The Dirac operator on $\mathrm{SU}_q(2)$

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Abstract

We construct a 3^+-summable spectral triple $(\mathcal{A}(\mathrm{SU}_q(2)), \mathcal{H}, D)$ over the quantum group $\mathrm{SU}_q(2)$ which is equivariant with respect to a left and a right action of $\mathcal{U}_q(\mathfrak{su}(2))$. The geometry is isospectral to the classical case since the spectrum of the operator D is the same as that of the usual Dirac operator on the 3-dimensional round sphere. The presence of an equivariant real structure J demands a modification in the axiomatic framework of spectral geometry, whereby the commutant and first-order properties need be satisfied only modulo infinitesimals of arbitrary high order.

1 Introduction

In this paper, we show how to successfully construct a (noncommutative) 3-dimensional spectral geometry on the manifold of the quantum group $\mathrm{SU}_q(2)$. This is done by building a 3^+-summable spectral triple $(\mathcal{A}(\mathrm{SU}_q(2)), \mathcal{H}, D)$ which is equivariant with respect to a left and a right action of $\mathcal{U}_q(\mathfrak{su}(2))$. The geometry is isospectral to the classical case in the sense that the spectrum of the operator D is the same as that of the usual Dirac operator on the 3-sphere $\mathbb{S}^3 \simeq \mathrm{SU}(2)$, with the "round" metric.

The possibility of such an isospectral deformation was suggested in [10] where the operator D was named the "true Dirac" operator. Subsequent investigations [13] seemed to rule out this deformation because some of the commutators $[D, x]$, with $x \in \mathcal{A}(\mathrm{SU}_q(2))$, failed to extend to bounded operators, a property which is essential to the definition of a spectral triple [7].

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These difficulties are overcome here by constructing on a Hilbert space of spinors $\mathcal{H}$ a spin representation of the algebra $\mathcal{A}(\mathrm{SU}_q(2))$ which differs slightly from the one used in [13]. Our spin representation is determined by requiring that it be equivariant with respect to a left and a right action of $\mathcal{U}_q(\mathfrak{su}(2))$, a condition which is not present in the previous approach. The role of Hopf-algebraic equivariance in producing interesting spectral triples has already met with some success [6, 12]; for a programmatic viewpoint, see [30].

Our construction of an isospectral noncommutative geometry on the manifold of $\mathrm{SU}_q(2)$, which deforms the usual geometry on the 3-dimensional sphere, belongs to an interesting terrain where noncommutative geometry meets the underlying "spaces" of quantum groups. Recent examples [11, 12, 26, 29] are concerned with the "two-dimensional" spheres of Podleś [27] and more general flag manifolds [22]. The left-equivariant spectral triple on $\mathrm{SU}_q(2)$ constructed in [6] and fully analyzed in [9] is not isospectral and does not have a good limit at the classical value of the deformation parameter.

► After a brief review in Section 2 of $\mathrm{SU}_q(2)$ and its symmetries, mainly to fix notation, we construct its left regular representation in Section 3 via equivariance, and transfer that construction to spinors in Section 4. On the Hilbert space of spinors, we consider in Section 5 a class of equivariant "Dirac" operators D. For such an operator D having a classical spectrum, that is, with eigenvalues depending linearly on "total angular momentum", we prove boundedness of the commutators $[D, x]$, for all $x \in \mathcal{A}(\mathrm{SU}_q(2))$. In fact, this equivariant Dirac operator is essentially determined by a modified first-order condition, as is shown later on.

Since the spectrum is classical, the deformation – from $\mathrm{SU}(2)$ to $\mathrm{SU}_q(2)$ – is isospectral, and in particular the metric dimension of the spectral geometry is 3.

The new feature of the spin geometry of $\mathrm{SU}_q(2)$ is the nature of the real structure J, whose existence is addressed in Section 6. An equivariant J is constructed by suitably lifting to the Hilbert space of spinors $\mathcal{H}$ the antiunitary Tomita conjugation operator for the left regular representation of $\mathcal{A}(\mathrm{SU}_q(2))$. However, this J is not the Tomita operator for the spin representation; for if it were, the spectral triple would inherit equivariance under the co-opposite symmetry algebra $\mathcal{U}_{1/q}(\mathfrak{su}(2))$, forcing it to be trivial. Therefore, the equivariant J we shall use does not intertwine the spin representation of $\mathcal{A}(\mathrm{SU}_q(2))$ with its commutant, and it is not possible to satisfy all the desirable properties of a real spectral triple as set forth in [8, 15]. This rupture was already observed in [11]; just as in that paper, we must also weaken the first-order requirement on D.

In Section 7, we rescue the formalism by showing that the commutant and first-order properties nevertheless do hold, up to infinitesimals of arbitrary high order. For that, we identify an ideal of trace-class operators containing all commutation defects; these defects vanish in the classical case. An appropriately modified first-order condition is given, which distinguishes Dirac operators with classical spectra.

A discussion of the Connes–Moscovici local index formula for the spectral geometry presented in this paper is currently under investigation and will be soon reported elsewhere.

2 Algebraic preliminaries

Definition 2.1. Let q be a real number with $0 < q < 1$, and let $\mathcal{A} = \mathcal{A}(\mathrm{SU}_q(2))$ be the $*$-algebra generated by a and b, subject to the following commutation rules:

$$\begin{gathered} ba = qab, \qquad b^*a = qab^*, \qquad bb^* = b^*b, \\ a^*a + q^2 b^*b = 1, \qquad aa^* + bb^* = 1. \end{gathered} \tag{2.1}$$

As a consequence, $a^*b = qba^*$ and $a^*b^* = qb^*a^*$. This becomes a Hopf $*$-algebra under the coproduct

$$\begin{aligned} \Delta a &:= a \otimes a - q\, b \otimes b^*, \\ \Delta b &:= b \otimes a^* + a \otimes b, \end{aligned}$$

counit $\varepsilon(a) = 1$, $\varepsilon(b) = 0$, and antipode $Sa = a^*$, $Sb = -qb$, $Sb^* = -q^{-1}b^*$, $Sa^* = a$.

Remark 2.2. Here we follow Majid's "lexicographic convention" [23, 24] (where, with $c = -qb^*$, $d = a^*$, a factor of q is needed to restore alphabetical order). Another much-used convention is related to ours by $a \leftrightarrow a^*$, $b \leftrightarrow -b$; see, for instance, [6, 9].

Definition 2.3. The Hopf $*$-algebra $\mathcal{U} = \mathcal{U}_q(\mathfrak{su}(2))$ is generated as an algebra by elements e, f, k, with k invertible, satisfying the relations

$$ek = qke, \qquad kf = qfk, \qquad k^2 - k^{-2} = (q - q^{-1})(fe - ef), \tag{2.2}$$

and its coproduct Δ is given by

$$\Delta k = k \otimes k, \qquad \Delta e = e \otimes k + k^{-1} \otimes e, \qquad \Delta f = f \otimes k + k^{-1} \otimes f.$$

Its counit ε, antipode S, and star structure * are given respectively by

$$\begin{aligned} &\varepsilon(k) = 1, && Sk = k^{-1}, && k^* = k, \\ &\varepsilon(f) = 0, && Sf = -qf, && f^* = e, \\ &\varepsilon(e) = 0, && Se = -q^{-1}e, && e^* = f. \end{aligned}$$

There is an automorphism ϑ of $\mathcal{U}_q(\mathfrak{su}(2))$ defined on the algebra generators by

$$\vartheta(k) := k^{-1}, \quad \vartheta(f) := -e, \quad \vartheta(e) := -f. \tag{2.3}$$

Remark 2.4. We recall that there is another convention for the generators of $\mathcal{U}_q(\mathfrak{su}(2))$ in widespread use: see [19], for instance. The handy compendium [21] gives both versions, denoting by $\breve{U}_q(su(2))$ the version which we adopt here. However, the parameter q of this paper corresponds to q^{-1} in [21], or alternatively, we keep the same q but exchange e and f of that book; the equivalence of these procedures is immediate from the above formulas (2.2).

The older literature uses the convention which we follow here, with generators usually written as $K = k$, $X^+ = f$, $X^- = e$.

We employ the so-called "q-integers", defined for each $n \in \mathbb{Z}$ as

$$[n] \equiv [n]_q := \frac{q^n - q^{-n}}{q - q^{-1}} \quad \text{provided} \quad q \neq 1. \tag{2.4}$$

Definition 2.5. There is a bilinear pairing between $\mathcal{U}$ and $\mathcal{A}$, defined on generators by

$$\langle k, a \rangle = q^{\frac{1}{2}}, \quad \langle k, a^* \rangle = q^{-\frac{1}{2}}, \quad \langle e, -qb^* \rangle = \langle f, b \rangle = 1,$$

with all other couples of generators pairing to 0. It satisfies

$$\langle (Sh)^*, x \rangle = \overline{\langle h, x^* \rangle}, \quad \text{for all} \quad h \in \mathcal{U},\ x \in \mathcal{A}. \tag{2.5}$$

We regard $\mathcal{U}$ as a subspace of the linear dual of $\mathcal{A}$ via this pairing. There are canonical left and right $\mathcal{U}$-module algebra structures on $\mathcal{A}$ [32] such that

$$\langle g, h \triangleright x \rangle := \langle gh, x \rangle, \quad \langle g, x \triangleleft h \rangle := \langle hg, x \rangle, \quad \text{for all} \quad g, h \in \mathcal{U},\ x \in \mathcal{A}.$$

They are given by $h \triangleright x := (\mathrm{id} \otimes h)\, \Delta x$ and $x \triangleleft h := (h \otimes \mathrm{id})\, \Delta x$, or equivalently by

$$h \triangleright x := x_{(1)} \langle h, x_{(2)} \rangle, \qquad x \triangleleft h := \langle h, x_{(1)} \rangle\, x_{(2)}, \tag{2.6}$$

using the Sweedler notation $\Delta x =: x_{(1)} \otimes x_{(2)}$ with implicit summation.

The right and left actions of $\mathcal{U}$ on $\mathcal{A}$ are mutually commuting:

$$(h \triangleright a) \triangleleft g = (a_{(1)} \langle h, a_{(2)} \rangle) \triangleleft g = \langle g, a_{(1)} \rangle\, a_{(2)} \langle h, a_{(3)} \rangle = h \triangleright (\langle g, a_{(1)} \rangle\, a_{(2)}) = h \triangleright (a \triangleleft g),$$

and it follows from (2.5) that the star structure is compatible with both actions:

$$h \triangleright x^* = ((Sh)^* \triangleright x)^*, \quad x^* \triangleleft h = (x \triangleleft (Sh)^*)^*, \quad \text{for all} \quad h \in \mathcal{U},\ x \in \mathcal{A}.$$

On the generators, the left action is given explicitly by

$$\begin{aligned}
&k \triangleright a = q^{\frac{1}{2}} a, && k \triangleright a^* = q^{-\frac{1}{2}} a^*, && k \triangleright b = q^{-\frac{1}{2}} b, && k \triangleright b^* = q^{\frac{1}{2}} b^*, \\
&f \triangleright a = 0, && f \triangleright a^* = -qb^*, && f \triangleright b = a, && f \triangleright b^* = 0, \\
&e \triangleright a = b, && e \triangleright a^* = 0, && e \triangleright b = 0, && e \triangleright b^* = -q^{-1} a^*,
\end{aligned} \tag{2.7}$$

and the right action is likewise given by

$$\begin{aligned}
&a \triangleleft k = q^{\frac{1}{2}} a, && a^* \triangleleft k = q^{-\frac{1}{2}} a^*, && b \triangleleft k = q^{\frac{1}{2}} b, && b^* \triangleleft k = q^{-\frac{1}{2}} b^*, \\
&a \triangleleft f = -qb^*, && a^* \triangleleft f = 0, && b \triangleleft f = a^*, && b^* \triangleleft f = 0, \\
&a \triangleleft e = 0, && a^* \triangleleft e = b, && b \triangleleft e = 0, && b^* \triangleleft e = -q^{-1} a.
\end{aligned} \tag{2.8}$$

We remark in passing that since $\mathcal{A}$ is also a Hopf algebra, the left and right actions are linked through the antipodes:

$$S(Sh \triangleright x) = Sx \triangleleft h.$$

Indeed, it is immediate from (2.6) and the duality relation $\langle Sh, y\rangle = \langle h, Sy\rangle$ that

$$S(Sh \triangleright x) = S(x_{(1)})\,\langle Sh, x_{(2)}\rangle = S(x_{(1)})\,\langle h, S(x_{(2)})\rangle = (Sx)_{(2)}\,\langle h, (Sx)_{(1)}\rangle = Sx \triangleleft h.$$

As noted in [14], for instance, the invertible antipode of $\mathcal{U}$ serves to transform the right action $\triangleleft$ into a second left action of $\mathcal{U}$ on $\mathcal{A}$, commuting with the first. Here we also use the automorphism ϑ of (2.3), and define

$$h \cdot x := x \triangleleft S^{-1}(\vartheta(h)).$$

Indeed, it is immediate that

$$g \cdot (h \cdot x) = (x \triangleleft S^{-1}(\vartheta h)) \triangleleft S^{-1}(\vartheta g) = x \triangleleft (S^{-1}(\vartheta h)S^{-1}(\vartheta g)) = x \triangleleft (S^{-1}(\vartheta(gh)) = gh \cdot x,$$

i.e., it is a left action. We tabulate this action directly from (2.8):

$$\begin{aligned}
&k \cdot a = q^{\frac{1}{2}} a, && k \cdot a^* = q^{-\frac{1}{2}} a^*, && k \cdot b = q^{\frac{1}{2}} b, && k \cdot b^* = q^{-\frac{1}{2}} b^*,\\
&f \cdot a = 0, && f \cdot a^* = qb, && f \cdot b = 0, && f \cdot b^* = -a,\\
&e \cdot a = -b^*, && e \cdot a^* = 0, && e \cdot b = q^{-1} a^*, && e \cdot b^* = 0.
\end{aligned} \tag{2.9}$$

In the "classical" case $q = 1$, we use the well-known identifications

$$\mathrm{SU}(2) \approx \mathbb{S}^3 \approx \mathrm{Spin}(4)/\mathrm{Spin}(3) = (\mathrm{SU}(2) \times \mathrm{SU}(2))/\mathrm{SU}(2);$$

on quotienting out the diagonal $\mathrm{SU}(2)$ subgroup of $\mathrm{Spin}(4)$, we realize $\mathrm{SU}(2)$ as the base space of the principal spin bundle $\mathrm{Spin}(4) \to \mathbb{S}^3$, with projection map $(g, h) \mapsto gh^{-1}$. The action of $\mathrm{Spin}(4)$ on $\mathrm{SU}(2)$ is given by $(g, h) \cdot x := gxh^{-1}$, and the stabilizer of 1 is the diagonal $\mathrm{SU}(2)$ subgroup. We may choose to regard this as a pair of commuting actions of $\mathrm{SU}(2)$ on the base space $\mathrm{SU}(2)$, apart from the nuance of switching one of them from a right to a left action via the group inversion map. The foregoing pair of actions of $\mathcal{U}_q(\mathfrak{su}(2))$ on $\mathcal{A}(\mathrm{SU}_q(2))$ extends this scheme to the case $q \neq 1$.

► We recall [21] that $\mathcal{A}$ has a vector-space basis consisting of matrix elements of its irreducible corepresentations, $\{\, t^l_{mn} : 2l \in \mathbb{N},\ m, n = -l, \dots, l-1, l \,\}$, where

$$t^0_{00} = 1, \qquad t^{\frac{1}{2}}_{\frac{1}{2},\frac{1}{2}} = a, \qquad t^{\frac{1}{2}}_{\frac{1}{2},-\frac{1}{2}} = b.$$

The coproduct has the matricial form $\Delta t^l_{mn} = \sum_k t^l_{mk} \otimes t^l_{kn}$, while the product is given by

$$t^j_{rs} t^l_{mn} = \sum_{k=|j-l|}^{j+l} C_q\begin{pmatrix} j & l & k \\ r & m & r+m \end{pmatrix} C_q\begin{pmatrix} j & l & k \\ s & n & s+n \end{pmatrix} t^k_{r+m,s+n}, \tag{2.10}$$

where the $C_q(-)$ factors are q-Clebsch–Gordan coefficients [3, 20].

The Haar state on the C^*-completion $C(\mathrm{SU}_q(2))$, which we shall denote by ψ, is faithful, and it is determined by setting $\psi(1) := 1$ and $\psi(t^l_{mn}) := 0$ if $l > 0$. (The Haar state is usually denoted by h, but here we use h for a generic element of $\mathcal{U}$ instead.) Let $\mathcal{H}_\psi = L^2(\mathrm{SU}_q(2), \psi)$ be the Hilbert space of its GNS representation; then the GNS map $\eta\colon C(\mathrm{SU}_q(2)) \to \mathcal{H}_\psi$ is injective and satisfies

$$\|\eta(t^l_{mn})\|^2 = \psi((t^l_{mn})^*\, t^l_{mn}) = \frac{q^{-2m}}{[2l+1]}, \tag{2.11}$$

and the vectors $\eta(t^l_{mn})$ are mutually orthogonal. From the formula

$$C_q\begin{pmatrix} l & l & 0 \\ -m & m & 0 \end{pmatrix} = (-1)^{l+m}\frac{q^{-m}}{[2l+1]^{\frac{1}{2}}},$$

we see that the involution in $C(\mathrm{SU}_q(2))$ is given by

$$(t^l_{mn})^* = (-1)^{2l+m+n} q^{n-m}\, t^l_{-m,-n}. \tag{2.12}$$

In particular, $t^{\frac{1}{2}}_{-\frac{1}{2},\frac{1}{2}} = -qb^*$ and $t^{\frac{1}{2}}_{-\frac{1}{2},-\frac{1}{2}} = a^*$, as expected.

An orthonormal basis of $\mathcal{H}_\psi$ is obtained by normalizing the matrix elements, using (2.11):

$$|lmn\rangle := q^m\,[2l+1]^{\frac{1}{2}}\,\eta(t^l_{mn}). \tag{2.13}$$

3 Equivariant representation of $\mathcal{A}(\mathrm{SU}_q(2))$

Let $\mathcal{U}$ be a Hopf algebra and let $\mathcal{A}$ be a left $\mathcal{U}$-module algebra. A representation of $\mathcal{A}$ on a vector space V is called $\mathcal{U}$-equivariant if there is also an algebra representation of $\mathcal{U}$ on V, satisfying the following compatibility relation:

$$h(x\xi) = (h_{(1)} \triangleright x)(h_{(2)}\xi), \qquad h \in \mathcal{U},\ x \in \mathcal{A},\ \xi \in V,$$

where $\triangleright$ denotes the Hopf action of $\mathcal{U}$ on $\mathcal{A}$. If $\mathcal{A}$ is instead a right $\mathcal{U}$-module algebra, the appropriate compatibility relation is $x(h\xi) = h_{(1)}((x \triangleleft h_{(2)})\xi)$. Also, if $\mathcal{A}$ is an $\mathcal{U}$-bimodule algebra (carrying commuting left and right Hopf actions of $\mathcal{U}$), one can demand both of these conditions simultaneously for pair of representations of $\mathcal{A}$ and $\mathcal{U}$ on the same vector space V.

In the present case, it turns out to be simpler to consider equivariance under two commuting left Hopf actions, as exemplified in the previous section. We shall first work out in detail a construction of the regular representation of the Hopf algebra $\mathcal{A}(\mathrm{SU}_q(2))$, showing how it is determined by its equivariance properties.

► We begin with the known representation theory [21] of $\mathcal{U}_q(\mathfrak{su}(2))$. The irreducible finite dimensional representations σ_l of $\mathcal{U}_q(\mathfrak{su}(2))$ are labelled by nonnegative half-integers $l = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \dots,$ and they are given by

$$\begin{aligned} \sigma_l(k)\,|lm\rangle &= q^m\,|lm\rangle, \\ \sigma_l(f)\,|lm\rangle &= \sqrt{[l-m][l+m+1]}\,|l,m+1\rangle, \\ \sigma_l(e)\,|lm\rangle &= \sqrt{[l-m+1][l+m]}\,|l,m-1\rangle, \end{aligned} \tag{3.1}$$

where the vectors $|lm\rangle$, for $m = -l, -l+1, \dots, l-1, l$, form a basis for the irreducible $\mathcal{U}$-module V_l, and the brackets denote q-integers as in (2.4). Moreover, σ_l is a $*$-representation of $\mathcal{U}_q(\mathfrak{su}(2))$, with respect to the hermitian scalar product on V_l for which the vectors $|lm\rangle$ are orthonormal.

Remark 3.1. The irreducible representations (3.1) coincide with those of $\breve{U}_q(su(2))$ in [21], after exchange of e and f (see Remark 2.4). Further results on the representation theory of $\mathcal{U}_q(\mathfrak{su}(2))$ are taken from [21, Chap. 3] without comment; in particular we use the q-Clebsch–Gordan coefficients found therein for the decomposition of tensor product representations. An alternative source for these coefficients is [3], although their $q^{\frac{1}{2}}$ is our q.

Definition 3.2. Let λ and ρ be mutually commuting representations of the Hopf algebra $\mathcal{U}$ on a vector space V. A representation π of the $*$-algebra $\mathcal{A}$ on V is (λ, ρ)-*equivariant* if the following compatibility relations hold:

$$\begin{aligned} \lambda(h)\,\pi(x)\xi &= \pi(h_{(1)} \cdot x)\,\lambda(h_{(2)})\xi, \\ \rho(h)\,\pi(x)\xi &= \pi(h_{(1)} \triangleright x)\,\rho(h_{(2)})\xi, \end{aligned} \tag{3.2}$$

for all $h \in \mathcal{U}$, $x \in \mathcal{A}$ and $\xi \in V$.

► We shall now exhibit an equivariant representation of $\mathcal{A}(\mathrm{SU}_q(2))$ on the prehilbert space which is the (algebraic) direct sum

$$V := \bigoplus_{2l=0}^{\infty} V_l \otimes V_l.$$

The two $\mathcal{U}_q(\mathfrak{su}(2))$ symmetries λ and ρ will act on the first and the second leg of the tensor product respectively; both actions will be via the irreps (3.1). In other words,

$$\lambda(h) = \sigma_l(h) \otimes \mathrm{id}, \qquad \rho(h) = \mathrm{id} \otimes \sigma_l(h) \qquad \text{on} \quad V_l \otimes V_l.$$

We abbreviate $|lmn\rangle := |lm\rangle \otimes |ln\rangle$, for $m, n = -l, \dots, l-1, l$; these form an orthonormal basis for $V_l \otimes V_l$, for each fixed l. (As we shall see, this is consistent with our labelling (2.13) of the orthonormal basis of $\mathcal{H}_\psi$ in the previous section.) Also, we adopt a shorthand notation:

$$l^\pm := l \pm \tfrac{1}{2}, \quad m^\pm := m \pm \tfrac{1}{2}, \quad n^\pm := n \pm \tfrac{1}{2}.$$

Proposition 3.3. *A (λ, ρ)-equivariant $*$-representation π of $\mathcal{A}(\mathrm{SU}_q(2))$ on the Hilbert space V of* (3.3) *must have the following form:*

$$\begin{aligned} \pi(a)\,|lmn\rangle &= A^+_{lmn}|l^+m^+n^+\rangle + A^-_{lmn}|l^-m^+n^+\rangle, \\ \pi(b)\,|lmn\rangle &= B^+_{lmn}|l^+m^+n^-\rangle + B^-_{lmn}|l^-m^+n^-\rangle, \\ \pi(a^*)\,|lmn\rangle &= \widetilde{A}^+_{lmn}|l^+m^-n^-\rangle + \widetilde{A}^-_{lmn}|l^-m^-n^-\rangle, \\ \pi(b^*)\,|lmn\rangle &= \widetilde{B}^+_{lmn}|l^+m^-n^+\rangle + \widetilde{B}^-_{lmn}|l^-m^-n^+\rangle, \end{aligned} \tag{3.3}$$

where the constants $A^\pm_{lmn}$ and $B^\pm_{lmn}$ are, up to phase factors depending only on l, given by

$$\begin{aligned} A^+_{lmn} &= q^{(-2l+m+n-1)/2}\left(\frac{[l+m+1]\,[l+n+1]}{[2l+1]\,[2l+2]}\right)^{\frac{1}{2}}, \\ A^-_{lmn} &= q^{(2l+m+n+1)/2}\left(\frac{[l-m]\,[l-n]}{[2l]\,[2l+1]}\right)^{\frac{1}{2}}, \\ B^+_{lmn} &= q^{(m+n-1)/2}\left(\frac{[l+m+1]\,[l-n+1]}{[2l+1]\,[2l+2]}\right)^{\frac{1}{2}}, \\ B^-_{lmn} &= -q^{(m+n-1)/2}\left(\frac{[l-m]\,[l+n]}{[2l]\,[2l+1]}\right)^{\frac{1}{2}}, \end{aligned} \tag{3.4}$$

and the other coefficients are complex conjugates of these, namely,

$$\widetilde{A}^\pm_{lmn} = (A^\mp_{l^\pm m^- n^-})^\star, \qquad \widetilde{B}^\pm_{lmn} = (B^\mp_{l^\pm m^- n^+})^\star. \tag{3.5}$$

Proof. First of all, notice that hermiticity of π entails the relations (3.5). We now use the covariance properties (3.2). When $h = k$, they simplify to

$$\lambda(k)\,\pi(x)\,\xi = \pi(k\cdot x)\,\lambda(k)\,\xi, \qquad \rho(k)\,\pi(x)\,\xi = \pi(k \triangleright x)\,\rho(k)\,\xi. \tag{3.6}$$

Thus, for instance, when $x = a$ we find the relations

$$\begin{aligned}\lambda(k)\,\pi(a)\,|lmn\rangle &= \pi(q^{\frac12}a)\big(q^m|lmn\rangle\big) = q^{m+\frac12}\pi(a)\,|lmn\rangle,\\ \rho(k)\,\pi(a)\,|lmn\rangle &= \pi(q^{\frac12}a)\big(q^n|lmn\rangle\big) = q^{n+\frac12}\pi(a)\,|lmn\rangle,\end{aligned}$$

where we have invoked $k\cdot a = k\triangleright a = q^{\frac12}a$. We conclude that $\pi(a)\,|lmn\rangle$ must lie in the closed span of the basis vectors $|l'm^+n^+\rangle$. A similar argument with $x = b$ in (3.6) shows that $\pi(b)$ increments n and decrements m by $\frac12$, since $k\cdot b = q^{\frac12}b$ while $k\triangleright b = q^{-\frac12}b$. The analogous behaviour for $x = a^*$ and $x = b^*$ follows in the same way from (2.7) and (2.9).

Thus, $\pi(a)|lmn\rangle$ is a (possibly infinite) sum

$$\pi(a)\,|lmn\rangle = \textstyle\sum_{l'} C_{l'lmn}\,|l'm^+n^+\rangle, \tag{3.7}$$

where the sum runs over nonnegative half-integers $l' = 0, \frac12, 1, \frac32, \dots$.

Next, we call on (3.2) with $h = f$, $x = a$, to get

$$\lambda(f)\,\pi(a)\xi = \pi(f\cdot a)\,\lambda(k)\xi + \pi(k^{-1}\cdot a)\,\lambda(f)\xi = q^{-\frac12}\pi(a)\,\lambda(f)\xi,$$

on account of (2.7). Consequently, $\lambda(f)^r\pi(a) = q^{-r/2}\pi(a)\,\lambda(f)^r$ for $r = 1,2,3,\dots$. On applying $\lambda(f)^r$ to both sides of (3.7), we obtain on the left hand side a multiple of $\pi(a)\,|l, m+r, n\rangle$, which vanishes for $m + r > l$; and on the right hand side we get $\sum_{l'} C_{l'lmn}\,D_{l'mr}\,|l', m^+ + r, n^+\rangle$, where $D_{l'mr} \neq 0$ as long as $m+r+\frac12 \leqslant l'$. We conclude that $C_{l'lmn} = 0$ for $l' > l+\frac12$, by linear independence of these summands.

To get a lower bound on the range of the index l' in (3.7), we consider the analogous expansion $\pi(a^*)\,|lmn\rangle = \sum_{l'} \widetilde{C}_{l'lmn}\,|l'm^-n^-\rangle$. Now $\lambda(e)^r\pi(a^*)\,|lmn\rangle = q^{r/2}\pi(a^*)\,\lambda(e)^r\,|lmn\rangle \propto \pi(a^*)\,|l, m-r, n\rangle$ vanishes for $m - r < -l$; while $\lambda(e)^r|l'm^-n^-\rangle = F_{l'mr}\,|l', m^- - r, n^-\rangle$ with $F_{l'mr} \neq 0$ for $m - r - \frac12 \geqslant -l'$. Again we conclude that $\widetilde{C}_{l'lmn} = 0$ for $l' > l + \frac12$. However, since π is a $*$-representation, the matrix element $\langle l'm'n' \mid \pi(a) \mid lmn\rangle$ is the complex conjugate of $\langle lmn|\pi(a^*)|l'm'n'\rangle$, which vanishes for $l > l'+\frac12$, so that the indices in (3.7) satisfy $l-\frac12 \leqslant l' \leqslant l+\frac12$. Clearly, $l' = l$ is ruled out because $l - m$ and $l' - m \pm \frac12$ must both be integers.

Therefore, $\pi(a)$ and also $\pi(a^*)$ have the structure indicated in (3.3). A parallel argument shows the corresponding result for $\pi(b)$ and $\pi(b^*)$.

The coefficients which appear in (3.4) may be determined by further application of the equivariance relations. Since $f\triangleright a = 0$ and $e\triangleright b = 0$, then by applying $\rho(f)$ and $\rho(e)$ to the first two relations of (3.3), we obtain the following recursion relations for the coefficients $A^\pm_{lmn}$, $B^\pm_{lmn}$:

$$\begin{aligned}A^+_{lmn}[l+n+2]^{\frac12} &= q^{-\frac12}A^+_{lm,n+1}[l+n+1]^{\frac12},\\ A^-_{lmn}[l-n-1]^{\frac12} &= q^{-\frac12}A^-_{lm,n+1}[l-n]^{\frac12},\\ B^+_{lmn}[l-n+2]^{\frac12} &= q^{\frac12}B^+_{lm,n-1}[l-n+1]^{\frac12},\\ B^-_{lmn}[l+n-1]^{\frac12} &= q^{\frac12}B^-_{lm,n-1}[l+n]^{\frac12}.\end{aligned}$$

Then, applying $\lambda(f)$ to the same pair of equations, we further find that

$$\begin{aligned}
A^+_{lmn}[l+m+2]^{\frac12} &= q^{-\frac12} A^+_{l,m+1,n}[l+m+1]^{\frac12}, \\
A^-_{lmn}[l-m-1]^{\frac12} &= q^{-\frac12} A^-_{l,m+1,n}[l-m]^{\frac12}, \\
B^+_{lmn}[l+m+2]^{\frac12} &= q^{-\frac12} B^+_{l,m+1,n}[l+m+1]^{\frac12}, \\
B^-_{lmn}[l-m-1]^{\frac12} &= q^{-\frac12} B^-_{l,m+1,n}[l-m]^{\frac12}.
\end{aligned} \tag{3.8a}$$

These recursions are explicitly solved by

$$\begin{aligned}
A^+_{lmn} &= q^{(m+n)/2}[l+m+1]^{\frac12}[l+n+1]^{\frac12}\, a^+_l, \\
A^-_{lmn} &= q^{(m+n)/2}[l-m]^{\frac12}[l-n]^{\frac12}\, a^-_l, \\
B^+_{lmn} &= q^{(m+n)/2}[l+m+1]^{\frac12}[l-n+1]^{\frac12}\, b^+_l, \\
B^-_{lmn} &= q^{(m+n)/2}[l-m]^{\frac12}[l+n]^{\frac12}\, b^-_l,
\end{aligned} \tag{3.8b}$$

where $a^\pm_l$, $b^\pm_l$ depend only on l.

Once more, we apply the equivariance relations (3.2); this time, we use

$$\rho(e)\pi(a) = \pi(e \triangleright a)\rho(k) + \pi(k^{-1} \triangleright a)\rho(e) = \pi(b)\rho(k) + q^{-\frac12}\pi(a)\rho(e). \tag{3.9}$$

Applied to $|lmn\rangle$, it yields an equation between linear combinations of $|l^+m^+n^-\rangle$ and $|l^-m^+n^-\rangle$; equating coefficients, we find

$$b^+_l = q^l a^+_l, \qquad b^-_l = -q^{-l-1} a^-_l.$$

Furthermore, applying also to $|lmn\rangle$ the relation

$$\begin{aligned}
\lambda(e)\pi(b) &= \pi(e \cdot b)\lambda(k) + \pi(k^{-1} \cdot b)\lambda(e) \\
&= q^{-1}\pi(a^*)\lambda(k) + q^{-\frac12}\pi(b)\lambda(e),
\end{aligned} \tag{3.10}$$

we get, after a little simplification and use of (3.5),

$$(a^-_{l+\frac12})^\star = q^{2l+\frac32}\, a^+_l.$$

It remains only to determine the parameters a^+_l. We turn to the algebra commutation relation $ba = qab$ and compare coefficients in the expansion of $\pi(b)\pi(a)\,|lmn\rangle = q\,\pi(a)\pi(b)\,|lmn\rangle$. Those of $|l+1,m+1,n\rangle$ and $|l-1,m+1,n\rangle$ already coincide; but from the $|l,m+1,n\rangle$ terms, we get the identity

$$q[2l+2]\,|a^+_l|^2 = [2l]\,|a^+_{l-\frac12}|^2.$$

This can be solved immediately, to give

$$a^+_l = \frac{C\zeta_l\, q^{-l}}{[2l+1]^{\frac12}[2l+2]^{\frac12}},$$

where C is a positive constant, and ζ_l is a phase factor which can be absorbed in the basis vectors $|lmn\rangle$; hereinafter we take $\zeta_l = 1$ (we comment on that choice at the end of the section).

Finally, from the relation $a^*a + q^2 b^*b = 1$ we obtain

$$1 = \langle 000 \mid \pi(a^*a + q^2 b^*b) \mid 000\rangle = |a_0^+|^2 + q^2|b_0^+|^2 = (1+q^2)C^2/[2] = q\,C^2,$$

and thus $C = q^{-\frac{1}{2}}$. We therefore find that

$$a_l^+ = \frac{q^{-l-\frac{1}{2}}}{[2l+1]^{\frac{1}{2}}[2l+2]^{\frac{1}{2}}}, \qquad a_l^- = \frac{q^{l+\frac{1}{2}}}{[2l]^{\frac{1}{2}}[2l+1]^{\frac{1}{2}}},$$
$$b_l^+ = \frac{q^{-\frac{1}{2}}}{[2l+1]^{\frac{1}{2}}[2l+2]^{\frac{1}{2}}}, \qquad b_l^- = -\frac{q^{-\frac{1}{2}}}{[2l]^{\frac{1}{2}}[2l+1]^{\frac{1}{2}}},$$

and substitution in (3.8b) yields the coefficients (3.4). □

It is easy to check that the formulas (3.3) give precisely the left regular representation π_ψ of $\mathcal{A}(\mathrm{SU}_q(2))$. Indeed, that representation was implicitly given already by the product rule (2.10). From [3, Eq. (3.53)] we obtain

$$\begin{aligned}
C_q\begin{pmatrix} \frac{1}{2} & l & l^+ \\ \frac{1}{2} & m & m^+ \end{pmatrix} &= q^{-\frac{1}{2}(l-m)}\,\frac{[l+m+1]^{\frac{1}{2}}}{[2l+1]^{\frac{1}{2}}}, \\
C_q\begin{pmatrix} \frac{1}{2} & l & l^+ \\ -\frac{1}{2} & m & m^- \end{pmatrix} &= q^{\frac{1}{2}(l+m)}\,\frac{[l-m+1]^{\frac{1}{2}}}{[2l+1]^{\frac{1}{2}}}, \\
C_q\begin{pmatrix} \frac{1}{2} & l & l^- \\ \frac{1}{2} & m & m^+ \end{pmatrix} &= q^{\frac{1}{2}(l+m+1)}\,\frac{[l-m]^{\frac{1}{2}}}{[2l+1]^{\frac{1}{2}}}, \\
C_q\begin{pmatrix} \frac{1}{2} & l & l^- \\ -\frac{1}{2} & m & m^- \end{pmatrix} &= -q^{-\frac{1}{2}(l-m+1)}\,\frac{[l+m]^{\frac{1}{2}}}{[2l+1]^{\frac{1}{2}}}.
\end{aligned} \tag{3.11}$$

By setting $j = r = s = \frac{1}{2}$ in (2.10), we find

$$\pi_\psi(a)\eta(t^l_{mn}) = \sum_\pm C_q\begin{pmatrix} \frac{1}{2} & l & l^\pm \\ \frac{1}{2} & m & m^+ \end{pmatrix} C_q\begin{pmatrix} \frac{1}{2} & l & l^\pm \\ \frac{1}{2} & n & n^+ \end{pmatrix} \eta(t^{l^\pm}_{m^+n^+}).$$

Taking the normalization (2.13) into account, this becomes

$$\begin{aligned}
\pi_\psi(a)|lmn\rangle &= q^{-\frac{1}{2}}\frac{[2l+1]^{\frac{1}{2}}}{[2l+2]^{\frac{1}{2}}} C_q\begin{pmatrix} \frac{1}{2} & l & l^+ \\ \frac{1}{2} & m & m^+ \end{pmatrix} C_q\begin{pmatrix} \frac{1}{2} & l & l^+ \\ \frac{1}{2} & n & n^+ \end{pmatrix} |l^+m^+n^+\rangle \\
&\quad + q^{-\frac{1}{2}}\frac{[2l+1]^{\frac{1}{2}}}{[2l]^{\frac{1}{2}}} C_q\begin{pmatrix} \frac{1}{2} & l & l^- \\ \frac{1}{2} & m & m^+ \end{pmatrix} C_q\begin{pmatrix} \frac{1}{2} & l & l^- \\ \frac{1}{2} & n & n^+ \end{pmatrix} |l^-m^+n^+\rangle \\
&= q^{\frac{1}{2}(-2l+m+n-1)}\,\frac{[l+m+1]^{\frac{1}{2}}[l+n+1]^{\frac{1}{2}}}{[2l+1]^{\frac{1}{2}}[2l+2]^{\frac{1}{2}}}\,|l^+m^+n^+\rangle \\
&\quad + q^{\frac{1}{2}(2l+m+n+1)}\,\frac{[l-m]^{\frac{1}{2}}[l-n]^{\frac{1}{2}}}{[2l]^{\frac{1}{2}}[2l+1]^{\frac{1}{2}}}\,|l^-m^+n^+\rangle \\
&= \pi(a)|lmn\rangle.
\end{aligned}$$

A similar calculation, using (3.11) again, shows that $\pi(b) = \pi_\psi(b)$. Since a and b generate $\mathcal{A}$ as a $*$-algebra, we conclude that $\pi = \pi_\psi$. (It should be noted that π_ψ has already been exhibited in [6] in the same way, albeit with a different convention for the algebra generators.)

The identification (2.13) embeds the prehilbert space V densely in the Hilbert space $\mathcal{H}_\psi$, and the representation π_ψ extends to the GNS representation of $C(\mathrm{SU}_q(2))$ on $\mathcal{H}_\psi$, as described by the Peter–Weyl theorem [21, 32]. In like manner, all other representations of $\mathcal{A}$ exhibited in this paper extend to C^*-algebra representations of $C(\mathrm{SU}_q(2))$ on the appropriate Hilbert spaces.

► The only lack of uniqueness in the proof of Proposition 3.3 involved the choice of the phase factors ζ_l; if Z is the linear operator on V which multiplies vectors in $V_l \otimes V_l$ by ζ_l, then Z commutes with each $\lambda(h)$ and $\rho(g)$, and extends to a unitary operator on $\mathcal{H}_\psi$. In other words, any (λ, ρ)-equivariant representation π extends to $\mathcal{H}_\psi$ and is unitarily equivalent to the left regular representation. The (standard) choice $\zeta_l = 1$ ensures that *all coefficients $A^\pm_{lmn}$ and $B^\pm_{lmn}$ are real*: it is indeed an extension of the Conden–Shortley phase convention [4].

4 The spin representation

The left regular representation π of $\mathcal{A}$, constructed in the previous section, can be amplified to $\pi' = \pi \otimes \mathrm{id}$ on $V \otimes \mathbb{C}^2$. In the commutative case when $q = 1$, this yields the spinor representation of $\mathrm{SU}(2)$, because the spinor bundle is parallelizable: $S \simeq \mathrm{SU}(2) \times \mathbb{C}^2$, although one needs to specify the trivialization. The representation theory of $\mathcal{U}$ (and the corepresentation theory of $\mathcal{A}$) follows the same pattern; only the Clebsch–Gordan coefficients need to be modified [20] when $q \neq 1$.

To fix notations, we take

$$W := V \otimes \mathbb{C}^2 = V \otimes V_{\frac12},$$

and its Clebsch–Gordan decomposition is the (algebraic) direct sum

$$W = \left(\bigoplus_{2l=0}^{\infty} V_l \otimes V_l \right) \otimes V_{\frac12} \simeq V_{\frac12} \oplus \bigoplus_{2j=1}^{\infty} (V_{j+\frac12} \otimes V_j) \oplus (V_{j-\frac12} \otimes V_j). \tag{4.1}$$

We rename the finite-dimensional spaces on the right hand side as

$$W = W_0^\uparrow \oplus \bigoplus_{2j\geqslant 1} W_j^\uparrow \oplus W_j^\downarrow, \tag{4.2}$$

where $W_j^\uparrow \simeq V_{j+\frac12} \otimes V_j$ and $W_j^\downarrow \simeq V_{j-\frac12} \otimes V_j$, so that

$$\begin{aligned} \dim W_j^\uparrow &= (2j+1)(2j+2), \quad &\text{for} \quad j = 0, \tfrac12, 1, \tfrac32, \dots, \\ \dim W_j^\downarrow &= 2j(2j+1), \quad &\text{for} \quad j = \tfrac12, 1, \tfrac32, \dots. \end{aligned} \tag{4.3}$$

Definition 4.1. We amplify the representation ρ of $\mathcal{U}$ on V to $\rho' = \rho \otimes \mathrm{id}$ on $W = V \otimes \mathbb{C}^2$. However, we replace λ on V by its tensor product with $\sigma_{\frac12}$ on $\mathbb{C}^2$:

$$\lambda'(h) := (\lambda \otimes \sigma_{\frac12})(\Delta h) = \lambda(h_{(1)}) \otimes \sigma_{\frac12}(h_{(2)}).$$

It is straightforward to check that the representations λ' and ρ' on W commute, and that the representation π' of $\mathcal{A}$ on W is (λ', ρ')-equivariant:

$$
\begin{aligned}
\lambda'(h)\,\pi'(x)\psi &= \pi'(h_{(1)} \cdot x)\,\lambda'(h_{(2)})\psi, \\
\rho'(h)\,\pi'(x)\psi &= \pi'(h_{(1)} \triangleright x)\,\rho'(h_{(2)})\psi,
\end{aligned}
\tag{4.4}
$$

for all $h \in \mathcal{U}$, $x \in \mathcal{A}$ and $\psi \in W$.

To determine an explicit basis for W which is well-adapted to (λ', ρ')-equivariance, consider the following vectors in $V \otimes \mathbb{C}^2$:

$$
\begin{aligned}
c_{lm}\,|lmn\rangle \otimes |\tfrac12, -\tfrac12\rangle + s_{lm}\,|l, m-1, n\rangle \otimes |\tfrac12, +\tfrac12\rangle, \\
-s_{lm}\,|lmn\rangle \otimes |\tfrac12, -\tfrac12\rangle + c_{lm}\,|l, m-1, n\rangle \otimes |\tfrac12, +\tfrac12\rangle,
\end{aligned}
$$

where

$$
c_{lm} := q^{-(l+m)/2}\,\frac{[l-m+1]^{\frac12}}{[2l+1]^{\frac12}}, \qquad s_{lm} := q^{(l-m+1)/2}\,\frac{[l+m]^{\frac12}}{[2l+1]^{\frac12}}
$$

are the q-Clebsch–Gordan coefficients corresponding to the above decomposition (4.1), satisfying $c_{lm}^2 + s_{lm}^2 = 1$. These are eigenvectors for $\lambda'(C_q)$, where $C_q := qk^2 + q^{-1}k^{-2} + (q-q^{-1})^2 ef$ is the Casimir element of $\mathcal{U}$, with respective eigenvalues $q^{2l+2} + q^{-2l-2}$ and $q^{2l} + q^{-2l}$. Thus, to get a good basis, one should offset the index l by $\pm\frac12$ (as is also suggested by the decomposition (4.2) of W).

For $j = l + \frac12$, $\mu = m - \frac12$, with $\mu = -j, \dots, j$ and $n = -j^-, \dots, j^-$, let

$$
|j\mu n{\downarrow}\rangle := C_{j\mu}\,|j^-\mu^+n\rangle \otimes |\tfrac12, -\tfrac12\rangle + S_{j\mu}\,|j^-\mu^-n\rangle \otimes |\tfrac12, +\tfrac12\rangle;
\tag{4.5a}
$$

and for $j = l - \frac12$, $\mu = m - \frac12$, with $\mu = -j, \dots, j$ and $n = -j^+, \dots, j^+$, let

$$
|j\mu n{\uparrow}\rangle := -S_{j+1,\mu}\,|j^+\mu^+n\rangle \otimes |\tfrac12, -\tfrac12\rangle + C_{j+1,\mu}\,|j^+\mu^-n\rangle \otimes |\tfrac12, +\tfrac12\rangle,
\tag{4.5b}
$$

where the coefficients are now

$$
C_{j\mu} := q^{-(j+\mu)/2}\,\frac{[j-\mu]^{\frac12}}{[2j]^{\frac12}}, \qquad S_{j\mu} := q^{(j-\mu)/2}\,\frac{[j+\mu]^{\frac12}}{[2j]^{\frac12}}.
\tag{4.5c}
$$

Notice that there are no $\downarrow$ vectors for $j = 0$. It is now straightforward, though tedious, to verify that these vectors are orthonormal bases for the respective subspaces $W_j^{\downarrow}$ and $W_j^{\uparrow}$.

The Hilbert space of spinors is $\mathcal{H} := \mathcal{H}_\psi \otimes \mathbb{C}^2$, which is just the completion of the algebraic direct sum (4.2). We may decompose it as $\mathcal{H} = \mathcal{H}^{\uparrow} \oplus \mathcal{H}^{\downarrow}$, where $\mathcal{H}^{\uparrow}$ and $\mathcal{H}^{\downarrow}$ are the respective completions of $\bigoplus_{2j\geqslant 0} W_j^{\uparrow}$ and $\bigoplus_{2j\geqslant 1} W_j^{\downarrow}$.

Lemma 4.2. *The basis vectors $|j\mu n{\uparrow}\rangle$ and $|j\mu n{\downarrow}\rangle$ are joint eigenvectors for $\lambda'(k)$ and $\rho'(k)$, and e, f are represented on them as ladder operators:*

$$
\begin{aligned}
\lambda'(k)|j\mu n{\uparrow}\rangle &= q^\mu |j\mu n{\uparrow}\rangle, \qquad & \rho'(k)|j\mu n{\uparrow}\rangle &= q^n |j\mu n{\uparrow}\rangle, \\
\lambda'(k)|j\mu n{\downarrow}\rangle &= q^\mu |j\mu n{\downarrow}\rangle, \qquad & \rho'(k)|j\mu n{\downarrow}\rangle &= q^n |j\mu n{\downarrow}\rangle.
\end{aligned}
\tag{4.6a}
$$

Moreover,

$$\begin{aligned}
\lambda'(f)|j\mu n\uparrow\rangle &= [j-\mu]^{\frac12}[j+\mu+1]^{\frac12}|j,\mu+1,n\uparrow\rangle,\\
\lambda'(e)|j\mu n\uparrow\rangle &= [j+\mu]^{\frac12}[j-\mu+1]^{\frac12}|j,\mu-1,n\uparrow\rangle,\\
\lambda'(f)|j\mu n\downarrow\rangle &= [j-\mu]^{\frac12}[j+\mu+1]^{\frac12}|j,\mu+1,n\downarrow\rangle,\\
\lambda'(e)|j\mu n\downarrow\rangle &= [j+\mu]^{\frac12}[j-\mu+1]^{\frac12}|j,\mu-1,n\downarrow\rangle,
\end{aligned} \tag{4.6b}$$

and

$$\begin{aligned}
\rho'(f)|j\mu n\uparrow\rangle &= [j-n+\tfrac12]^{\frac12}[j+n+\tfrac32]^{\frac12}|j\mu,n+1,\uparrow\rangle,\\
\rho'(e)|j\mu n\uparrow\rangle &= [j+n+\tfrac12]^{\frac12}[j-n+\tfrac32]^{\frac12}|j\mu,n-1,\uparrow\rangle,\\
\rho'(f)|j\mu n\downarrow\rangle &= [j-n-\tfrac12]^{\frac12}[j+n+\tfrac12]^{\frac12}|j\mu,n+1,\downarrow\rangle,\\
\rho'(e)|j\mu n\downarrow\rangle &= [j+n-\tfrac12]^{\frac12}[j-n+\tfrac12]^{\frac12}|j\mu,n-1,\downarrow\rangle.
\end{aligned} \tag{4.6c}$$

► The representation π' can now be computed in the new spinor basis by conjugating the form of $\pi\otimes\mathrm{id}$ found in Proposition 3.3 by the basis transformation (4.5). However, it is more instructive to derive these formulas from the property of (λ',ρ')-equivariance. First, we introduce a handy notation.

Definition 4.3. For $j=0,\frac12,1,\frac32,\dots$, with $\mu=-j,\dots,j$ and $n=-j-\frac12,\dots,j+\frac12$, we juxtapose the pair of spinors

$$|j\mu n\rangle\!\rangle := \begin{pmatrix} |j\mu n\uparrow\rangle \\ |j\mu n\downarrow\rangle \end{pmatrix},$$

with the convention that the lower component is zero when $n=\pm(j+\frac12)$ or $j=0$. Furthermore, a matrix with scalar entries,

$$A = \begin{pmatrix} A_{\uparrow\uparrow} & A_{\uparrow\downarrow} \\ A_{\downarrow\uparrow} & A_{\downarrow\downarrow} \end{pmatrix},$$

is understood to act on $|j\mu n\rangle\!\rangle$ by the rule:

$$\begin{aligned}
A|j\mu n\uparrow\rangle &= A_{\uparrow\uparrow}|j\mu n\uparrow\rangle + A_{\downarrow\uparrow}|j\mu n\downarrow\rangle,\\
A|j\mu n\downarrow\rangle &= A_{\downarrow\downarrow}|j\mu n\downarrow\rangle + A_{\uparrow\downarrow}|j\mu n\uparrow\rangle.
\end{aligned} \tag{4.7}$$

Proposition 4.4. *The representation* $\pi' := \pi\otimes\mathrm{id}$ *of* $\mathcal{A}$ *is given by*

$$\begin{aligned}
\pi'(a)\,|j\mu n\rangle\!\rangle &= \alpha^+_{j\mu n}|j^+\mu^+n^+\rangle\!\rangle + \alpha^-_{j\mu n}|j^-\mu^+n^+\rangle\!\rangle,\\
\pi'(b)\,|j\mu n\rangle\!\rangle &= \beta^+_{j\mu n}|j^+\mu^+n^-\rangle\!\rangle + \beta^-_{j\mu n}|j^-\mu^+n^-\rangle\!\rangle,\\
\pi'(a^*)\,|j\mu n\rangle\!\rangle &= \tilde\alpha^+_{j\mu n}|j^+\mu^-n^-\rangle\!\rangle + \tilde\alpha^-_{j\mu n}|j^-\mu^-n^-\rangle\!\rangle,\\
\pi'(b^*)\,|j\mu n\rangle\!\rangle &= \tilde\beta^+_{j\mu n}|j^+\mu^-n^+\rangle\!\rangle + \tilde\beta^-_{j\mu n}|j^-\mu^-n^+\rangle\!\rangle,
\end{aligned} \tag{4.8}$$

where $\alpha^{\pm}_{j\mu n}$ and $\beta^{\pm}_{j\mu n}$ are, up to phase factors depending only on j, the following triangular 2×2 matrices:

$$\begin{aligned}
\alpha^{+}_{j\mu n} &= q^{(\mu+n-\frac12)/2}[j+\mu+1]^{\frac12} \begin{pmatrix} q^{-j-\frac12}\, \frac{[j+n+\frac32]^{1/2}}{[2j+2]} & 0 \\ q^{\frac12}\, \frac{[j-n+\frac12]^{1/2}}{[2j+1]\,[2j+2]} & q^{-j}\, \frac{[j+n+\frac12]^{1/2}}{[2j+1]} \end{pmatrix}, \\
\alpha^{-}_{j\mu n} &= q^{(\mu+n-\frac12)/2}[j-\mu]^{\frac12} \begin{pmatrix} q^{j+1}\, \frac{[j-n+\frac12]^{1/2}}{[2j+1]} & -q^{\frac12}\, \frac{[j+n+\frac12]^{1/2}}{[2j]\,[2j+1]} \\ 0 & q^{j+\frac12}\, \frac{[j-n-\frac12]^{1/2}}{[2j]} \end{pmatrix}, \\
\beta^{+}_{j\mu n} &= q^{(\mu+n-\frac12)/2}[j+\mu+1]^{\frac12} \begin{pmatrix} \frac{[j-n+\frac32]^{1/2}}{[2j+2]} & 0 \\ -q^{-j-1}\, \frac{[j+n+\frac12]^{1/2}}{[2j+1]\,[2j+2]} & q^{-\frac12}\, \frac{[j-n+\frac12]^{1/2}}{[2j+1]} \end{pmatrix}, \qquad (4.9) \\
\beta^{-}_{j\mu n} &= q^{(\mu+n-\frac12)/2}[j-\mu]^{\frac12} \begin{pmatrix} -q^{-\frac12}\, \frac{[j+n+\frac12]^{1/2}}{[2j+1]} & -q^{j}\, \frac{[j-n+\frac12]^{1/2}}{[2j]\,[2j+1]} \\ 0 & -\frac{[j+n-\frac12]^{1/2}}{[2j]} \end{pmatrix},
\end{aligned}$$

and the remaining matrices are the hermitian conjugates

$$\tilde\alpha^{\pm}_{j\mu n} = (\alpha^{\mp}_{j^\pm \mu^- n^-})^\dagger, \qquad \tilde\beta^{\pm}_{j\mu n} = (\beta^{\mp}_{j^\pm \mu^- n^+})^\dagger.$$

Proof. The proof of Proposition 3.3 applies with minor changes. From the analogues of (3.6) and the relations $\lambda'(f)\pi'(a) = q^{-\frac12}\pi'(a)\,\lambda'(f)$ and $\lambda'(e)\pi'(a^*) = q^{\frac12}\pi'(a^*)\,\lambda'(e)$, applied to the spinors $|j\mu n\rangle\!\rangle$, together with the formulas (4.6a) and (4.6b), we determine that $\pi'(a)$ has the indicated form, where the $\alpha^{\pm}_{j\mu n}$ are 2×2 matrices. The other cases of (4.8) are handled similarly.

To compute these matrices, we again use the commutation relations of $\lambda'(f)$ with $\pi'(a)$ and $\pi'(b)$ to establish recurrence relations, analogous to (3.8a), which yield

$$\begin{aligned}
\alpha^{+}_{j\mu n} &= q^{(\mu+n-\frac12)/2}[j+\mu+1]^{\frac12}\, A^{+}_{jn}, \qquad & \alpha^{-}_{j\mu n} &= q^{(\mu+n-\frac12)/2}[j-\mu]^{\frac12}\, A^{-}_{jn}, \\
\beta^{+}_{j\mu n} &= q^{(\mu+n-\frac12)/2}[j+\mu+1]^{\frac12}\, B^{+}_{jn}, \qquad & \beta^{-}_{j\mu n} &= q^{(\mu+n-\frac12)/2}[j-\mu]^{\frac12}\, B^{-}_{jn}.
\end{aligned}$$

The new matrices $A^{\pm}_{jn}$, $B^{\pm}_{jn}$ may be further refined by using commutation relations involving $\rho'(f)$ and $\rho'(e)$. For instance, $\rho'(f)\pi'(a) = q^{-\frac12}\pi'(a)\,\rho'(f)$ entails

$$\begin{aligned}
&\begin{pmatrix} [j-n+\frac12]^{\frac12}[j+n+\frac52]^{\frac12} & 0 \\ 0 & [j-n-\frac12]^{\frac12}[j+n+\frac32]^{\frac12} \end{pmatrix} A^{+}_{jn} \\
&\qquad = A^{+}_{j,n+1} \begin{pmatrix} [j-n+\frac12]^{\frac12}[j+n+\frac32]^{\frac12} & 0 \\ 0 & [j-n-\frac12]^{\frac12}[j+n+\frac12]^{\frac12} \end{pmatrix}.
\end{aligned}$$

This yields four recurrence relations for the entries of A^{+}_{jn}, one of which has only the trivial solution; we conclude that

$$A^{+}_{jn} = \begin{pmatrix} [j+n+\frac32]^{\frac12} a^{+}_{j\uparrow\uparrow} & 0 \\ [j-n+\frac12]^{\frac12} a^{+}_{j\downarrow\uparrow} & [j+n+\frac12]^{\frac12} a^{+}_{j\downarrow\downarrow} \end{pmatrix},$$

where the $a^+_{j\updownarrow\updownarrow}$ are scalars depending only on j. In a similar fashion, we arrive at

$$A^-_{jn} = \begin{pmatrix} [j-n+\frac12]^{\frac12} a^-_{j\uparrow\uparrow} & [j+n+\frac12]^{\frac12} a^-_{j\uparrow\downarrow} \\ 0 & [j-n-\frac12]^{\frac12} a^-_{j\downarrow\downarrow} \end{pmatrix},$$

$$B^+_{jn} = \begin{pmatrix} [j-n+\frac32]^{\frac12} b^+_{j\uparrow\uparrow} & 0 \\ [j+n+\frac12]^{\frac12} b^+_{j\downarrow\uparrow} & [j-n+\frac12]^{\frac12} b^+_{j\downarrow\downarrow} \end{pmatrix},$$

$$B^-_{jn} = \begin{pmatrix} [j+n+\frac12]^{\frac12} b^-_{j\uparrow\uparrow} & [j-n+\frac12]^{\frac12} b^-_{j\uparrow\downarrow} \\ 0 & [j+n-\frac12]^{\frac12} b^-_{j\downarrow\downarrow} \end{pmatrix}.$$

The analogue of (3.9) leads quickly to the relations

$$\begin{aligned} b^+_{j\uparrow\uparrow} &= q^{j+\frac12} a^+_{j\uparrow\uparrow}, \qquad & b^+_{j\downarrow\uparrow} &= -q^{-j-\frac32} a^+_{j\downarrow\uparrow}, \qquad & b^+_{j\downarrow\downarrow} &= q^{j-\frac12} a^+_{j\downarrow\downarrow}, \\ b^-_{j\uparrow\uparrow} &= -q^{-j-\frac32} a^-_{j\uparrow\uparrow}, \qquad & b^-_{j\uparrow\downarrow} &= q^{j-\frac12} a^-_{j\uparrow\downarrow}, \qquad & b^-_{j\downarrow\downarrow} &= -q^{-j-\frac12} a^-_{j\downarrow\downarrow}. \end{aligned} \tag{4.10}$$

Next, from the analogue of (3.10) we get

$$(a^-_{j+\frac12,\uparrow\uparrow})^\star = q^{2j+2} a^+_{j\uparrow\uparrow}, \qquad (a^-_{j+\frac12,\uparrow\downarrow})^\star = -a^+_{j\downarrow\uparrow}, \qquad (a^-_{j+\frac12,\downarrow\downarrow})^\star = q^{2j+1} a^+_{j\downarrow\downarrow}.$$

The $a^+_{j\updownarrow\updownarrow}$ parameters may be determined from $\pi'(b)\pi'(a)\,|j\mu n\rangle\rangle = q\,\pi'(a)\pi'(b)\,|j\mu n\rangle\rangle$. The coefficients of $|j\pm 1, \mu+1, n\rangle\rangle$ yield only the relation

$$[2j+1]\; a^+_{j+\frac12,\downarrow\downarrow} a^+_{j\downarrow\uparrow} = [2j+3]\; a^+_{j+\frac12,\downarrow\uparrow} a^+_{j\uparrow\uparrow}. \tag{4.11}$$

From the $|j, \mu+1, n\rangle\rangle$ terms, we obtain

$$B^-_{j^+n^+} A^+_{jn} + B^+_{j^-n^+} A^-_{jn} = q^{\frac12} (A^-_{j^+n^-} B^+_{jn} + A^+_{j^-n^-} B^-_{jn}).$$

Comparison of the diagonal entries on both sides gives two more relations:

$$\begin{aligned} [2j+1]\,|a^+_{j\downarrow\uparrow}|^2 &= q^{2j+1}\big([2j+1]\,|a^+_{j-\frac12,\uparrow\uparrow}|^2 - q[2j+3]\,|a^+_{j\uparrow\uparrow}|^2\big), \\ [2j+1]\,|a^+_{j-\frac12,\downarrow\uparrow}|^2 &= q^{2j}\big(q[2j+1]\,|a^+_{j\downarrow\downarrow}|^2 - [2j-1]\,|a^+_{j-\frac12,\downarrow\downarrow}|^2\big). \end{aligned}$$

Finally, the expectation of $\pi'(a^*a + q^2 b^*b) = 1$ in the vector states for $|j\mu n\uparrow\rangle$ and $|j\mu n\downarrow\rangle$ leads to the relations

$$q^{2j}\,[2j+1]^2 |a^+_{j-\frac12,\uparrow\uparrow}|^2 = 1, \qquad q^{2j}\,[2j+1]^2 |a^+_{j\downarrow\downarrow}|^2 = 1.$$

Thus all coefficients are now determined, up to a few j-dependent phases:

$$a^+_{j\uparrow\uparrow} = \zeta_j \frac{q^{-j-\frac12}}{[2j+2]}, \qquad a^+_{j\downarrow\uparrow} = \eta_j \frac{q^{\frac12}}{[2j+1]\,[2j+2]}, \qquad a^+_{j\downarrow\downarrow} = \xi_j \frac{q^{-j}}{[2j+1]}, \tag{4.12}$$

with $|\zeta_j| = |\eta_j| = |\xi_j| = 1$. The relation (4.11) also implies $\zeta_{j+\frac12}\eta_j = \eta_{j+\frac12}\xi_j$. As before, we may reset these phases to 1 by redefining $|j\mu n\uparrow\rangle$ and $|j\mu n\downarrow\rangle$, without breaking the (λ', ρ')-equivariance. Substituting (4.12) back in previous formulas then gives (4.9). □

As already mentioned, formulas (4.9) for the matrices $\alpha^{\pm}_{j\mu n}$ and $\beta^{\pm}_{j\mu n}$ could have been obtained also from a direct but tedious computation using equations (4.5) and their inverses.

Remark 4.5. Were we to consider a representation of $\mathcal{A}$ that need not be (λ', ρ')-equivariant, we could as well have defined our spinor space, like in [13], as $\mathbb{C}^2 \otimes V$, instead of $V \otimes \mathbb{C}^2$. The Clebsch–Gordan decomposition of $\mathbb{C}^2 \otimes V$ would be that of equation (4.1), but the q-Clebsch–Gordan coefficients appearing in (4.5a) and (4.5b) would be different due to the rule for exchanging the first two columns in q-Clebsch–Gordan coefficients [21]:

$$C_q\begin{pmatrix} j & l & m \\ r & s & t \end{pmatrix} = C_q\begin{pmatrix} l & j & m \\ -s & -r & -t \end{pmatrix},$$

which results in a substitution of q by q^{-1} in (4.5c).

However, this is not the correct lifting of the (λ, ρ)-equivariant representation π of $\mathcal{A}$ to a (λ', ρ')-equivariant representation of $\mathcal{A}$ on spinor space. We already noted that π' as defined by $\pi \otimes \mathrm{id}$ on $V \otimes \mathbb{C}^2$ is (λ', ρ')-equivariant, directly from (λ, ρ)-equivariance of π. One checks, simply by working out both sides of equation (4.4), that the noncocommutativity of $\mathcal{U}_q(\mathfrak{su}(2))$ spoils (λ'', ρ'')-equivariance of the representation $\pi'' := \mathrm{id} \otimes \pi$ of $\mathcal{A}$ on the tensor product $\mathbb{C}^2 \otimes V$, where we now define $\rho'' := \mathrm{id} \otimes \rho$, and

$$\lambda''(h) := (\sigma_{\frac{1}{2}} \otimes \lambda)(\Delta h) = \sigma_{\frac{1}{2}}(h_{(1)}) \otimes \lambda(h_{(2)}).$$

5 The equivariant Dirac operator

Recall the central Casimir element $C_q = qk^2 + q^{-1}k^{-2} + (q - q^{-1})^2 ef \in \mathcal{U}$. The symmetric operators $\lambda'(C_q)$ and $\rho'(C_q)$ on $\mathcal{H}$, initially defined with dense domain W, extend to selfadjoint operators on $\mathcal{H}$. The finite-dimensional subspaces $W^{\uparrow}_j$ and $W^{\downarrow}_j$ are their joint eigenspaces:

$$\begin{aligned}
\lambda'(C_q)|j\mu n\!\uparrow\rangle &= (q^{2j+1} + q^{-2j-1})\,|j\mu n\!\uparrow\rangle, & \rho'(C_q)|j\mu n\!\uparrow\rangle &= (q^{2j+2} + q^{-2j-2})\,|j\mu n\!\uparrow\rangle, \\
\lambda'(C_q)|j\mu n\!\downarrow\rangle &= (q^{2j+1} + q^{-2j-1})\,|j\mu n\!\downarrow\rangle, & \rho'(C_q)|j\mu n\!\downarrow\rangle &= (q^{2j} + q^{-2j})\,|j\mu n\!\downarrow\rangle,
\end{aligned}$$

directly from (4.6).

Let D be a selfadjoint operator on $\mathcal{H}$ which commutes strongly with $\lambda'(C_q)$ and $\rho'(C_q)$; then the finite-dimensional subspaces $W^{\uparrow}_j$ and $W^{\downarrow}_j$ reduce D. We look for the general form of such a selfadjoint operator D which is moreover (λ', ρ')-invariant in the sense that it commutes with $\lambda'(h)$ and $\rho'(h)$, for each $h \in \mathcal{U}_q(\mathfrak{su}(2))$.

Lemma 5.1. *The subspaces $W^{\uparrow}_j$ and $W^{\downarrow}_j$ are eigenspaces for D.*

Proof. We may restrict to either the subspace $W^{\uparrow}_j$ or $W^{\downarrow}_j$. Since $\lambda'(k)$ and $\rho'(k)$ are required to commute with D and moreover have distinct eigenvalues on these subspaces, it follows that D has a diagonal matrix with respect to the basis $|j\mu n\!\uparrow\rangle$, respectively $|j\mu n\!\downarrow\rangle$. If we provisionally write $D|j\mu n\!\uparrow\rangle = d^{\uparrow}_{j\mu n}\,|j\mu n\!\uparrow\rangle$, then the vanishing of

$$[D, \lambda'(f)]\,|j\mu n\!\uparrow\rangle = (d^{\uparrow}_{j,\mu+1,n} - d^{\uparrow}_{j\mu n})\,[j - \mu]^{\frac{1}{2}}[j + \mu + 1]^{\frac{1}{2}}\,|j, \mu + 1, n\!\uparrow\rangle,$$

for $\mu = -j, \dots, j-1$, shows that $d^{\uparrow}_{j\mu n}$ is independent of μ; and $[D, \rho'(f)] = 0$ likewise shows that $d^{\uparrow}_{j\mu n}$ does not depend on n. The same goes for $d^{\downarrow}_{j\mu n}$, too. Thus we may write

$$D|j\mu n\!\uparrow\rangle = d^{\uparrow}_{j}\,|j\mu n\!\uparrow\rangle, \qquad D|j\mu n\!\downarrow\rangle = d^{\downarrow}_{j}\,|j\mu n\!\downarrow\rangle, \tag{5.1}$$

where $d^{\uparrow}_{j}$ and $d^{\downarrow}_{j}$ are real eigenvalues of D. The respective multiplicities are $(2j+1)(2j+2)$ and $2j(2j+1)$, in view of (4.3). □

One of the conditions for the triple $(\mathcal{A}, \mathcal{H}, D)$ to be a spectral triple is boundedness of the commutators $[D, \pi'(x)]$ for $x \in \mathcal{A}$. This naturally imposes certain restrictions on the eigenvalues $d^{\uparrow}_{j}, d^{\downarrow}_{j}$ of the operator D.

For convenience, we recall the representation π' of a in the basis $|j\mu n\rangle\rangle$, written explicitly on $|j\mu n\!\uparrow\rangle$ and $|j\mu n\!\downarrow\rangle$ as in (4.7):

$$\begin{aligned}\pi'(a)|j\mu n\!\uparrow\rangle &= \sum_{\pm} \alpha^{\pm}_{j\mu n\uparrow\uparrow}|j^{\pm}\mu^{+}n^{+}\!\uparrow\rangle + \alpha^{+}_{j\mu n\downarrow\uparrow}|j^{+}\mu^{+}n^{+}\!\downarrow\rangle,\\ \pi'(a)|j\mu n\!\downarrow\rangle &= \sum_{\pm} \alpha^{\pm}_{j\mu n\downarrow\downarrow}|j^{\pm}\mu^{+}n^{+}\!\downarrow\rangle + \alpha^{-}_{j\mu n\uparrow\downarrow}|j^{-}\mu^{+}n^{+}\!\uparrow\rangle.\end{aligned}$$

Then, a straightforward computation shows that

$$\begin{aligned}[D, \pi'(a)]\,|j\mu n\!\uparrow\rangle &= \sum_{\pm} \alpha^{\pm}_{j\mu n\uparrow\uparrow}(d^{\uparrow}_{j^{\pm}} - d^{\uparrow}_{j})|j^{\pm}\mu^{+}n^{+}\!\uparrow\rangle + \alpha^{+}_{j\mu n\downarrow\uparrow}(d^{\downarrow}_{j^{+}} - d^{\uparrow}_{j})|j^{+}\mu^{+}n^{+}\!\downarrow\rangle,\\ [D, \pi'(a)]\,|j\mu n\!\downarrow\rangle &= \sum_{\pm} \alpha^{\pm}_{j\mu n\downarrow\downarrow}(d^{\downarrow}_{j^{\pm}} - d^{\downarrow}_{j})|j^{\pm}\mu^{+}n^{+}\!\downarrow\rangle + \alpha^{-}_{j\mu n\uparrow\downarrow}(d^{\uparrow}_{j^{-}} - d^{\downarrow}_{j})|j^{-}\mu^{+}n^{+}\!\uparrow\rangle.\end{aligned} \tag{5.2}$$

Recall that the standard Dirac operator $\not{D}$ on the sphere $\mathbb{S}^3$, with the round metric, has eigenvalues $(2j+\frac{3}{2})$ for $j = 0, \frac{1}{2}, 1, \frac{3}{2}$, with respective multiplicities $(2j+1)(2j+2)$; and $-(2j+\frac{1}{2})$ for $j = \frac{1}{2}, 1, \frac{3}{2}$, with respective multiplicities $2j(2j+1)$: see [1, 18], for instance. Notice that its spectrum is symmetric about 0.

In [2] a "q-Dirac" operator D was proposed, which in our notation corresponds to taking $d^{\uparrow}_{j} = 2[2j+1]/(q+q^{-1})$ and $d^{\downarrow}_{j} = -d^{\uparrow}_{j}$; these are q-analogues of the classical eigenvalues of $\not{D} - \frac{1}{2}$. For this particular choice of eigenvalues, it follows directly from the explicit form (4.9) of the matrices $\alpha^{\pm}_{j\mu n}$ that then the right hand sides of (5.2) diverge, and therefore $[D, \pi'(a)]$ is unbounded. This was already noted in [10] and it was suggested that one should instead consider an operator D whose spectrum matches that of the classical Dirac operator. In fact, Proposition 7.3 below shows that this is essentially the only possibility for a Dirac operator satisfying a (modified) first-order condition.

Let us then consider any operator D given by (5.1) – that is, a bi-equivariant one – with eigenvalues of the following form:

$$d^{\uparrow}_{j} = c^{\uparrow}_{1} j + c^{\uparrow}_{2}, \qquad d^{\downarrow}_{j} = c^{\downarrow}_{1} j + c^{\downarrow}_{2}, \tag{5.3}$$

where $c^{\uparrow}_{1}, c^{\uparrow}_{2}, c^{\downarrow}_{1}, c^{\downarrow}_{2}$ are independent of j. For brevity, we shall say that the eigenvalues are "linear in j". On the right hand side of (5.2), the "diagonal" coefficients simplify to

$$\alpha^{\pm}_{j\mu n\uparrow\uparrow}(d^{\uparrow}_{j^{\pm}} - d^{\uparrow}_{j}) = \tfrac{1}{2}\alpha^{\pm}_{j\mu n\uparrow\uparrow}c^{\uparrow}_{1}, \qquad \alpha^{\pm}_{j\mu n\downarrow\downarrow}(d^{\downarrow}_{j^{\pm}} - d^{\downarrow}_{j}) = \tfrac{1}{2}\alpha^{\pm}_{j\mu n\downarrow\downarrow}c^{\downarrow}_{1}, \tag{5.4}$$

which can be uniformly bounded with respect to j – see expressions (4.9). For the off-diagonal terms, involving $\alpha^+_{j\mu n\downarrow\uparrow}$ and $\alpha^-_{j\mu n\uparrow\downarrow}$, the differences between the "up" and "down" eigenvalues are linear in j. Since $0 < q < 1$, it is clear that $[N] \sim (q^{-1})^{N-1}$ for large N, and thus $\alpha^+_{j\mu n\downarrow\uparrow} \sim q^{3j+n+\frac{3}{2}} \leqslant q^{2j+1}$ for large j. Similar easy estimates yield

$$\alpha^+_{j\mu n\downarrow\uparrow} = O(q^{2j+1}), \qquad \beta^+_{j\mu n\downarrow\uparrow} = O(q^{2j+\frac{1}{2}}), \tag{5.5}$$

$$\alpha^-_{j\mu n\uparrow\downarrow} = O(q^{2j}), \qquad \beta^-_{j\mu n\uparrow\downarrow} = O(q^{2j+\frac{1}{2}}), \quad \text{as } j \to \infty. \tag{5.6}$$

We therefore arrive at

$$|\alpha^+_{j\mu n\downarrow\uparrow}(d^\downarrow_{j^+} - d^\uparrow_j - 1)| \leqslant C j q^{2j}, \qquad |\alpha^-_{j\mu n\uparrow\downarrow}(d^\uparrow_{j^-} - d^\downarrow_j - 1)| \leqslant C' j q^{2j}, \tag{5.7}$$

for some $C > 0, C' > 0$, independent of j; and similar estimates hold for the off-diagonal coefficients of $\pi'(b)$.

Proposition 5.2. *Let D be any selfadjoint operator with eigenspaces $W^\uparrow_j$ and $W^\downarrow_j$, and eigenvalues* (5.1)*. If the eigenvalues $d^\uparrow_j$ and $d^\downarrow_j$ are linear in j as in* (5.3)*, then $[D, \pi'(x)]$ is a bounded operator for all $x \in \mathcal{A}$.*

Proof. Since a and b generate $\mathcal{A}$ as a $*$-algebra, it is enough to consider the cases $x = a$ and $x = b$. For $x = a$ and any $\xi \in \mathcal{H}$, the relations (5.2) and (5.4), together with the Schwarz inequality, give the estimate

$$\|[D, \pi'(a)]\,\xi\|^2 \leqslant \tfrac{1}{4}\max\{(c^\uparrow_1)^2, (c^\downarrow_1)^2\}\,\|\pi'(a)\xi\|^2 + \|\xi\|^2\|\eta\|^2,$$

where η is a vector whose components are estimated by (5.7), which establishes finiteness of $\|\eta\|$ since $0 < q < 1$. Therefore, $[D, \pi'(a)]$ is norm bounded. In the same way, we find that $[D, \pi'(b)]$ is bounded. □

Now, if D is a selfadjoint operator as in Proposition 5.2, and if the eigenvalues of D satisfy (5.3) and, moreover,

$$c^\downarrow_1 = -c^\uparrow_1, \qquad c^\downarrow_2 = -c^\uparrow_2 + c^\uparrow_1, \tag{5.8}$$

then the spectrum of D coincides with that of the classical Dirac operator $\not{D}$ on the round sphere $\mathbb{S}^3$, up to rescaling and addition of a constant. Thus, we can regard our spectral triple as an isospectral deformation of $(C^\infty(\mathbb{S}^3), \mathcal{H}, \not{D})$, and in particular, its spectral dimension is 3. We summarize our conclusions in the following theorem.

Theorem 5.3. *The triple $(\mathcal{A}(\mathrm{SU}_q(2)), \mathcal{H}, D)$, where the eigenvalues of D satisfy* (5.3) *and* (5.8)*, is a 3^+-summable spectral triple.* □

At this point, it is appropriate to comment on the relation of our construction with that of [13]. There, a spinor representation is constructed by tensoring the left regular representation of $\mathcal{A}(\mathrm{SU}_q(2))$ by $\mathbb{C}^2$ on the left. This spinor space is then decomposed into two subspaces, similar to our "up" and "down" subspaces, on which D acts diagonally with eigenvalues linear in the total spin number j. The corresponding decomposition of the representation π' of $\mathcal{A}(\mathrm{SU}_q(2))$ on spinor space is obtained by using the appropriate Clebsch–Gordan coefficients. However, contrary to what we have established above, in [13] it is found that a certain commutator $[D, \pi'(x)]$

is an unbounded operator. In particular, the off-diagonal terms in the representation of [13] do not have the compact nature we encountered in (5.6). They can be bounded from below by a positive constant, which leads, when multiplied by a term linear in j, to an unbounded operator.

The origin of this notable contrast is the following. Since in [13] no condition of $\mathcal{U}_q(\mathfrak{su}(2))$-equivariance is imposed *a priori* on the representation of $\mathcal{A}(\mathrm{SU}_q(2))$, the spinor space W could be identified either with $V \otimes \mathbb{C}^2$ or $\mathbb{C}^2 \otimes V$, according to convenience. However, as we noted in Remark 4.5, the choice of $\mathbb{C}^2 \otimes V$ is not allowed by the condition of (λ', ρ')-equivariance, because $\mathcal{U}_q(\mathfrak{su}(2))$ is not cocommutative. Indeed, repeating the construction of a spinor representation and Dirac operator on the spinor space $\mathbb{C}^2 \otimes V$ instead of $V \otimes \mathbb{C}^2$ – hence ignoring equivariance – results eventually in unbounded commutators.

6 The real structure

The next issue we address is the real structure J on the spectral triple $(\mathcal{A}(\mathrm{SU}_q(2)), \mathcal{H}, D)$. We shall see that by requiring equivariance of J it is not possible to satisfy all usual properties of a real spectral triple like in [8] or [15]. Among other things, these conditions entail for J that it intertwine a left action and a commuting right action of the algebra on the Hilbert space, which then gets a bimodule structure (the commutant property); and that the bounded commutators $[D, a]$, for any element a in the algebra, commute with the opposite action by any b in the algebra (the first order condition on D). However, we shall be able to satisfy these two conditions only up to certain compact operators.

6.1 The Tomita operator of the regular representation

On the GNS representation space $\mathcal{H}_\psi$, there is a natural involution $T_\psi : \eta(x) \mapsto \eta(x^*)$, with domain $\eta(C(\mathrm{SU}_q(2)))$, which may be regarded as an unbounded (antilinear) operator on $\mathcal{H}_\psi$. The Tomita–Takesaki theory [31] shows that this operator is closable (we denote its closure also by T_ψ) and that the polar decomposition $T_\psi =: J_\psi \Delta_\psi^{1/2}$ defines both the positive "modular operator" Δ_ψ and the antiunitary "modular conjugation" J_ψ. It has already been noted by Chakraborty and Pal [5] that this J_ψ has a simple expression in terms of the matrix elements of our chosen orthonormal basis for $\mathcal{H}_\psi$. Indeed, it follows immediately from (2.12) and (2.13) that

$$T_\psi \, |lmn\rangle = (-1)^{2l+m+n} q^{m+n} \, |l, -m, -n\rangle.$$

One checks, using (3.3), that

$$T_\psi \pi(a) \, |000\rangle = \pi(a^*) \, |000\rangle, \qquad T_\psi \pi(b) \, |000\rangle = \pi(b^*) \, |000\rangle.$$

Since π is the GNS representation for the state ψ, this is enough to conclude that

$$T_\psi \eta(x) = \eta(x^*) \quad \text{for all} \quad x \in \mathcal{A}. \tag{6.1}$$

The adjoint antilinear operator, satisfying $\langle \eta \mid T_\psi^* \mid \xi \rangle = \langle \xi \mid T_\psi \mid \eta \rangle$, is given by $T_\psi^* \, |lmn\rangle = (-1)^{2l+m+n} q^{-m-n} \, |l, -m, -n\rangle$, and since $\Delta_\psi = T_\psi^* T_\psi$, we see that every $|lmn\rangle$ lies in $\operatorname{Dom} \Delta_\psi$ with $\Delta_\psi \, |lmn\rangle = q^{2m+2n} \, |lmn\rangle$. Consequently,

$$J_\psi \, |lmn\rangle = (-1)^{2l+m+n} \, |l, -m, -n\rangle. \tag{6.2}$$

It is clear that $J_\psi^2 = 1$ on $\mathcal{H}_\psi$.

Definition 6.1. Let $\pi^\circ(x) := J_\psi\,\pi(x^*)\,J_\psi^{-1}$, so that π° is a $*$-antirepresentation of $\mathcal{A}$ on $\mathcal{H}_\psi$. Equivalently, π° is a $*$-representation of the opposite algebra $\mathcal{A}(\mathrm{SU}_{1/q}(2))$. By Tomita's theorem [31], π and π° are commuting representations.

As an example, we compute

$$\begin{aligned}
\pi^\circ(a)\,|lmn\rangle &= (-1)^{2l+m+n} J_\psi \pi(a^*)\,|l,-m,-n\rangle \\
&= (-1)^{2l+m+n} J_\psi\big(\widetilde{A}^+_{l,-m,-n}|l^+,-m^+,-n^+\rangle + \widetilde{A}^-_{l,-m,-n}|l^-,-m^+,-n^+\rangle\big) \\
&= \widetilde{A}^+_{l,-m,-n}|l^+m^+n^+\rangle + \widetilde{A}^-_{l,-m,-n}|l^-m^+n^+\rangle \\
&= A^-_{l^+,-m^+,-n^+}|l^+m^+n^+\rangle + A^+_{l^-,-m^+,-n^+}|l^-m^+n^+\rangle,
\end{aligned}$$

where, explicitly,

$$\begin{aligned}
A^-_{l^+,-m^+,-n^+} &= q^{(2l-m-n+1)/2}\left(\frac{[l+m+1]\,[l+n+1]}{[2l+1]\,[2l+2]}\right)^{\frac12}, \\
A^+_{l^-,-m^+,-n^+} &= q^{-(2l+m+n+1)/2}\left(\frac{[l-m]\,[l-n]}{[2l]\,[2l+1]}\right)^{\frac12}.
\end{aligned}$$

A glance back at (3.4) shows that these coefficients are identical with those of $\pi(a)\,|lmn\rangle$, *after substituting* $q \mapsto q^{-1}$. A similar phenomenon occurs with the coefficients of $\pi^\circ(b)$. We find, indeed, that

$$\begin{aligned}
\pi^\circ(a)\,|lmn\rangle &= A^{\circ+}_{lmn}|l^+m^+n^+\rangle + A^{\circ-}_{lmn}|l^-m^+n^+\rangle, \\
\pi^\circ(b)\,|lmn\rangle &= B^{\circ+}_{lmn}|l^+m^+n^-\rangle + B^{\circ-}_{lmn}|l^-m^+n^-\rangle,
\end{aligned}$$

where

$$A^{\circ\pm}_{lmn}(q) = A^\pm_{lmn}(q^{-1}), \qquad B^{\circ\pm}_{lmn}(q) = q^{-1}B^\pm_{lmn}(q^{-1}). \tag{6.3}$$

We can now verify directly that the representations π and π° commute, without need to appeal to the theorem of Tomita. For instance,

$$\begin{aligned}
\langle l+1,m+1,n+1 \mid [\pi(a),\pi^\circ(a)] \mid lmn\rangle &= A^{\circ+}_{l^+m^+n^+}A^+_{lmn} - A^+_{l^+m^+n^+}A^{\circ+}_{lmn} \\
&= Q\left(\frac{[l+m+1]\,[l+m+2]\,[l+n+1]\,[l+n+2]}{[2l+1]\,[2l+2]^2\,[2l+3]}\right)^{\frac12},
\end{aligned}$$

where

$$Q = q^{\frac12(2l^+-m^+-n^++1)}q^{\frac12(-2l+m+n-1)} - q^{\frac12(-2l^++m^++n^+-1)}q^{\frac12(2l-m-n+1)} = 0.$$

Likewise, $\langle l-1,m+1,n+1 \mid [\pi(a),\pi^\circ(a)] \mid lmn\rangle = 0$, and one checks that the matrix element $\langle l,m+1,n+1 \mid [\pi(a),\pi^\circ(a)] \mid lmn\rangle$ vanishes, too.

The (λ,ρ)-equivariance of π is reflected in an analogous equivariance condition for π°. We now identify this condition explicitly.

Lemma 6.2. *The symmetry of the antirepresentation π° of $\mathcal{A}$ on $\mathcal{H}_\psi$ is given by the equivariance conditions:*

$$\begin{aligned}
\lambda(h)\,\pi^\circ(x)\xi &= \pi^\circ(\tilde h_{(2)}\cdot x)\,\lambda(h_{(1)})\xi, \\
\rho(h)\,\pi^\circ(x)\xi &= \pi^\circ(\tilde h_{(2)} \triangleright x)\,\rho(h_{(1)})\xi,
\end{aligned} \tag{6.4}$$

for all $h \in \mathcal{U}$, $x \in \mathcal{A}$ and $\xi \in V$, and $h \mapsto \tilde h$ is the automorphism of $\mathcal{U}$ determined on generators by $\tilde k := k$, $\tilde f := q^{-1}f$, and $\tilde e := qe$.

Proof. We work only on the dense subspace V. From (3.1) and (6.2), we get at once

$$J_\psi \lambda(k)^* J_\psi^{-1} = \lambda(k^{-1}), \quad J_\psi \lambda(f)^* J_\psi^{-1} = -\lambda(f), \quad J_\psi \lambda(e)^* J_\psi^{-1} = -\lambda(e), \tag{6.5}$$

and identical relations with ρ instead of λ. Write α for the antiautomorphism of $\mathcal{U}$ determined by $\alpha(k) := k^{-1}$, $\alpha(f) := -f$, and $\alpha(e) := -e$; so that $J_\psi \lambda(h)^* J_\psi^{-1} = \lambda(\alpha(h))$ for $h \in \mathcal{U}$, and similarly with ρ instead of λ.

Next, the first relation of (3.2) is equivalent to

$$\pi(x)\, \lambda(Sh) = \lambda(Sh_{(1)})\, \pi(h_{(2)} \cdot x). \tag{6.6}$$

Indeed, the left hand side can be expanded as

$$\pi(x)\, \lambda(\varepsilon(h_{(1)})\, Sh_{(2)}) = \lambda(Sh_{(1)}\, h_{(2)})\, \pi(x)\, \lambda(Sh_{(3)}) = \lambda(Sh_{(1)})\, \pi(h_{(2)} \cdot x)\, \lambda(h_{(3)})\, \lambda(Sh_{(4)})$$

on applying (3.2); and the rightmost expression equals the right-hand side of (6.6). Taking hermitian adjoints and conjugating by J_ψ, we get

$$\lambda(\alpha(Sh))\, \pi^\circ(x) = \pi^\circ(h_{(2)} \cdot x)\, \lambda(\alpha(Sh_{(1)})).$$

It remains only to note that $S\alpha = \alpha S$ is an automorphism of $\mathcal{U}$, whose inverse is the map $h \mapsto \tilde{h}$ above; and to repeat the argument with ρ instead of λ, changing only the left action of $\mathcal{U}$ in concordance with (3.2). □

An independent check of (6.4) is afforded by the following argument. We may ask which antirepresentations π° of $\mathcal{H}_\psi$ satisfy these equivariance conditions. It suffices to run the proof of Proposition 3.3, *mutatis mutandis*, to determine the possible form of such a π° on the basis vectors $|lmn\rangle$. For instance, (3.9) is replaced by

$$\rho(e)\pi^\circ(a) = \pi^\circ(\tilde{e} \triangleright a)\rho(k^{-1}) + \pi^\circ(\tilde{k} \triangleright a)\rho(e) = q\, \pi^\circ(b)\rho(k^{-1}) + q^{\frac{1}{2}}\pi^\circ(a)\rho(e).$$

One finds that all formulas in that proof are reproduced, except for changes in the powers of q that appear; and, apart from the aforementioned phase ambiguities, one recovers precisely the form of π° given by (6.3).

► Before proceeding, we indicate also the symmetry of the Tomita operator T_ψ, analogous to (6.5) above. Combining (6.1) with (3.2), and recalling that $\eta(x) = \pi(x)\,|000\rangle$, we find that for generators h of $\mathcal{U}$,

$$T_\psi \lambda(h)\pi(x)\,|000\rangle = \pi(x^* \triangleleft \vartheta(h)^*)\,|000\rangle.$$

On the other hand,

$$\lambda(\vartheta^{-1} S(\vartheta(h^*)))T_\psi \pi(x)\,|000\rangle = \pi(x^* \triangleleft \vartheta(h)^*)\,|000\rangle.$$

One checks easily on generators that $\vartheta^{-1}S(\vartheta(h)^*) = S(h)^*$. Since the vector $|000\rangle$ is separating for the GNS representation, we conclude that

$$T_\psi\, \lambda(h)\, T_\psi^{-1} = \lambda(Sh)^*.$$

Similarly, we find that

$$T_\psi\, \rho(h)\, T_\psi^{-1} = \rho(Sh)^*.$$

In other words, the antilinear involutory automorphism $h \mapsto (Sh)^*$ of the Hopf $*$-algebra $\mathcal{U}$ is implemented by the Tomita operator for the Haar state of the dual Hopf $*$-algebra $\mathcal{A}$. This is a known feature of quantum-group duality in the C^*-algebra framework; for this and several other implementations by spatial operators, see [25].

6.2 The real structure on spinors

We are now ready to come back to spinors. Notice that J_ψ does not appear explicitly in the equivariance conditions (6.4) for the right regular representation π° of $\mathcal{A}$ on $\mathcal{H}_\psi$. Thus, we are now able to construct the "right multiplication" representation of $\mathcal{A}$ on spinors from its symmetry alone, and to deduce the conjugation operator J on spinors after the fact.

Proposition 6.3. *Let π'° be an antirepresentation of $\mathcal{A}$ on $\mathcal{H} = \mathcal{H}_\psi \oplus \mathcal{H}_\psi$ satisfying the following equivariance conditions:*

$$\begin{aligned}\lambda'(h)\,\pi'^\circ(x)\xi &= \pi'^\circ(\tilde{h}_{(2)} \cdot x)\,\lambda'(h_{(1)})\xi,\\ \rho'(h)\,\pi'^\circ(x)\xi &= \pi'^\circ(\tilde{h}_{(2)} \triangleright x)\,\rho'(h_{(1)})\xi.\end{aligned} \tag{6.7}$$

Then, up to some phase factors depending only on the index j in the decomposition (4.2)*, π'° is given on the spinor basis by*

$$\begin{aligned}\pi'^\circ(a)\,|j\mu n\rangle\rangle &= \alpha^{\circ+}_{j\mu n}|j^+\mu^+n^+\rangle\rangle + \alpha^{\circ-}_{j\mu n}|j^-\mu^+n^+\rangle\rangle,\\ \pi'^\circ(b)\,|j\mu n\rangle\rangle &= \beta^{\circ+}_{j\mu n}|j^+\mu^+n^-\rangle\rangle + \beta^{\circ-}_{j\mu n}|j^-\mu^+n^-\rangle\rangle,\\ \pi'^\circ(a^*)\,|j\mu n\rangle\rangle &= \tilde{\alpha}^{\circ+}_{j\mu n}|j^+\mu^-n^-\rangle\rangle + \tilde{\alpha}^{\circ-}_{j\mu n}|j^-\mu^-n^-\rangle\rangle,\\ \pi'^\circ(b^*)\,|j\mu n\rangle\rangle &= \tilde{\beta}^{\circ+}_{j\mu n}|j^+\mu^-n^+\rangle\rangle + \tilde{\beta}^{\circ-}_{j\mu n}|j^-\mu^-n^+\rangle\rangle,\end{aligned} \tag{6.8}$$

where $\alpha^{\circ\pm}_{j\mu n}$ and $\beta^{\circ\pm}_{j\mu n}$ are the triangular 2×2 matrices given by $\alpha^{\circ\pm}_{j\mu n}(q) = \alpha^{\pm}_{j\mu n}(q^{-1})$ and $\beta^{\circ\pm}_{j\mu n}(q) = q^{-1}\beta^{\pm}_{j\mu n}(q^{-1})$, with $\alpha^{\pm}_{j\mu n}$ and $\beta^{\pm}_{j\mu n}$ given by (4.9).

Proof. We retrace the steps of the proof of Proposition 4.4, *mutatis mutandis*. Since $\tilde{k}\cdot a = k\cdot a = q^{\frac12}a$, the relations involving $\lambda'(k)$ and $\rho'(k)$ are unchanged. We quickly conclude that π'° must have the form (6.8), and it remains to determine the coefficient matrices.

The commutation relations of $\lambda'(f)$ with $\pi'^\circ(a)$ and $\pi'^\circ(b)$ give:

$$\begin{aligned}\alpha^{\circ+}_{j\mu n} &= q^{-\frac12(\mu+n-\frac12)}[j+\mu+1]^{\frac12}\;A^{\circ+}_{jn}, &\qquad \alpha^{\circ-}_{j\mu n} &= q^{-\frac12(\mu+n-\frac12)}[j-\mu]^{\frac12}\;A^{\circ-}_{jn},\\ \beta^{\circ+}_{j\mu n} &= q^{-\frac12(\mu+n-\frac12)}[j+\mu+1]^{\frac12}\;B^{\circ+}_{jn}, &\qquad \beta^{\circ-}_{j\mu n} &= q^{-\frac12(\mu+n-\frac12)}[j-\mu]^{\frac12}\;B^{\circ-}_{jn}.\end{aligned}$$

The matrices $A^{\circ\pm}_{jn}$, $B^{\circ\pm}_{jn}$ may be determined, as before, by the commutation relations involving $\rho'(f)$ and $\rho'(e)$. One finds that the n-dependent factors such as $[j+n+\frac32]^{\frac12}$ and so on, are the same as the respective entries of $A^{\pm}_{jn}$, $B^{\pm}_{jn}$; let $a^{\circ+}_{j\uparrow\uparrow}$, etc., be the remaining factors which depend on j only. Then (4.10) is replaced by

$$\begin{aligned}b^{\circ+}_{j\uparrow\uparrow} &= q^{-j-\frac32}a^{\circ+}_{j\uparrow\uparrow}, &\qquad b^{\circ+}_{j\downarrow\uparrow} &= -q^{j+\frac12}a^{\circ+}_{j\downarrow\uparrow}, &\qquad b^{\circ+}_{j\downarrow\downarrow} &= q^{-j-\frac12}a^{\circ+}_{j\downarrow\downarrow},\\ b^{\circ-}_{j\uparrow\uparrow} &= -q^{j+\frac12}a^{\circ-}_{j\uparrow\uparrow}, &\qquad b^{\circ-}_{j\uparrow\downarrow} &= q^{-j-\frac12}a^{\circ-}_{j\uparrow\downarrow}, &\qquad b^{\circ-}_{j\downarrow\downarrow} &= -q^{j-\frac12}a^{\circ-}_{j\downarrow\downarrow}.\end{aligned}$$

Next, we find

$$(a^{\circ-}_{j+\frac12,\uparrow\uparrow})^\star = q^{-2j-2}a^{\circ+}_{j\uparrow\uparrow}, \qquad (a^{\circ-}_{j+\frac12,\uparrow\downarrow})^\star = -a^{\circ+}_{j\downarrow\uparrow}, \qquad (a^{\circ-}_{j+\frac12,\downarrow\downarrow})^\star = q^{-2j-1}a^{\circ+}_{j\downarrow\downarrow}.$$

Since π'° is an antirepresentation, $ab = q^{-1}ba$ implies $\pi'^{\circ}(b)\pi'^{\circ}(a) = q^{-1}\,\pi'^{\circ}(a)\pi'^{\circ}(b)$. The matrix elements of both sides lead to three relations:

$$[2j+1]\, a^{\circ+}_{j+\frac{1}{2},\downarrow\downarrow} a^{\circ+}_{j\downarrow\uparrow} = [2j+3]\, a^{\circ+}_{j+\frac{1}{2},\downarrow\uparrow} a^{\circ+}_{j\uparrow\uparrow}, \tag{6.9}$$

which is formally identical to (4.11), and

$$\begin{aligned} [2j+1]\, |a^{\circ+}_{j\downarrow\uparrow}|^2 &= q^{-2j-1}\big([2j+1]\, |a^{\circ+}_{j-\frac{1}{2},\uparrow\uparrow}|^2 - q^{-1}[2j+3]\, |a^{\circ+}_{j\uparrow\uparrow}|^2\big), \\ [2j+1]\, |a^{\circ+}_{j-\frac{1}{2},\downarrow\uparrow}|^2 &= q^{-2j}\big(q^{-1}[2j+1]\, |a^{\circ+}_{j\downarrow\downarrow}|^2 - [2j-1]\, |a^{\circ+}_{j-\frac{1}{2},\downarrow\downarrow}|^2\big). \end{aligned}$$

Finally, the relation $aa^* + bb^* = 1$ yields $\pi'^{\circ}(a^*)\pi'^{\circ}(a) + \pi'^{\circ}(b^*)\pi'^{\circ}(b) = 1$; its diagonal matrix elements gives the last two relations:

$$q^{-2j}\,[2j+1]^2 |a^{\circ+}_{j-\frac{1}{2},\uparrow\uparrow}|^2 = 1, \qquad q^{-2j}\,[2j+1]^2 |a^{\circ+}_{j\downarrow\downarrow}|^2 = 1.$$

All coefficients are now determined except for their phases:

$$a^{\circ+}_{j\uparrow\uparrow} = \zeta^{\circ}_j \frac{q^{j+\frac{1}{2}}}{[2j+2]}, \quad a^{\circ+}_{j\downarrow\uparrow} = \eta^{\circ}_j \frac{q^{-\frac{1}{2}}}{[2j+1]\,[2j+2]}, \quad a^{\circ+}_{j\downarrow\downarrow} = \xi^{\circ}_j \frac{q^{j}}{[2j+1]}, \tag{6.10}$$

and (6.9) also entails the phase relations $\zeta^{\circ}_j \eta^{\circ}_{j+\frac{1}{2}} = \eta^{\circ}_j \xi^{\circ}_{j+\frac{1}{2}}$. Once more, we choose all phases to be $+1$ by convention. Substituting (6.10) back in previous formulas, we find

$$\alpha^{\circ\pm}_{j\mu n}(q) = \alpha^{\pm}_{j\mu n}(q^{-1}), \qquad \beta^{\circ\pm}_{j\mu n}(q) = q^{-1}\beta^{\pm}_{j\mu n}(q^{-1}). \tag{6.11}$$

in perfect analogy with (6.3). □

Definition 6.4. The conjugation operator J is the antilinear operator on $\mathcal{H}$ which is defined explicitly on the orthonormal spinor basis by

$$\begin{aligned} J\,|j\mu n\!\uparrow\rangle &:= i^{2(2j+\mu+n)}\,|j,-\mu,-n,\uparrow\rangle, \\ J\,|j\mu n\!\downarrow\rangle &:= i^{2(2j-\mu-n)}\,|j,-\mu,-n,\downarrow\rangle. \end{aligned} \tag{6.12}$$

It is immediate from this presentation that J is antiunitary and that $J^2 = -1$, since each $4j \pm 2(\mu+n)$ is an odd integer.

Proposition 6.5. *The invariant operator D of Section 5 commutes with the conjugation operator J:*

$$JDJ^{-1} = D. \tag{6.13}$$

Proof. This is clear from the diagonal form of both D and J on their common eigenspaces $W^{\uparrow}_j$ and $W^{\downarrow}_j$, given by the respective equations (5.1) and (6.12). □

Remark 6.6. Proposition 6.5 is a minimal requirement for $(\mathcal{A}(\mathrm{SU}_q(2)), \mathcal{H}, D, J)$ to constitute a real spectral triple. However, here is where we part company with the axiom scheme for real spectral triples proposed in [8]. Indeed, the conjugation operator J that we have defined by (6.12) is *not* the modular conjugation for the spinor representation of $\mathcal{A}$. That modular operator is $J_\psi \oplus J_\psi$, which does not have a diagonal form in our chosen spinor basis (unless $q = 1$). It is clear that conjugation of $\pi'(\mathcal{A}(\mathrm{SU}_q(2))$ by the modular operator would yield a representation of the opposite algebra $\mathcal{A}(\mathrm{SU}_{1/q}(2))$, and the commutation relation analogous to (6.13) would then force D to be equivariant under the corresponding symmetry of $U_{1/q}(su(2))$, denoted by (λ'', ρ'') in our earlier Remark 4.5. It is not hard to check that this extra equivariance condition would force D to be merely a scalar operator, thereby negating the possibility of an equivariant 3^+-summable real spectral triple based on $\mathcal{A}(\mathrm{SU}_q(2))$ with the modular conjugation operator. This result is consonant with the "no-go theorem" of Schmüdgen [28] for nontrivial commutator representations of Woronowicz differential calculi on $\mathrm{SU}_q(2)$.

The remedy that we propose here is to modify J, in keeping with the symmetry of the spinor representation, to a non-Tomita conjugation operator. We shall see, however, that the expected properties of real spectral triples do hold "up to compact perturbations".

It should be noted that J satisfies the analogue of (6.5) for the representations λ' and ρ':

$$\begin{aligned} J\lambda'(k)J^{-1} &= \lambda'(k^{-1}), \qquad & J\lambda'(e)J^{-1} &= -\lambda'(f), \\ J\rho'(k)J^{-1} &= \rho'(k^{-1}), \qquad & J\rho'(e)J^{-1} &= -\rho'(f), \end{aligned} \tag{6.14}$$

which follows directly from the definition (6.12) and the relations (4.6).

Proposition 6.7. *The antiunitary operator J intertwines the left and right spinor representations:*

$$J\,\pi'(x^*)\,J^{-1} = \pi'^{\circ}(x), \quad \textit{for all} \quad x \in \mathcal{A}. \tag{6.15}$$

Proof. It follows directly from the proof of Lemma 6.2, using the relations (6.14) instead of (6.5), that the antirepresentation $x \mapsto J\,\pi'(x^*)\,J^{-1}$ complies with the equivariance conditions (6.7). By Proposition 6.3, it coincides with π'° up to an equivalence obtained by resetting the phase factors in (6.10). It remains only to check that $\zeta_j^\circ = \eta_j^\circ = \xi_j^\circ = 1$ for the aforementioned antirepresentation. This check is easily effected by calculating $J\,\pi'(a^*)\,J^{-1}$ directly on the basis vectors $|j\mu n\uparrow\rangle$. We compute

$$\begin{aligned} J\pi'(a^*)J^{-1}\,|j\mu n\uparrow\rangle &= i^{-2(2j-\mu-n)} J\pi'(a^*)\,|j,-\mu,-n,\uparrow\rangle \\ &= i^{-2(2j-\mu-n)} J\big(\tilde\alpha^+_{j,-\mu,-n,\uparrow\uparrow}\,|j^+,-\mu^+,-n^+\uparrow\rangle + \tilde\alpha^+_{j,-\mu,-n,\downarrow\uparrow}\,|j^+,-\mu^+,-n^+\downarrow\rangle \\ &\qquad + \tilde\alpha^-_{j,-\mu,-n,\uparrow\uparrow}\,|j^-,-\mu^+,-n^+\uparrow\rangle\big) \\ &= \tilde\alpha^+_{j,-\mu,-n,\uparrow\uparrow}\,|j^+\mu^+n^+\uparrow\rangle - \tilde\alpha^+_{j,-\mu,-n,\downarrow\uparrow}\,|j^+\mu^+n^+\downarrow\rangle + \tilde\alpha^-_{j,-\mu,-n,\uparrow\uparrow}\,|j^-\mu^+n^+\uparrow\rangle \\ &= \alpha^-_{j^+,-\mu^+,-n^+,\uparrow\uparrow}\,|j^+\mu^+n^+\uparrow\rangle - \alpha^-_{j^+,-\mu^+,-n^+,\downarrow\uparrow}\,|j^+\mu^+n^+\downarrow\rangle + \alpha^+_{j^-,-\mu^+,-n^+,\uparrow\uparrow}\,|j^-\mu^+n^+\uparrow\rangle \\ &= q^{-\frac12(\mu+n-\frac12)}\bigg(q^{j+\frac12}\frac{[j+\mu+1]^{\frac12}[j+n+\frac32]^{\frac12}}{[2j+2]}\,|j^+\mu^+n^+\uparrow\rangle \\ &\qquad + q^{-\frac12}\frac{[j+\mu+1]^{\frac12}[j-n+\frac12]^{\frac12}}{[2j+1][2j+2]}\,|j^+\mu^+n^+\downarrow\rangle + q^{-j-1}\frac{[j-\mu]^{\frac12}[j-n+\frac12]^{\frac12}}{[2j+1]}\,|j^-\mu^+n^+\uparrow\rangle\bigg) \\ &= \alpha^{\circ+}_{j\mu n\uparrow\uparrow}\,|j^+\mu^+n^+\uparrow\rangle + \alpha^{\circ+}_{j\mu n\downarrow\uparrow}\,|j^+\mu^+n^+\downarrow\rangle + \alpha^{\circ-}_{j\mu n\uparrow\uparrow}\,|j^-\mu^+n^+\uparrow\rangle \\ &= \pi'^{\circ}(a)\,|j\mu n\uparrow\rangle, \end{aligned}$$

where the $\alpha^{\circ\pm}_{j\mu n}$ coefficients are taken according to (6.11).

In the same way, one finds that $J\pi'(b^*)J^{-1}\,|j\mu n\!\uparrow\rangle = \pi'^{\circ}(b)\,|j\mu n\!\uparrow\rangle$, again using (6.11) for $\beta^{\circ\pm}_{j\mu n}$; and similar calculations show that both sides of (6.15) coincide on the basis vector $|j\mu n\!\downarrow\rangle$. (These four calculations, taken together, afford a direct proof of (6.15) without need to consider the symmetries of J.) □

7 Algebraic properties of the spectral triple

In this section, we discuss the properties of the real spectral triple $(\mathcal{A}(\mathrm{SU}_q(2)), \mathcal{H}, D, J)$, in particular its commutant property and its first-order condition. We shall see that these are only satisfied up to certain compact operators, quite similarly to [11].

We can simplify our discussion somewhat by replacing the spinor representation π' of $\mathcal{A} = \mathcal{A}(\mathrm{SU}_q(2))$ of Proposition 4.4 by a so-called approximate representation $\underline{\pi}'\colon \mathcal{A} \to \mathcal{B}(\mathcal{H})$, such that $\pi'(x) - \underline{\pi}'(x)$ is a compact operator for each $x \in \mathcal{A}$. In other words, although $\underline{\pi}'$ need not preserve the algebra relations of $\mathcal{A}$, the mappings π' and $\underline{\pi}'$ have the same image in the Calkin algebra $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$, that is, they define the same $*$-homomorphism of $\mathcal{A}$ into the Calkin algebra.

We denote by L_q the positive trace-class operator given by

$$L_q|j\mu n\rangle\!\rangle := q^j\,|j\mu n\rangle\!\rangle \quad \text{for} \quad j \in \tfrac{1}{2}\mathbb{N},$$

and let $\mathcal{K}_q$ be the two-sided ideal of $\mathcal{B}(\mathcal{H})$ generated by L_q; it consists of trace-class operators. The ideal $\mathcal{K}_q$ is indeed contained in the ideal of infinitesimals of order α, that is, compact operators whose n-th singular value μ_n satisfies $\mu_n = O(n^{-\alpha})$, for all $\alpha > 0$. Thus the following analysis holds modulo infinitesimals of arbitrary high order.

Proposition 7.1. *The following equations define a mapping $\underline{\pi}'\colon \mathcal{A} \to \mathcal{B}(\mathcal{H})$ on generators, which is a $*$-representation modulo $\mathcal{K}_q$, and is approximate to the spin representation π' of Proposition 4.4 in the sense that $\pi'(x) - \underline{\pi}'(x) \in \mathcal{K}_q$ for each $x \in \mathcal{A}$:*

$$\begin{aligned}
\underline{\pi}'(a)\,|j\mu n\rangle\!\rangle &= \underline{\alpha}^+_{j\mu n}|j^+\mu^+n^+\rangle\!\rangle + \underline{\alpha}^-_{j\mu n}|j^-\mu^+n^+\rangle\!\rangle, \\
\underline{\pi}'(b)\,|j\mu n\rangle\!\rangle &= \underline{\beta}^+_{j\mu n}|j^+\mu^+n^-\rangle\!\rangle + \underline{\beta}^-_{j\mu n}|j^-\mu^+n^-\rangle\!\rangle, \\
\underline{\pi}'(a^*)\,|j\mu n\rangle\!\rangle &= \underline{\tilde\alpha}^+_{j\mu n}|j^+\mu^-n^-\rangle\!\rangle + \underline{\tilde\alpha}^-_{j\mu n}|j^-\mu^-n^-\rangle\!\rangle, \\
\underline{\pi}'(b^*)\,|j\mu n\rangle\!\rangle &= \underline{\tilde\beta}^+_{j\mu n}|j^+\mu^-n^+\rangle\!\rangle + \underline{\tilde\beta}^-_{j\mu n}|j^-\mu^-n^+\rangle\!\rangle,
\end{aligned} \tag{7.1}$$

where

$$\begin{aligned}
\underline{\alpha}^+_{j\mu n} &:= \sqrt{1-q^{2j+2\mu+2}}\begin{pmatrix}\sqrt{1-q^{2j+2n+3}} & 0 \\ 0 & \sqrt{1-q^{2j+2n+1}}\end{pmatrix}, \\
\underline{\alpha}^-_{j\mu n} &:= q^{2j+\mu+n+\frac{1}{2}}\sqrt{1-q^{2j-2\mu}}\begin{pmatrix}q\sqrt{1-q^{2j-2n+1}} & 0 \\ 0 & \sqrt{1-q^{2j-2n-1}}\end{pmatrix}, \\
\underline{\beta}^+_{j\mu n} &:= q^{j+n-\frac{1}{2}}\sqrt{1-q^{2j+2\mu+2}}\begin{pmatrix}q\sqrt{1-q^{2j-2n+3}} & 0 \\ 0 & \sqrt{1-q^{2j-2n+1}}\end{pmatrix}, \\
\underline{\beta}^-_{j\mu n} &:= -q^{j+\mu}\sqrt{1-q^{2j-2\mu}}\begin{pmatrix}\sqrt{1-q^{2j+2n+1}} & 0 \\ 0 & \sqrt{1-q^{2j+2n-1}}\end{pmatrix},
\end{aligned} \tag{7.2}$$

and

$$\underline{\tilde{\alpha}}^{\pm}_{j\mu n} = \underline{\alpha}^{\mp}_{j^{\pm}\mu^{-}n^{-}}, \qquad \underline{\tilde{\beta}}^{\pm}_{j\mu n} = \underline{\alpha}^{\mp}_{j^{\pm}\mu^{-}n^{+}}. \tag{7.3}$$

Proof. First of all, we claim that the defining relations (2.1) are preserved by $\underline{\pi}'$ modulo the ideal $\mathcal{K}_q$ of $\mathcal{B}(\mathcal{H})$, that is, $\underline{\pi}'(b)\underline{\pi}'(a) - q\,\underline{\pi}'(a)\underline{\pi}'(b) \in \mathcal{K}_q$, and so on. Indeed, it can be verified by a direct but tedious check on the spinor basis that $\underline{\pi}'(b)\underline{\pi}'(a) - q\,\underline{\pi}'(a)\underline{\pi}'(b) = L_q^4 A$ where A is a bounded operator; the same is true for each of the other relations listed in (2.1).

It is well known, and easily checked from (2.1), that $\mathcal{A}$ is generated as a vector space by the products $a^k b^l b^{*m}$ and $b^l b^{*m} a^{*n}$, for $k, l, m, n \in \mathbb{N}$. We may thus define $\underline{\pi}'(x)$ for any $x \in \mathcal{A}$ by extending (7.1) multiplicatively on such products, and then extending further by linearity. With this convention, we conclude that

$$\underline{\pi}'(xy) - \underline{\pi}'(x)\underline{\pi}'(y) \in \mathcal{K}_q \quad \text{for all} \quad x, y \in \mathcal{A}. \tag{7.4}$$

The defining formulas also entail that $\underline{\pi}'(x)^* = \underline{\pi}'(x^*)$ for each $x \in \mathcal{A}$.

If $\pi'(x) - \underline{\pi}'(x) \in \mathcal{K}_q$ and $\pi'(y) - \underline{\pi}'(y) \in \mathcal{K}_q$, then

$$\pi'(xy) - \underline{\pi}'(x)\underline{\pi}'(y) = \pi'(x)\big(\pi'(y) - \underline{\pi}'(y)\big) + \big(\pi'(x) - \underline{\pi}'(x)\big)\underline{\pi}'(y) \in \mathcal{K}_q,$$

and therefore $\pi'(xy) - \underline{\pi}'(xy)$ lies in $\mathcal{K}_q$ also; thus, it suffices to verify this property in the cases $x = a, b$.

On comparing the coefficients (7.2) with the corresponding ones of $\pi'(a)$ and $\pi'(b)$ from (4.9), we get, for instance,

$$\begin{aligned}
\alpha^+_{j\mu n\uparrow\uparrow} - \underline{\alpha}^+_{j\mu n\uparrow\uparrow} &= \frac{q^{4j+4}\sqrt{1-q^{2j+2\mu+2}}\sqrt{1-q^{2j+2n+3}}}{1-q^{4j+4}} = q^{4j+4}\,\alpha^+_{j\mu n\uparrow\uparrow}\,, \\
\alpha^+_{j\mu n\downarrow\downarrow} - \underline{\alpha}^+_{j\mu n\downarrow\downarrow} &= \frac{q^{4j+2}\sqrt{1-q^{2j+2\mu+2}}\sqrt{1-q^{2j+2n+1}}}{1-q^{4j+2}} = q^{4j+2}\,\alpha^+_{j\mu n\downarrow\downarrow}\,.
\end{aligned} \tag{7.5a}$$

and similarly,

$$\alpha^-_{j\mu n\uparrow\uparrow} - \underline{\alpha}^-_{j\mu n\uparrow\uparrow} = q^{4j+2}\,\alpha^-_{j\mu n\uparrow\uparrow}\,, \qquad \alpha^-_{j\mu n\downarrow\downarrow} - \underline{\alpha}^-_{j\mu n\downarrow\downarrow} = q^{4j}\,\alpha^-_{j\mu n\downarrow\downarrow}\,. \tag{7.5b}$$

We estimate the off-diagonal terms, using the inequalities $q^{\pm\mu} \leqslant q^{-j}$, $q^{\pm n} \leqslant q^{-j-\frac12}$, $[N]^{-1} < q^{N-1}$:

$$\begin{aligned}
|\alpha^+_{j\mu n\downarrow\uparrow}| &= q^{(\mu+n+\frac12)/2}\,\frac{[j+\mu+1]^{\frac12}\,[j-n+\frac12]^{\frac12}}{[2j+1]\,[2j+2]} \leqslant \frac{q^{-2j-2}}{[2j+1]\,[2j+2]} < q^{2j-1}, \\
|\alpha^-_{j\mu n\uparrow\downarrow}| &= q^{(\mu+n+\frac12)/2}\,\frac{[j-\mu]^{\frac12}\,[j+n+\frac12]^{\frac12}}{[2j]\,[2j+1]} \leqslant \frac{q^{-2j-1}}{[2j]\,[2j+1]} < q^{2j-2}.
\end{aligned}$$

On account of (7.5) and analogous relations for the coefficients of $\underline{\pi}'(b)$, we find that

$$\begin{aligned}
\pi'(a) - \underline{\pi}'(a) &\equiv T\pi'(a)T \mod \mathcal{K}_q, \\
\pi'(b) - \underline{\pi}'(b) &\equiv T\pi'(b)T \mod \mathcal{K}_q,
\end{aligned}$$

where T is the operator defined by

$$T\,|j\mu n\rangle\!\rangle := \begin{pmatrix} q^{2j+\frac32} & 0 \\ 0 & q^{2j+\frac12} \end{pmatrix} |j\mu n\rangle\!\rangle = \begin{pmatrix} q^{\frac32} & 0 \\ 0 & q^{\frac12} \end{pmatrix} L_q^2\,|j\mu n\rangle\!\rangle. \tag{7.6}$$

Clearly, $T \in \mathcal{K}_q$, so that by boundedness of $\pi'(x)$ it follows that $\pi'(x) - \underline{\pi}'(x) \in \mathcal{K}_q$ for $x = a, b$. □

Using the conjugation operator J, we can also define an approximate antirepresentation of $\mathcal{A}$ by $\underline{\pi}'^{\circ}(x) := J\underline{\pi}'(x)J^{-1}$. It is immediate that $\pi'^{\circ}(x) - \underline{\pi}'^{\circ}(x) \in \mathcal{K}_q$, with π'° as defined in Proposition 6.3. Explicitly, we can write

$$
\begin{aligned}
\underline{\pi}'^{\circ}(a)\,|j\mu n\rangle\!\rangle &= \underline{\alpha}^{\circ+}_{j\mu n}|j^+\mu^+n^+\rangle\!\rangle + \underline{\alpha}^{\circ-}_{j\mu n}|j^-\mu^+n^+\rangle\!\rangle, \\
\underline{\pi}'^{\circ}(b)\,|j\mu n\rangle\!\rangle &= \underline{\beta}^{\circ+}_{j\mu n}|j^+\mu^+n^-\rangle\!\rangle + \underline{\beta}^{\circ-}_{j\mu n}|j^-\mu^+n^-\rangle\!\rangle, \\
\underline{\pi}'^{\circ}(a^*)\,|j\mu n\rangle\!\rangle &= \underline{\tilde{\alpha}}^{\circ+}_{j\mu n}|j^+\mu^-n^-\rangle\!\rangle + \underline{\tilde{\alpha}}^{\circ-}_{j\mu n}|j^-\mu^-n^-\rangle\!\rangle, \\
\underline{\pi}'^{\circ}(b^*)\,|j\mu n\rangle\!\rangle &= \underline{\tilde{\beta}}^{\circ+}_{j\mu n}|j^+\mu^-n^+\rangle\!\rangle + \underline{\tilde{\beta}}^{\circ-}_{j\mu n}|j^-\mu^-n^+\rangle\!\rangle,
\end{aligned}
$$

where

$$
\underline{\alpha}^{\circ\pm}_{j\mu n} = \underline{\tilde{\alpha}}^{\pm}_{j,-\mu,-n}\,, \quad \underline{\tilde{\alpha}}^{\circ\pm}_{j\mu n} = \underline{\alpha}^{\pm}_{j,-\mu,-n}\,, \quad \underline{\beta}^{\circ\pm}_{j\mu n} = -\underline{\tilde{\beta}}^{\pm}_{j,-\mu,-n}, \quad \underline{\tilde{\beta}}^{\circ\pm}_{j\mu n} = -\underline{\beta}^{\pm}_{j,-\mu,-n}.
$$

► It turns out that the approximate representations $\underline{\pi}'$ and $\underline{\pi}'^{\circ}$ almost commute, in the following sense.

Proposition 7.2. *For each $x, y \in \mathcal{A}$, the commutant $[\underline{\pi}'^{\circ}(x), \underline{\pi}'(y)]$ lies in $\mathcal{K}_q$.*

Proof. In view of our earlier remarks on the almost-multiplicativity of $\underline{\pi}'$, and thus also of $\underline{\pi}'^{\circ}$, it is enough to check this for the cases $x, y = a, a^*, b, b^*$. We omit the detailed calculation, which we have performed with a symbolic computer program. In each case, the commutator $[\underline{\pi}'^{\circ}(x), \underline{\pi}'(y)]$ decomposes as a direct sum of operators in the subspaces $W_j^{\uparrow}$ and $W_j^{\downarrow}$ separately, in view of (7.2) and (6.12), and the explicit calculation shows that for each pair of generators x, y, we obtain $[\underline{\pi}'^{\circ}(x), \underline{\pi}'(y)] = L_q^2 A$ where A is a bounded operator. □

If we further impose the first-order condition up to compact operators in the ideal $\mathcal{K}_q$, it turns out that this (almost) determines the Dirac operator.

Proposition 7.3. *Up to rescaling, adding constants, and adding elements of $\mathcal{K}_q$, there is only one operator D of the form* (5.1) *which satisfies the first order condition modulo $\mathcal{K}_q$, that is, each $[D, \underline{\pi}'(y)]$ is bounded, and*

$$
[\underline{\pi}'^{\circ}(x), [D, \underline{\pi}'(y)]] \in \mathcal{K}_q \quad \text{for all} \quad x, y \in \mathcal{A}. \tag{7.7}
$$

This operator D has eigenvalues that are linear in j.

Proof. Suppose first that D is an equivariant selfadjoint operator of the type considered in Section 5, with eigenvalues linear in j; that is, D is determined by (5.1) and (5.3). Since each operator appearing in (7.7) decomposes into a pair of operators on the "up" and "down" spinor subspaces, it is clear that the nested commutators are independent of the parameters $c_2^{\uparrow}$ and $c_2^{\downarrow}$; and that $c_1^{\uparrow}$ and $c_1^{\downarrow}$ are merely scale factors on both subspaces. Again we take x and y to be generators: explicit calculations show that in each case, $[\underline{\pi}'^{\circ}(x), [D, \underline{\pi}'(y)]] = L_q^2 B$ with B a bounded operator.

To prove the converse, assume only that D satisfies the equivariance condition (5.1), and that $[D, \underline{\pi}'(a)]$ and $[D, \underline{\pi}'(b)]$ are bounded.

We may decompose $\underline{\pi}'(a) = \underline{\pi}'(a)^+ + \underline{\pi}'(a)^-$ according to whether the index j in (7.1) is raised or lowered; and similarly for $\underline{\pi}'(b)$, $\underline{\pi}'^\circ(a)$, and $\underline{\pi}'^\circ(b)$. Proposition 7.2 shows that, modulo $\mathcal{K}_q$:

$$\begin{aligned}\underline{\pi}'(a)^+\underline{\pi}'^\circ(a)^+ &\equiv \underline{\pi}'^\circ(a)^+\underline{\pi}'(a)^+,\\ \underline{\pi}'(a)^-\underline{\pi}'^\circ(a)^- &\equiv \underline{\pi}'^\circ(a)^-\underline{\pi}'(a)^-,\\ \underline{\pi}'(a)^+\underline{\pi}'^\circ(a)^- + \underline{\pi}'(a)^-\underline{\pi}'^\circ(a)^+ &\equiv \underline{\pi}'^\circ(a)^+\underline{\pi}'(a)^- + \underline{\pi}'^\circ(a)^-\underline{\pi}'(a)^+.\end{aligned}$$

By (7.2), the operators $\underline{\pi}'(a)$ and $\underline{\pi}'(b)$, as well as D, are diagonal for the decomposition $\mathcal{H} = \mathcal{H}^\uparrow \oplus \mathcal{H}^\downarrow$. On the subspace $\mathcal{H}^\uparrow$, we obtain

$$\begin{aligned}&[[D, \underline{\pi}'(a)], \underline{\pi}'^\circ(a)]\,|j\mu n\uparrow\rangle\\ &\quad= \big(D\underline{\pi}'(a)\underline{\pi}'^\circ(a) + \underline{\pi}'^\circ(a)\underline{\pi}'(a)D - \underline{\pi}'(a)D\underline{\pi}'^\circ(a) - \underline{\pi}'^\circ(a)D\underline{\pi}'(a)\big)\,|j\mu n\uparrow\rangle\\ &\quad= \Big((d^\uparrow_{j+1} + d^\uparrow_j - 2d^\uparrow_{j^+})\,\underline{\pi}'(a)^+\underline{\pi}'^\circ(a)^+ + (d^\uparrow_{j-1} + d^\uparrow_j - 2d^\uparrow_{j^-})\,\underline{\pi}'(a)^-\underline{\pi}'^\circ(a)^-\\ &\qquad + 2d^\uparrow_j(\underline{\pi}'(a)^+\underline{\pi}'^\circ(a)^- + \underline{\pi}'(a)^-\underline{\pi}'^\circ(a)^+) - d^\uparrow_{j^+}(\underline{\pi}'(a)^-\underline{\pi}'^\circ(a)^+ + \underline{\pi}'^\circ(a)^-\underline{\pi}'(a)^+)\\ &\qquad - d^\uparrow_{j^-}(\underline{\pi}'(a)^+\underline{\pi}'^\circ(a)^- + \underline{\pi}'^\circ(a)^+\underline{\pi}'(a)^-) + R\Big)\,|j\mu n\uparrow\rangle,\end{aligned} \tag{7.8}$$

where $R \in \mathcal{K}_q$. On the subspace $\mathcal{H}^\downarrow$, we get the precisely analogous expression with the arrows reversed.

In order that the expression on the right hand side of (7.8) come from an element of $\mathcal{K}_q$ applied to $|j\mu n\uparrow\rangle$, and likewise for $|j\mu n\downarrow\rangle$, it is necessary and sufficient that the scalars

$$w^\uparrow_j := d^\uparrow_{j+1} + d^\uparrow_j - 2d^\uparrow_{j^+}, \qquad w^\downarrow_j := d^\downarrow_{j+1} + d^\downarrow_j - 2d^\downarrow_{j^+} \tag{7.9}$$

satisfy $w^\uparrow_j = O(q^j)$ and $w^\downarrow_j = O(q^j)$ as $j \to \infty$.

In the particular case where $w^\uparrow_j = 0$ and $w^\downarrow_j = 0$ for all j, (7.9) gives elementary recurrence relations for $d^\uparrow_j$ and $d^\downarrow_j$, whose solutions are precisely the expressions (5.3) that are linear in j, namely,

$$d^\uparrow_j = c^\uparrow_1 j + c^\uparrow_2, \qquad d^\downarrow_j = c^\downarrow_1 j + c^\downarrow_2.$$

The general case gives a pair of perturbed recurrence relations, that may be treated by generating-function methods [16]; their solutions differ from the linear case by terms that are $O(q^j)$ as $j \to \infty$. Thus, the corresponding operator D differs from one whose eigenvalues are linear in j by an element of $\mathcal{K}_q$. □

We finish by summarizing the implications of the above Propositions 7.1, 7.2 and 7.3 for the spectral triple $(\mathcal{A}(\mathrm{SU}_q(2)), \mathcal{H}, D, J)$, where $\mathcal{A}(\mathrm{SU}_q(2))$ acts on $\mathcal{H}$ via the spinor representation π'.

The representations π' and π'° do not commute, since the conjugation operator J differs from the Tomita conjugation for π'. However, we do obtain commutation "up to infinitesimals"; since $[\pi'^\circ(x), \pi'(y)] \equiv [\underline{\pi}'^\circ(x), \underline{\pi}'(y)] \bmod \mathcal{K}_q$, Proposition 7.2 entails the analogous result for the exact representations:

$$[\pi'^\circ(x), \pi'(y)] \in \mathcal{K}_q \quad \text{for all} \quad x, y \in \mathcal{A}.$$

To examine the first-order property, we note first that if $x, y \in \mathcal{A}$ and $[D, \pi'(y) - \underline{\pi}'(y)]$ lies in $\mathcal{K}_q$, then

$$\begin{aligned}[\pi'^\circ(x), [D, \pi'(y)]] &= \left[\underline{\pi}'^\circ(x) + (\pi'^\circ(x) - \underline{\pi}'^\circ(x)),\ [D, \underline{\pi}'(y) + (\pi'(y) - \underline{\pi}'(y))]\right] \\ &\equiv [\underline{\pi}'^\circ(x), [D, \underline{\pi}'(y)]] \equiv 0 \bmod \mathcal{K}_q. \end{aligned} \tag{7.10}$$

Since D commutes with the positive operator T defined in (7.6), we find in the case of a generator $y = a$, a^*, b or b^*, that

$$[D, \pi'(y) - \underline{\pi}'(y)] = [D, T\pi'(y)T] = T[D, \pi'(y)]T,$$

which lies in $\mathcal{K}_q$ since $[D, \pi'(y)]$ is bounded, by Proposition 5.2. Thus, $[D, \underline{\pi}'(y)]$ is bounded, too – as required by Proposition 7.3. The general case of $[D, \pi'(y) - \underline{\pi}'(y)] \in \mathcal{K}_q$ then follows from (7.4). Thus (7.10) holds for general $x, y \in \mathcal{A}$. Combining that with Proposition 7.3, we arrive at the following characterization of our spectral triple over $\mathcal{A}(\mathrm{SU}_q(2))$.

Theorem 7.4. *The real spectral triple* $(\mathcal{A}(\mathrm{SU}_q(2)), \mathcal{H}, D, J)$ *defined here, with* $\mathcal{A}(\mathrm{SU}_q(2))$ *acting on* $\mathcal{H}$ *via the spinor representation* π'*, satisfies both the commutant property and the first order condition up to infinitesimals:*

$$\begin{aligned}[\pi'^\circ(x), \pi'(y)] &\in \mathcal{K}_q, \\ [\pi'^\circ(x), [D, \pi'(y)]] &\in \mathcal{K}_q, \end{aligned} \qquad \textit{for all} \quad x, y \in \mathcal{A}(\mathrm{SU}_q(2)). \qquad \square$$

In [17] it was argued that there are obstructions to the construction of "deformed spectral triples" satisfying a type of first order condition for the Dirac operator. Theorem 7.4 above shows a way to overcome these obstructions.

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