{i<j}(-)^{i+j} x_i\cdot\Theta^{-1}x_j}\ \delta(x - x_1 + x_2 - x_3 + \dots - x_{2m+1}). \qquad (5.4b)
\end{aligned}
$$

These functionals of $x_1, \dots, x_{2m}$ or $x_1, \dots, x_{2m+1}$ belong to $(\mathcal{D}'_{L^2})^{2m}$ or respectively $(\mathcal{D}'_{L^2})^{2m+1}$ – recall that the space of rapidly decreasing distributions $\mathcal{O}'_C$ is a subspace of $\mathcal{D}'_{L^2}$ [77].

There can be no question of making ${:}\varphi(x_1) \cdots \varphi(x_l){:}$ "noncommutative"; so, how are we to define the Moyal–Wick products ${:}\varphi^{\star_\Theta l}{:}\,(x)$?

A "quantum Wick product" was recently introduced in [4]; but it is at variance with Moyal NCFT, and so is unsuitable for our present purposes. A different course is suggested by the older duality theory of [43, 90] and the discussion in the previous subsection. Our declared tactics are to construct $:\varphi^{\star_\Theta l}:(x)$ on the very same Fock space of the real scalar field. This would seem to run against the spirit of noncommutative geometry, but is in fact demanded by our results here so far, and the treatment in the previous subsection. We posit

$$:\varphi^{\star_\Theta l}:(x) := \langle :\varphi(x_1)\cdots\varphi(x_l):, \delta(x-x_1)\star_\Theta\cdots\star_\Theta\delta(x-x_l)\rangle_{x_1,\dots,x_l}, \tag{5.5}$$

to be compared with (5.2).

We may also define Moyal products of Moyal–Wick monomials with suitable scalar functions or distributions on configuration space:

$$:\varphi^{\star_\Theta l}: \star_\Theta h(x) = \langle :\varphi(x_1)\cdots\varphi(x_l):, \delta(x-x_1)\star_\Theta\cdots\star_\Theta\delta(x-x_l)\star_\Theta h(x)\rangle_{x_1,\dots,x_l}$$
$$h\star_\Theta :\varphi^{\star_\Theta l}:(x) = \langle :\varphi(x_1)\cdots\varphi(x_l):, h(x)\star_\Theta\delta(x-x_1)\star_\Theta\cdots\star_\Theta\delta(x-x_l)\rangle_{x_1,\dots,x_l}.$$

What it is required is that the functional $\delta(x-x_1)\star_\Theta\cdots\star_\Theta\delta(x-x_l)\star_\Theta h(x)$, in the $x_1,\dots,x_l$ variables, belong to $(\mathcal{H}^{-\infty}(\underline{D}))^l$. A seemingly alternative definition is given by

$$\langle :\varphi^{\star_\Theta l}: \star_\Theta h(x), g(x)\rangle = \langle :\varphi^{\star_\Theta l}:(x), h\star_\Theta g(x)\rangle,$$
$$\langle h\star_\Theta :\varphi^{\star_\Theta l}:(x), g(x)\rangle = \langle :\varphi^{\star_\Theta l}:(x), g\star_\Theta h(x)\rangle,$$

in the spirit of [43, 90], for suitable spaces of functions g and distributions h. The verification that both kinds of definition coincide is immediate.

Note that the identity

$$\langle :\varphi^{\star_\Theta l}:(x), h(x)\rangle = \langle :\varphi(x_1)\cdots\varphi(x_l):, h(x_1)\star_\Theta\delta(x_1-x_2)\star_\Theta\cdots\star_\Theta\delta(x_1-x_l)\rangle_{x_1,\dots,x_l}$$

affords a definition of the Moyal–Wick monomials *à la* Wightman and Gårding.

Using now (5.4) together with (5.3) and (5.5), we obtain the completely explicit formula on the boson Fock space

$$\begin{aligned}
&(2\pi)^{Nl}\left[\varphi^{\star_\Theta l}(x)\Phi\right]^{(n)}(p_1,\dots,p_n)\\
&= \sum_{j=0}^{l}\int\cdots\int\sum_{|X|=l-j}\frac{1}{j!\,(l-j)!}\sum_P\Big[P\,e^{ix(\eta_1+\cdots+\eta_j-\eta_{j+1}-\cdots-\eta_l)}e^{\mp\frac{i}{2}\sum_{m<r}\eta_m\cdot\Theta\eta_r}\Big]\\
&\qquad\qquad\times\Phi^{(n-l+2j)}(\eta_1,\dots,\eta_j,p_1,\dots,\widehat{\eta_{j+1}},\dots,\widehat{\eta_l},\dots,p_n)\prod_{k=1}^{j}d^{2N}\eta_k.
\end{aligned}$$

Here in the exponent quadratic in the η's the $-$ sign applies when $r\leqslant j$ or $m>j$, the $+$ sign otherwise. In the simplest instance, we get

$$\begin{aligned}
&(2\pi)^{3N}\left[\langle\varphi^{\star 2}(x),h(x)\rangle\Phi\right]^{(n)}(k_1,\dots,k_n)\\
&= \iint \hat h(\kappa_1+\kappa_2)\cos\tfrac12\kappa_1\Theta\kappa_2\,\Phi^{(n+2)}(\kappa_1,\kappa_2,k_1,\dots,k_n)\,d^{2N}\kappa_1\,d^{2N}\kappa_2\\
&\quad+\sum_{j=1}^{n}\int\left[\hat h(\kappa-k_j)e^{\frac{i}{2}\kappa\Theta k_j}+\hat h(k_j-\kappa)e^{\frac{i}{2}k_j\Theta\kappa}\right]\Phi^{(n)}(\kappa,k_1,\dots,\widehat{k_j},\dots,k_n)\,d^{2N}\kappa\\
&\quad+\sum_{1\leqslant j\neq l\leqslant n}\hat h(-k_j-k_l)\cos\tfrac12 k_j\Theta k_l\,\Phi^{(n-2)}(k_1,\dots,\widehat{k_j},\dots,\widehat{k_l},\dots,k_n).
\end{aligned}$$

(We underline again that the $:\varphi(x_1)\dots\varphi(x_l):$, here as in (5.2), are the usual commutative boson products of fields, with all creation operators to the left of the annihilation operators – see [9, Sec. 4.1] – as for instance in

$$
\begin{aligned}
&(2\pi)^{3N} :\varphi(x_1)\varphi(x_2)\varphi(x_3): \\
&= \int\cdots\int \big[e^{i(k_1x_1+k_2x_2+k_3x_3)}a(k_1)a(k_2)a(k_3) + e^{-i(k_1x_1-k_2x_2-k_3x_3)}a^\dagger(k_1)a(k_2)a(k_3) \\
&\quad + e^{i(k_1x_1-k_2x_2+k_3x_3)}a^\dagger(k_2)a(k_1)a(k_3) + e^{i(k_1x_1+k_2x_2-k_3x_3)}a^\dagger(k_3)a(k_1)a(k_2) \\
&\quad + e^{-i(k_1x_1+k_2x_2-k_3x_3)}a^\dagger(k_1)a^\dagger(k_2)a(k_3) + e^{-i(k_1x_1-k_2x_2+k_3x_3)}a^\dagger(k_1)a^\dagger(k_3)a(k_2) \\
&\quad + e^{i(k_1x_1-k_2x_2-k_3x_3)}a^\dagger(k_2)a^\dagger(k_3)a(k_1) + e^{-i(k_1x_1+k_2x_2+k_3x_3)}a^\dagger(k_1)a^\dagger(k_2)a^\dagger(k_3)\big] \prod_{i=1}^{3} d^{2N}k_i.
\end{aligned}
$$

In turn we are assured that $\varphi^{\star_\Theta l}(x)$ is normally ordered. Had we tried to use in (5.5) the operator product instead of the normal product, we would have been punished by extra divergent terms of the type $\int \delta(k_1-k_2)\, d^{2N}k_1\, d^{2N}k_2$, just as in the commutative case. Thus, as anticipated, the twisted product does not help with the ordering problem.)

The previous formulae have been obtained under the assumption that $\det\Theta > 0$. For $k = 2N$, the set of nonsingular skewsymmetric $k\times k$ matrices is open and dense in the set of all skewsymmetric $k\times k$ matrices, and the same formulae are valid when $\det\Theta = 0$ by continuity. We also conclude their validity in the case that k is odd, by consideration of an extra dimension with trivial commutation relations.

6 The functional action

The functional action plays a great role in the applications to physics of noncommutative geometry, because it reproduces not only the Yang–Mills action but also the full Yang–Mills–Higgs [55, 91] and even, in its more general incarnation [13], the Einstein–Hilbert action.

Here we choose, for the reasons indicated in the introduction, to compute the Connes–Lott action [16, 21, 22, 63], which notionally is $\mathrm{Tr}^+(F^2\, D\!\!\!/^{-2N})$, for F the field strength or curvature associated to a vector potential α. Due to some ambiguity in the transition to F from α, unimportant for the general theory but crucial for physics, we need to deal with "junk": that is, to quotient by an ideal living in the representation π_D of the universal differential algebra on $\mathcal{H}$. Then we show that the action coincides with the noncommutative Yang–Mills action currently used in Moyal gauge theory. Of course, physicists have not waited for formal developments of this kind before forging ahead (see [42], for instance); but for our purposes this is an indispensable check.

We first make some necessary remarks on the bimodule nature of the image of π_D.

6.1 Connes–Terashima fermions

That bimodule nature is completely familiar to customers of Connes' noncommutative geometry, and basically means that $\mathcal{H}^\infty$ can sustain a bimodule action of two algebras. The reconstruction of the Standard Model Lagrangian in [18] uses actions of this type, exchanged by the charge conjugation operator.

Independently, in the traditional context of Lie algebras, Terashima [83] summarized to similar effect some natural methods and restrictions, that were scattered in practice, to introduce noncommutative gauge fields.

First of all, assume an infinitesimal gauge variation given by

$$\delta_\lambda A_\mu(x) = \partial_\mu \lambda(x) - i[A_\mu, \lambda]_{\star_\theta}(x),$$

or explicitly,

$$\delta_\lambda A_\mu(x) = T^a \partial_\mu \lambda_a(x) - \tfrac{i}{2}\big([T^a, T^b]\,\{A^a_\mu, \lambda^b\}_{\star_\theta}(x) + \{T^a, T^b\}\,[A^a_\mu, \lambda^b]_{\star_\theta}(x)\big)$$

where the T^a denote the gauge "group" generators, normalized as

$$\mathrm{Tr}(T^a T^b) = \tfrac{1}{2}\,\delta^{ab},$$

closing to a Lie algebra

$$[T^a, T^b] = i\, f^{ab}_c\, T^c,$$

and $[\cdot,\cdot]_{\star_\theta}$, $\{\cdot,\cdot\}_{\star_\theta}$ denote the Moyal commutator and anticommutator brackets, respectively. Let us think of the Lie algebra of $\mathrm{SU}(n)$, to fix ideas. Then

$$\{T^a, T^b\} = \frac{1}{n}\,\delta^{ab} + d^{ab}_c\, T^c,$$

where the d^{ab}_c are totally symmetric and real. This is not a linear combination of T^d's. Therefore noncommutative gauge transformations are consistent only for unitary groups (there are some ways round this obstacle; but they are not very appealing). But then the gauge group of unitary transformations is identified to the unitary endomorphism group of a module, and we are back in Connes' context.

The second remark by Terashima is that the same closure requirement and consideration of the covariant derivative forces the representation of $\mathrm{U}(n)$ to be fundamental or antifundamental. It is possible, however, for a gauge group to act from the left, say in the fundamental representation, and (perhaps a different one) from the right in the antifundamental one, with gauge transformations given by

$$\Psi \mapsto U_{(1)} \star_\theta \Psi \star_\theta U^*_{(2)}.$$

Again, this is completely natural in the context of algebra bimodules. We remark that already the chiral anomaly for these fermions has been calculated [62].

We want to add that, even in the context of pure group theory, the concept of bimodule is called for. We formalize this remark, in the spirit of [96]. By definition, a linear space V is a (G,H)-bimodule if it carries a left action $\triangleright$ of the group G (with the usual continuity or smoothness conditions) and a right action $\triangleleft$ of the group H which are compatible, that is to say,

$$g \triangleright (v \triangleleft h) = (g \triangleright v) \triangleleft h,$$

for $g \in G$, $h \in H$, $v \in V$. A bimodule is irreducible if there is no proper subspace of V stable under both actions. If V is a G left-module and W is an H right-module, then $V \otimes W$ is a (G,H)-bimodule. When $G = H$, interesting bimodules usually have a conjugation operator that exchanges the actions. In that case, the bimodule is a very canonical object in harmonic analysis: the space of functions on a group G is a (G,G)-bimodule; and if G possesses a representation on a space W, then $V = \mathrm{End}\, W$ is a (G,G)-bimodule; but, strangely enough, it does not seem to be in use.

6.2 The differential algebra

In the Connes–Lott approach one works with the tensor product of some finite dimensional Eigenschaften algebra and a spacetime algebra (that here is no longer commutative). We disregard the Eigenschaften algebra in what follows; in other words, we concentrate on the analytical details of the U(1) Moyal gauge theory.

In this subsection we do not need to consider the preferred unitization of $\mathcal{A}_\theta$. As in [17,21,22], let $\Omega^\bullet\mathcal{A}_\theta := \bigoplus_{p\in\mathbb{N}}\Omega^p\mathcal{A}_\theta$ be the universal differential graded algebra over $\mathcal{A}_\theta$, where $\Omega^p\mathcal{A}_\theta := \{\, f_0\,\delta f_1 \dots \delta f_p : f_i \in \mathcal{A}_\theta \,\}$ and the only constraint on δ is to satisfy the Leibniz rule $\delta(f_1 \star_\theta f_2) = \delta f_1\, f_2 + f_1\, \delta f_2$ so δ can be extended on $\Omega^\bullet\mathcal{A}_\theta$. Since $\mathcal{A}_\theta$ has no unit, we define [17, III.1.α] $\Omega^0\mathcal{A}_\theta := \mathcal{A}_\theta \oplus \mathbb{C}$, which is the minimal unitization of $\mathcal{A}_\theta$, and $\delta(0\oplus 1) := 0$. Moreover $(\delta f)^* := \delta f^*$.

The representation π^θ of $\mathcal{A}_\theta$ by elements of $\mathcal{L}(\mathcal{H})$ extends naturally to $\Omega^\bullet\mathcal{A}_\theta$, by

$$\tilde\pi^\theta : \Omega^p\mathcal{A}_\theta \to \mathcal{L}(\mathcal{H}) : f_0\,\delta f_1 \dots \delta f_p \mapsto i^p\, \pi^\theta(f_0)\, [D\!\!\!/, \pi^\theta(f_1)] \dots [D\!\!\!/, \pi^\theta(f_p)].$$

Lemma 6.1. *If $f_i \in \mathcal{A}_\theta$, then $\tilde\pi^\theta(f_0\,\delta f_1 \dots \delta f_p) = L^\theta(f_0 \star_\theta \partial_{\mu_1} f_1 \star_\theta \dots \star_\theta \partial_{\mu_p} f_p) \otimes \gamma^{\mu_1} \dots \gamma^{\mu_p}$.*

Proof. This follows from $[D\!\!\!/, L^\theta_f \otimes 1_{2^N}] = -iL^\theta(\partial_\mu f) \otimes \gamma^\mu$ and $L^\theta_f\, L^\theta_g = L^\theta(f \star_\theta g)$. □

To overcome the unfaithfulness of $\tilde\pi^\theta$ (even if π^θ is faithful), one introduces a graded two-sided ideal of $\Omega^\bullet\mathcal{A}_\theta$, namely Junk $:= \bigoplus_{p\in\mathbb{N}} J^p = \bigoplus_{p\in\mathbb{N}} J^p_0 + \delta J^{p-1}_0$, $J^p_0 := \{\, \omega \in \Omega^p\mathcal{A}_\theta : \tilde\pi^\theta(\omega) = 0 \,\}$, and finally

$$\Omega_{D\!\!\!/}\, \mathcal{A}_\theta := \tilde\pi^\theta(\Omega^\bullet\mathcal{A}_\theta)/\tilde\pi^\theta(\text{Junk}).$$

Here, the 2-junk is particularly simple since it is isomorphic to $\pi^\theta(\mathcal{A}_\theta)$, as we now show.

Proposition 6.2. *There is a natural identification $\tilde\pi^\theta(J^2) \simeq \pi^\theta(\mathcal{A}_\theta) = L^\theta(\mathcal{A}_\theta) \otimes 1_{2^N}$.*

Proof. Any $\omega \in \tilde\pi^\theta(J^2) \subset \tilde\pi^\theta(\Omega^2\mathcal{A}_\theta)$ can be written as $\omega = \sum_{j\in I} L^\theta(\partial_\mu f_j) L^\theta(\partial_\nu g_j) \otimes \gamma^\mu\gamma^\nu$ where I is a finite set, and satisfies $\sum_{j\in I} L^\theta(f_j \star_\theta \partial_\mu g_j) \otimes \gamma^\mu = 0$. By the Leibniz rule,

$$\begin{aligned}\omega &= \sum_{j\in I} L^\theta(\partial_\mu(f_j \star_\theta \partial_\nu g_j) - f_j \star_\theta \partial_\mu\partial_\nu g_j) \otimes \gamma^\mu\gamma^\nu \\ &= -\sum_{j\in I} L^\theta(f_j \star_\theta \partial_\mu\partial_\nu g_j) \otimes \gamma^\mu\gamma^\nu = -\sum_{j\in I} L^\theta(f_j \star_\theta \partial_\mu\partial_\nu g_j) \otimes \eta^{\mu\nu}\, 1_{2^N}.\end{aligned}$$

Hence $\tilde\pi^\theta(J^2) \subset \pi^\theta(\mathcal{A}_\theta) = L^\theta(\mathcal{A}_\theta) \otimes 1_{2^N}$.

Consider $\omega_{mnkl} := f_{mk}\,\delta f_{kn} - f_{ml}\,\delta f_{ln}$ (no summation) in $\Omega^1\mathcal{A}_\theta$. In subsection A.2, it is shown that $\tilde\pi^\theta(\omega_{mnkl}) = 0$ and $\tilde\pi^\theta(\delta\omega_{mnkl}) = \frac{2}{\theta}\sum_{j=1}^N (k_j - l_j)\, L^\theta(f_{mn}) \otimes 1_{2^N}$, which is nonzero if $|l| \neq |k|$. Thus, $L^\theta(f_{mn}) \otimes 1_{2^N}$ lies in $\tilde\pi^\theta(J^2)$ for all $m, n \in \mathbb{N}^N$. Since $\{f_{mn}\}$ is a basis for $\mathcal{A}_\theta$, we conclude that $\pi^\theta(\mathcal{A}_\theta) \simeq L^\theta(\mathcal{A}_\theta) \otimes 1_{2^N} \subset \tilde\pi^\theta(J^2)$. □

It is easy to generalize the above proof, to get the next Corollary.

Corollary 6.3. *For $p \geqslant 2$, $\tilde\pi^\theta(J^p)$ is the linear span of the elements in $\tilde\pi^\theta(\Omega^p\mathcal{A}_\theta)$ of the form $L^\theta_f \otimes \gamma^{\mu_1} \dots \gamma^{\mu_k}$, with $k \leqslant p-2$ and of the same parity as p.*

6.3 The action

Let $\widetilde{\mathcal{H}}_p$ be the Hilbert space obtained by completion of $\tilde{\pi}^\theta(\Omega^p\mathcal{A}_\theta)$ under the scalar product

$$\langle \tilde{\pi}^\theta(\omega) \mid \tilde{\pi}^\theta(\omega')\rangle_p := \mathrm{Tr}^+\big(\tilde{\pi}^\theta(\omega)^*\, \tilde{\pi}^\theta(\omega')\, (D\!\!\!/^2 + \varepsilon^2)^{-N}\big),$$

for $\omega, \omega' \in \Omega^p\mathcal{A}_\theta$. This defines a natural pre-action $I(\eta)$ when $p = 2$ and $\omega' = \omega = \delta\eta + \eta^2$:

$$I(\eta) := \mathrm{Tr}^+\big(\tilde{\pi}^\theta(\omega)^*\, \tilde{\pi}^\theta(\omega)\, (D\!\!\!/^2 + \varepsilon^2)^{-N}\big). \tag{6.1}$$

Let P be the orthogonal projector on $\widetilde{\mathcal{H}}_p$ whose range is the orthogonal complement of $\tilde{\pi}^\theta(\delta J_0^{p-1})$, and define $\mathcal{H}_p := P\widetilde{\mathcal{H}}_p$. Then P extends the quotient map from $\tilde{\pi}^\theta(\Omega^p\mathcal{A}_\theta)$ onto $\Omega^p_{D\!\!\!/}\,\mathcal{A}_\theta$, which is identified with a dense subspace of $\mathcal{H}_p$. The possible ambiguity in (6.1) due to the unfaithfulness of $\tilde{\pi}^\theta$ disappears if we define the functional action (noncommutative Yang–Mills action) as:

$$YM(\alpha) := \frac{N!\,(2\pi)^N}{8g^2}\,\langle P\tilde{\pi}^\theta(F) \mid P\tilde{\pi}^\theta(F)\rangle_2 \tag{6.2}$$

where $\Omega^1_{D\!\!\!/}\,\mathcal{A}_\theta \ni \alpha = \tilde{\pi}^\theta(\eta)$ and $F = \delta\eta + \eta^2$ is the curvature of the 1-form η and g is the coupling constant. It is shown in [21,91] that $YM(\alpha)$ is equal to the infimum of the preaction on all $\eta \in \Omega^1\mathcal{A}_\theta$ with the same image in $\Omega^1_{D\!\!\!/}\,\mathcal{A}_\theta$:

$$YM(\alpha) = \frac{N!\,(2\pi)^N}{8g^2}\,\inf\{\, I(\eta) : \tilde{\pi}^\theta(\eta) = \alpha \,\}.$$

This result justifies the notation $YM(\alpha)$, because this positive quartic functional of η depends only on its equivalence class in $\Omega^1_{D\!\!\!/}\mathcal{A}_\theta$, namely α.

Theorem 6.4. *Let $\eta = -\eta^* \in \Omega^1\mathcal{A}_\theta$. Then the Yang–Mills action $YM(\alpha)$ of the universal connection $\delta + \eta$, with $\alpha = \tilde{\pi}^\theta(\eta)$, is equal to*

$$YM(\alpha) = -\frac{1}{4g^2}\int F^{\mu\nu} \star_\theta F_{\mu\nu}(x)\, d^{2N}x = -\frac{1}{4g^2}\int F^{\mu\nu}(x)\, F_{\mu\nu}(x)\, d^{2N}x,$$

where $F_{\mu\nu} := \frac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]_{\star_\theta})$ and A_μ is defined by $\alpha = L^\theta(A_\mu) \otimes \gamma^\mu$.

Proof. If $\eta = \sum_{j\in I} f_j\, \delta g_j$ for some $f_j, g_j \in \mathcal{S}$ and a finite set I, then $\alpha = \sum_{j\in I} L^\theta_{f_j} L^\theta_{\partial_\mu g_j} \otimes \gamma^\mu = \sum_{j\in I} L^\theta(f_j \star_\theta \partial_\mu g_j) \otimes \gamma^\mu$. Thus $A_\mu := \sum_{j\in I} f_j \star_\theta \partial_\mu g_j$ and, with a sum over $j, k \in I$ understood,

$$\begin{aligned}
\tilde{\pi}^\theta(\delta\eta + \eta^2) &= \tilde{\pi}^\theta(\delta f_j\, \delta g_j + (f_j\, \delta g_j)(f_k\, \delta g_k)) \\
&= \tilde{\pi}^\theta(\delta f_j\, \delta g_j + f_j\, \delta(g_j \star_\theta f_k)\delta g_k - (f_j \star_\theta g_j)\, \delta f_k\, \delta g_k) \\
&= L^\theta(\partial_\mu f_j \star_\theta \partial_\nu g_j + f_j \star_\theta \partial_\mu(g_j \star_\theta f_k) \star_\theta \partial_\nu g_k - f_j \star_\theta g_j \star_\theta \partial_\mu f_k \star_\theta \partial_\nu g_k) \otimes \gamma^\mu\gamma^\nu \\
&= L^\theta(\partial_\mu f_j \star_\theta \partial_\nu g_k + f_j \star_\theta \partial_\mu g_j \star_\theta f_k \star_\theta \partial_\nu g_k) \otimes \gamma^\mu\gamma^\nu \\
&= L^\theta(\partial_\mu(f_j \star_\theta \partial_\nu g_j) + f_j \star_\theta \partial_\mu g_j \star_\theta f_k \star_\theta \partial_\nu g_k) \otimes \gamma^\mu\gamma^\nu - L^\theta(f_j \star_\theta \partial_\mu\partial_\nu g_j) \otimes \eta^{\mu\nu}\, 1_{2^N} \\
&= L^\theta(\partial_\mu A_\nu + A_\mu \star_\theta A_\nu) \otimes \tfrac{1}{2}[\gamma^\mu, \gamma^\nu] + \eta^{\mu\nu} L^\theta(\partial_\mu A_\nu + A_\mu \star_\theta A_\nu) \otimes 1_{2^N} \\
&\qquad - \eta^{\mu\nu} L^\theta(f \star_\theta \partial_\mu\partial_\nu g) \otimes 1_{2^N}.
\end{aligned}$$

The two last terms are in $\tilde{\pi}^\theta(J^2)$. Thus,

$$\begin{aligned} P(\tilde{\pi}^\theta(F)) &= P\big(L^\theta(\partial_\mu A_\nu + A_\mu \star_\theta A_\nu) \otimes \tfrac{1}{2}[\gamma^\mu, \gamma^\nu]\big) \\ &= P\big(L^\theta(\tfrac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]_{\star_\theta}) \otimes \gamma^\mu\gamma^\nu\big) \\ &= P(L^\theta(F_{\mu\nu}) \otimes \gamma^\mu\gamma^\nu) = L^\theta(F_{\mu\nu}) \otimes \gamma^\mu\gamma^\nu, \end{aligned}$$

where the last equality follows because the junk affects only the scalar part of $\tilde{\pi}^\theta(\Omega^\bullet\mathcal{A}_\theta)$. To repeat: each $\omega = \omega_{\mu\nu} \otimes \gamma^\mu\gamma^\nu \in \tilde{\pi}^\theta(\Omega^2\mathcal{A}_\theta)$ can be uniquely decomposed as

$$\omega = \omega_{\mu\nu} \otimes \tfrac{1}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu) + \omega_{\mu\nu} \otimes \tfrac{1}{2}(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu)$$

in $\tilde{\pi}^\theta(\Omega^2\mathcal{A}_\theta)_a \oplus \tilde{\pi}^\theta(\Omega^2\mathcal{A}_\theta)_s = \tilde{\pi}^\theta(\Omega^2\mathcal{A}_\theta)_a \oplus \tilde{\pi}^\theta(J^2)$, the direct sum of its alternating and symmetric parts.

Since $A_\mu = -A_\mu^*$, we also find $F_{\mu\nu}^* = -F_{\mu\nu}$ and therefore $P(\tilde{\pi}^\theta(F))^* = L^\theta(F_{\mu\nu}) \otimes \gamma^\mu\gamma^\nu$. Then

$$\begin{aligned} &\operatorname{Tr}^+\big(L^\theta(F_{\mu\nu} \star_\theta F_{\rho\sigma})\,(-\Delta + \varepsilon^2)^{-N} \otimes \gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\big) \\ &\quad = \operatorname{Tr}^+\big(L^\theta(F_{\mu\nu} \star_\theta F_{\rho\sigma})\,(-\Delta + \varepsilon^2)^{-N}\big)\ \operatorname{tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma). \end{aligned}$$

But $\operatorname{tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 2^N(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho})$, so since $F_{\mu\nu} = -F_{\nu\mu}$, the Proposition 4.17, computed with $-i\nabla$ instead of $\not{D}$, yields

$$YM(\alpha) = -\frac{2\,N!\,(4\pi)^N}{8g^2} \operatorname{Tr}^+\big(L^\theta(F_{\mu\nu} \star_\theta F^{\mu\nu})\,(-\Delta + \varepsilon^2)^{-N}\big) = -\frac{1}{4g^2} \int (F_{\mu\nu} \star_\theta F^{\mu\nu})(x)\, d^{2N}x,$$

and according to Lemma 2.1(v) the pointwise product can replace the Moyal product. ◻

Remark 6.5. The action as we have defined it is positive definite, since

$$YM(\alpha) = \frac{1}{4g^2} \int \sum_{\mu,\nu=1}^{2N} |F_{\mu\nu}(x)|^2\, d^{2N}x.$$

7 Conclusions and outlook

We have shown in detail how to build noncompact noncommutative spin geometries. As a consequence, the classical background of present-day NCFTs is recast in the framework of the rigorous Connes formalism for geometrical noncommutative spaces.

One can wonder about the uniqueness of the constructions presented here. Our detailed scrutiny shows that appropriate algebras for the spectral triples are to a large extent "selected" by the Dirac operator itself. The choice of $\mathcal{A} = \mathcal{S}$ for the original nonunital algebra, made in the flat space cases, has much to recommend it, not least Fourier invariance and the existence of a body of tempered distribution analysis. However, an outcome of the study in this paper is that, both in the commutative and the Moyal-algebra example, a more canonical 'arrival' point is the bigger algebra $\mathcal{A}_1 := \mathcal{D}_{L^2}(\mathbb{R}^{2N})$; we found that nearly everything that works for $\mathcal{A}$ works also for $\mathcal{A}_1$, with significant improvement of the finiteness axiom; also, $\mathcal{A}_1$ yields the most advantageous framework for quantization.

We can accommodate an $\mathcal{A}_1$-triple instead of an $\mathcal{A}$-triple, provided we make a slight modification of the summability axiom. That is, we use the data $(\mathcal{A}_1, \widetilde{\mathcal{A}}, \mathcal{H}, D, J, \chi)$, and the suggested new version of the noncompact noncommutative geometry postulates runs as follows.

1′. *Spectral dimension, 2nd version*:
There is a unique nonnegative integer k, the spectral or "classical" dimension of the geometry, for which $a(|D| + \varepsilon)^{-1}$ belongs to the Schatten class $\mathcal{L}^p$ for $p > k$ whenever $a \in \mathcal{A}_1$, for any $\varepsilon > 0$; and moreover, for a in a dense ideal $\mathcal{A}$ of $\mathcal{A}_1$, $a(|D| + \varepsilon)^{-1}$ lies in the generalized Schatten class $\mathcal{L}^{k+}$ and the trace $a \mapsto \mathrm{Tr}^+(a(|D| + \varepsilon)^{-k})$ is finite and not identically zero. This k is even if and only if the spectral triple is even.

3′. *Finiteness, 2nd version*:
The algebras $\mathcal{A}_1$ and its preferred unitization $\widetilde{\mathcal{A}}$ are pre-C^*-algebras. The space of smooth vectors $\mathcal{H}^\infty$ is the $\mathcal{A}_1$-pullback of a finite projective $\widetilde{\mathcal{A}}$-module. Moreover, an $\mathcal{A}_1$-valued hermitian structure $(\cdot \mid \cdot)$ is implicitly defined on $\mathcal{H}^\infty$ with the noncommutative integral, as follows:

$$\mathrm{Tr}^+\big((a\xi \mid \eta)(|D| + \varepsilon)^{-k}\big) = \langle \eta \mid a\xi \rangle,$$

where $a \in \widetilde{\mathcal{A}}$ and $\langle \cdot \mid \cdot \rangle$ denotes the standard inner product on $\mathcal{H}$.

In the other postulates $\mathcal{A}$ is replaced by $\mathcal{A}_1$; they are otherwise unchanged. In our case $\mathcal{A}$ could be taken equal to $\mathcal{S}$ or larger: $\mathrm{Tr}^+(a(|D| + \varepsilon)^{-k}) < \infty$ is valid for a belonging to a larger ideal of $\mathcal{D}_{L^2}$.

Support for enrollment of $\mathcal{A}_1$ comes from physics, on one hand, and abstract nonsense, on the other. Langmann and Mickelsson [58] found existence of the quantum scattering matrix for quantized fermions in external gauge potentials with components precisely in the sibling $\mathcal{N}$ of $\mathcal{D}_{L^2}$; this is a both strong and significant result. Also, as exploited in Section 5, the more correct and general approach to the construction of Wick monomials makes use precisely of the smooth domain of the Dirac operator. The close relation of $\mathcal{A}_1$ to this smooth domain points to generalizations of the pseudodifferential calculus in the fully noncommutative context [50, 58].

The orientation condition and the required boundedness of the operators $[D, a]$ give rather tight lower and upper bounds (so to speak) on what the preferred compactification of $\mathcal{A}_1$ should be. It would be good to know whether these two conditions determine such a unitization uniquely. The following conjecture is strengthened by the result of [64].

Conjecture 7.1. $\widetilde{\mathcal{A}} = \mathcal{B}(\mathbb{R}^{2N})$ *is the largest Moyal multiplier algebra of* $\mathcal{A}_1 = \mathcal{D}_{L^2}(\mathbb{R}^{2N})$ *such that* $[D\!\!\!/, a]$ *is bounded for each* $a \in \widetilde{\mathcal{A}}$.

The clever argument in [11] leads one to ponder what kind of boundary conditions one would impose on $D\!\!\!/$ (without presumably changing the leading term behaviour of its spectral density) in order to obtain a *compact* spectral triple canonically associated to the given noncompact one. This should allow the anomaly calculations in [42, 62] to be made more rigorous. The subject of noncommutative manifolds with boundary is still in its infancy, however, and we shall not elaborate the point.

Apart from eventually proving a reconstruction theorem (a rather strenuous task), much remains to be done. There are probably already enough examples of noncommutative spaces around for consideration of the "category" of spectral triples to be promising. For instance, NC tori are quotients of the spaces considered in this paper. A mathematically important question is the computation of the Hochschild cohomology of $\widetilde{\mathcal{A}}_\theta$. Another is the explicit lifting of (a central extension of) the group of (nonlinear, in general) symplectomorphisms (or at least, of those connected to the identity) to a group of inner automorphisms of M^θ (or of A_θ), which should be irreducibly represented on $\mathcal{H}$. In this context, work on the geometry of the gauge algebra in noncommutative Yang–Mills theories [60] can be pursued.

A A few explicit formulas

A.1 On the oscillator basis functions

For $N = 1$ and $m, n \in \mathbb{N}$, the basic eigentransition $f_{mn}(x_1, x_2)$ is explicitly given by

$$2(-1)^{\min(m,n)} \frac{\sqrt{n!m!}}{\max(m,n)!} (4H_1/\theta)^{|m-n|/2} e^{i(n-m)\arctan(x_2/x_1)} \exp(-2H_1/\theta)\, L^{|m-n|}_{\min(m,n)}(4H_1/\theta),$$

with L^r_j being the generalized Laguerre polynomials of order j and $H_1 = \frac{1}{2}(x_1^2 + x_2^2)$. In general,

$$f_{mn}(x_1, \dots, x_N) = f_{m_1n_1}(x_1, x_{1+N}) \dots f_{m_Nn_N}(x_N, x_{2N}).$$

Also, using the coalgebra formula for the Laguerre polynomials

$$L^{r+s+1}_n(u + v) = \sum_{j+l=n} L^r_j(u) L^s_l(v),$$

one obtains [7] eigenstates for $H = H_1 + \dots + H_N$:

$$H \star_\theta f_M = f_M \star_\theta H = \theta\Big(M + \frac{N}{2}\Big) f_M,$$

where

$$f_M(x_1, \dots, x_{2N}) := \sum_{|m|=M} f_{m_1m_1} \dots f_{m_Nm_N}(x_1, \dots, x_{2N}) = 2^N (-)^M \exp(-2H/\theta)\, L^{N-1}_M(4H/\theta).$$

It is known that $\int |f_{nn}(x_1, x_2)|\, dx_1\, dx_2 \sim \sqrt{n}$ as $n \to \infty$. From this, using the closed graph theorem, it easy to show that there are non-absolutely integrable functions in $\mathcal{I}_{00}$ [28].

A.2 More junk

Lemma A.1. *For $m, n, k, l \in \mathbb{N}^N$, let $\omega_{mnkl} := f_{mk}\, \delta f_{kn} - f_{ml}\, \delta f_{ln} \in \Omega^1\mathcal{A}_\theta$ (no summation on k or l). Then*

$$\tilde{\pi}^\theta(\omega_{mnkl}) = 0 \quad \textit{and} \quad \tilde{\pi}^\theta(\delta\omega_{mnkl}) = \tfrac{2}{\theta}(|k| - |l|)\, L^\theta(f_{mn}) \otimes 1_{2^N}.$$

Proof. Using the creation and annihilation functions (2.11) we may rewrite the Dirac operator as follows; we adopt the convention that $j = 1, \dots, N$, and write $\partial_{a_j} = \partial/\partial a_j$ and $\partial_{a_j^*} = \partial/\partial a_j^*$:

$$D\!\!\!/ = -\frac{i}{\sqrt{2}} \sum_j \gamma^j (\partial_{a_j} + \partial_{a_j^*}) + i\gamma^{j+N}(\partial_{a_j} - \partial_{a_j^*}) = -i \sum_j (\gamma^{a_j}\, \partial_{a_j} + \gamma^{a_j^*}\, \partial_{a_j^*}),$$

where $\gamma^{a_j} := \frac{1}{\sqrt{2}}(\gamma^j + i\gamma^{j+N})$ and $\gamma^{a_j^*} := \frac{1}{\sqrt{2}}(\gamma^j - i\gamma^{j+N})$.

Lemma 2.1(iv), applied to a_j and a_j^* respectively, yields

$$\partial_{a_j} = -\frac{1}{\theta}\,\mathrm{ad}_{\star_\theta}\, a_j^* := -\frac{1}{\theta}[a_j^*,\cdot\,]_{\star_\theta}, \qquad \partial_{a_j^*} = \frac{1}{\theta}\,\mathrm{ad}_{\star_\theta}\, a_j := \frac{1}{\theta}[a_j,\cdot\,]_{\star_\theta}$$

and hence

$$\not{D} = -\frac{i}{\theta}\sum_j(\gamma^{a_j^*}\,\mathrm{ad}_{\star_\theta}\, a_j - \gamma^{a_j}\,\mathrm{ad}_{\star_\theta}\, a_j^*).$$

Let $u_j := (0,0,\dots,1,\dots,0)$ be the j-th standard basis vector of $\mathbb{R}^N$. From the definition (2.10) of f_{mn}, we directly compute:

$$a_j^* \star_\theta f_{mn} = \sqrt{\theta(m_j+1)}\, f_{m+u_j,n}, \qquad f_{mn}\star_\theta a_j^* = \sqrt{\theta n_j}\, f_{m,n-u_j},$$
$$a_j \star_\theta f_{mn} = \sqrt{\theta m_j}\, f_{m-u_j,n}, \qquad f_{mn}\star_\theta a_j = \sqrt{\theta(n_j+1)}\, f_{m,n+u_j}.$$

Consequently,

$$\begin{aligned}\not{D}(f_{mn}) = -\frac{i}{\theta}\sum_j \gamma^{a_j}\big(\sqrt{\theta n_j}\, f_{m,n-u_j} - \sqrt{\theta(m_j+1)}\, f_{m+u_j,n}\big)\\ + \gamma^{a_j^*}\big(\sqrt{\theta m_j}\, f_{m-u_j,n} - \sqrt{\theta(n_j+1)}\, f_{m,n+u_j}\big).\end{aligned} \tag{A.1}$$

We are now able to compute $\tilde{\pi}^\theta(\omega_{mnkl})$ and $\tilde{\pi}^\theta(\delta\omega_{mnkl})$. Firstly,

$$\begin{aligned}\tilde{\pi}^\theta(\omega_{mnkl}) &= \tilde{\pi}^\theta(f_{mk}\,\delta f_{kn} - f_{ml}\,\delta f_{ln}) = L^\theta(f_{mk}\star_\theta \partial_\mu f_{kn} - f_{ml}\star_\theta\partial_\mu f_{ln})\otimes\gamma^\mu\\ &= \frac{1}{\theta}\sum_j\big(\sqrt{\theta n_j}\, L^\theta(f_{mk}\star_\theta f_{k,n-u_j}) - \sqrt{\theta(k_j+1)}\, L^\theta(f_{mk}\star_\theta f_{k+u_j,n})\\ &\qquad - \sqrt{\theta n_j}\, L^\theta(f_{ml}\star_\theta f_{l,n-u_j}) + \sqrt{\theta(l_j+1)}\, L^\theta(f_{ml}\star_\theta f_{l+u_j,n})\big)\otimes\gamma^{a_j}\\ &\quad + \big(\sqrt{\theta k_j}\, L^\theta(f_{mk}\star_\theta f_{k-u_j,n}) - \sqrt{\theta(n_j+1)}\, L^\theta(f_{mk}\star_\theta f_{k,n+u_j})\\ &\qquad - \sqrt{\theta l_j}\, L^\theta(f_{ml}\star_\theta f_{l-u_j,n}) + \sqrt{\theta(n_j+1)}\, L^\theta(f_{ml}\star_\theta f_{l,n+u_j})\big)\otimes\gamma^{a_j^*} = 0.\end{aligned}$$

Secondly, we calculate that

$$\tilde{\pi}^\theta(\delta\omega_{mnkl}) = \tilde{\pi}^\theta(\delta f_{mk}\,\delta f_{kn} - \delta f_{ml}\,\delta f_{ln}) = L^\theta(\partial_\mu f_{mk}\star_\theta\partial_\nu f_{kn} - \partial_\mu f_{ml}\star_\theta\partial_\nu f_{ln})\otimes\gamma^\mu\gamma^\nu$$

equals

$$\begin{aligned}\frac{1}{\theta^2}\Big\{&\sum_j\Big(\big(\sqrt{\theta k_j}\, L^\theta(f_{m,k-u_j}) - \sqrt{\theta(m_j+1)}\, L^\theta(f_{m+u_j,k})\big)\otimes\gamma^{a_j}\\ &\qquad + \big(\sqrt{\theta m_j}\, L^\theta(f_{m-u_j,k}) - \sqrt{\theta(k_j+1)}\, L^\theta(f_{m,k+u_j})\big)\otimes\gamma^{a_j^*}\Big)\\ &\sum_p\Big(\big(\sqrt{\theta n_p}\, L^\theta(f_{k,n-u_p}) - \sqrt{\theta(k_p+1)}\, L^\theta(f_{k+u_p,n})\big)\otimes\gamma^{a_p}\\ &\qquad + \big(\sqrt{\theta k_p}\, L^\theta(f_{k-u_p,n}) - \sqrt{\theta(n_p+1)}\, L^\theta(f_{k,n+u_p})\big)\otimes\gamma^{a_p^*}\Big)\end{aligned}$$

$$
\begin{aligned}
&- \sum_j \Big(\big(\sqrt{\theta l_j}\, L^\theta(f_{m,l-u_j}) - \sqrt{\theta(m_j+1)}\, L^\theta(f_{m+u_j,l})\big) \otimes \gamma^{a_j} \\
&\qquad + \big(\sqrt{\theta m_j}\, L^\theta(f_{m-u_j,l}) - \sqrt{\theta(l_j+1)}\, L^\theta(f_{m,l+u_j})\big) \otimes \gamma^{a_j^*} \Big) \\
&\sum_p \Big(\big(\sqrt{\theta n_p}\, L^\theta(f_{l,n-u_p}) - \sqrt{\theta(l_p+1)}\, L^\theta(f_{l+u_p,n})\big) \otimes \gamma^{a_p} \\
&\qquad + \big(\sqrt{\theta l_p}\, L^\theta(f_{l-u_p,n}) - \sqrt{\theta(n_p+1)}\, L^\theta(f_{l,n+u_p})\big) \otimes \gamma^{a_p^*} \Big) \Big\}.
\end{aligned}
$$

Using the elementary properties of the f_{mn} from Lemma 2.3, this simplifies to

$$
\begin{aligned}
\tilde{\pi}^\theta(\delta\omega_{mnkl}) &= \frac{1}{\theta} \sum_j \big(k_j L^\theta(f_{mn}) \otimes \gamma^{a_j}\gamma^{a_j^*} + (k_j+1) L^\theta(f_{mn}) \otimes \gamma^{a_j^*}\gamma^{a_j} \\
&\qquad - l_j L^\theta(f_{mn}) \otimes \gamma^{a_j}\gamma^{a_j^*} - (l_j+1) L^\theta(f_{mn}) \otimes \gamma^{a_j^*}\gamma^{a_j}\big) \\
&= \frac{1}{\theta} L^\theta(f_{mn}) \otimes \sum_j (k_j - l_j)\,(\gamma^{a_j}\gamma^{a_j^*} + \gamma^{a_j^*}\gamma^{a_j}) \\
&= \frac{2}{\theta} \sum_j (k_j - l_j)\, L^\theta(f_{mn}) \otimes 1_{2^N}. \qquad \square
\end{aligned}
$$

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