The local index formula for $\mathrm{SU}_q(2)$

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Abstract

We discuss the local index formula of Connes–Moscovici for the isospectral noncommutative geometry that we have recently constructed on quantum SU(2). We work out the cosphere bundle and the dimension spectrum as well as the local cyclic cocycles yielding the index formula.

1 Introduction

Recent investigations show that the "quantum space" underlying the quantum group $\mathrm{SU}_q(2)$ is an important arena for testing and implementing ideas coming from noncommutative differential geometry. In [8] it has been endowed with an isospectral tridimensional geometry via a bi-equivariant 3^+-summable spectral triple $(\mathcal{A}(\mathrm{SU}_q(2)), \mathcal{H}, D)$. Earlier, a "singular" (in the sense of not admitting a commutative limit) spectral triple was constructed in [2]. The latter geometry was put in the general theory of Connes–Moscovici [7] by a systematic discussion of the local index formula [5]. In this paper, we present a similar analysis for the former geometry. It turns out that most of the results coincide with those of [5].

The main idea of that paper is to construct a (quantum) cosphere bundle $\mathbb{S}_q^*$ on $\mathrm{SU}_q(2)$, that considerably simplifies the computations concerning the local index formula. Essentially, with the operator derivation δ defined by $\delta(T) := |D|T - T|D|$, one considers an operator x in the algebra $\mathcal{B} = \bigcup_{n=0}^{\infty} \delta^n(\mathcal{A})$ up to smoothing operators; these give no contribution to the residues appearing in the local cyclic cocycle giving the local index formula. The removal of the irrelevant smoothing operators is accomplished by introducing a symbol map from $\mathrm{SU}_q(2)$ to the cosphere bundle $\mathbb{S}_q^*$. The latter is defined by its algebra $C^\infty(\mathbb{S}_q^*)$ of "smooth functions" which is, by definition, the image of a map

$$\rho : \mathcal{B} \to C^\infty(D_{q+}^2 \times D_{q-}^2 \times \mathbb{S}^1)$$

where $D^2_{q\pm}$ are two quantum disks. One finds that an element x in the algebra $\mathcal{B}$ can be determined up to smoothing operators by $\rho(x)$.

In our present case, the cosphere bundle coincides with the one obtained in [5]; the same being true for the dimension spectrum. Indeed, using this much simpler form of operators up to smoothing ones, it is not difficult to compute the dimension spectrum and obtain simple expressions for the residues appearing in the local index formula. We find that the dimension spectrum is simple and given by the set $\{1, 2, 3\}$.

The cyclic cohomology of the algebra $\mathcal{A}(\mathrm{SU}_q(2))$ has been computed explicitly in [10], where it was found to be given in terms of a single generator. We express this element in terms of a single local cocycle similarly to the computations in [5]. But contrary to the latter, we get an extra term involving $P|D|^{-3}$ which drops in [5], being traceclass for the case considered there. Here $P = \frac{1}{2}(1 + F)$ with $F = \mathrm{Sign}\, D$, the sign of the operator D.

Finally as a simple example, we compute the Fredholm index of D coupled with the unitary representative of the generator of $K_1(\mathcal{A}(\mathrm{SU}_q(2)))$.

2 The isospectral geometry of $\mathrm{SU}_q(2)$

We recall the construction of the spectral triple $(\mathcal{A}(\mathrm{SU}_q(2)), \mathcal{H}, D)$ of [8]. Let $\mathcal{A} = \mathcal{A}(\mathrm{SU}_q(2))$ be the $*$-algebra generated by a and b, subject to the following commutation rules:

$$\begin{gathered} ba = qab, \qquad b^*a = qab^*, \qquad bb^* = b^*b, \\ a^*a + q^2b^*b = 1, \qquad aa^* + bb^* = 1. \end{gathered} \tag{2.1}$$

In the following we shall take $0 < q < 1$. Note that we have exchanged $a \leftrightarrow a^*$, $b \leftrightarrow -b$ with respect to the notation of [2] and [5].

The Hilbert space of spinors $\mathcal{H}$ has an orthonormal basis labelled as follows. For each $j = 0, \frac{1}{2}, 1, \dots$, we abbreviate $j^+ = j + \frac{1}{2}$ and $j^- = j - \frac{1}{2}$. The orthonormal basis consists of vectors $|j\mu n\uparrow\rangle$ for $j = 0, \frac{1}{2}, 1, \dots$, $\mu = -j, \dots, j$ and $n = -j^+, \dots, j^+$; together with $|j\mu n\downarrow\rangle$ for $j = \frac{1}{2}, 1, \dots$, $\mu = -j, \dots, j$ and $n = -j^-, \dots, j^-$. We adopt a vector notation by juxtaposing the pair of spinors

$$|j\mu n\rangle\!\rangle := \begin{pmatrix} |j\mu n\uparrow\rangle \\ |j\mu n\downarrow\rangle \end{pmatrix}, \tag{2.2}$$

and with the convention that the lower component is zero when $n = \pm(j + \frac{1}{2})$ or $j = 0$. In this way, we get a decomposition $\mathcal{H} = \mathcal{H}^\uparrow \oplus \mathcal{H}^\downarrow$ into subspaces spanned by the "up" and "down" kets, respectively.

The spinor representation is the $*$-representation π of $\mathcal{A}$ on $\mathcal{H}$ – denoted by π' in [8] – defined as follows. We set $\pi(a) := a_+ + a_-$ and $\pi(b) := b_+ + b_-$, where $a_\pm$ and $b_\pm$ are the following operators in $\mathcal{H}$:

$$a_+ |j\mu n\rangle\rangle := q^{(\mu+n-\frac12)/2}[j+\mu+1]^{\frac12} \begin{pmatrix} q^{-j-\frac12}\,\frac{[j+n+\frac32]^{1/2}}{[2j+2]} & 0 \\ q^{\frac12}\,\frac{[j-n+\frac12]^{1/2}}{[2j+1]\,[2j+2]} & q^{-j}\,\frac{[j+n+\frac12]^{1/2}}{[2j+1]} \end{pmatrix} |j^+\mu^+n^+\rangle\rangle,$$

$$a_- |j\mu n\rangle\rangle := q^{(\mu+n-\frac12)/2}[j-\mu]^{\frac12} \begin{pmatrix} q^{j+1}\,\frac{[j-n+\frac12]^{1/2}}{[2j+1]} & -q^{\frac12}\,\frac{[j+n+\frac12]^{1/2}}{[2j]\,[2j+1]} \\ 0 & q^{j+\frac12}\,\frac{[j-n-\frac12]^{1/2}}{[2j]} \end{pmatrix} |j^-\mu^+n^+\rangle\rangle,$$

$$b_+ |j\mu n\rangle\rangle := q^{(\mu+n-\frac12)/2}[j+\mu+1]^{\frac12} \begin{pmatrix} \frac{[j-n+\frac32]^{1/2}}{[2j+2]} & 0 \\ -q^{-j-1}\,\frac{[j+n+\frac12]^{1/2}}{[2j+1]\,[2j+2]} & q^{-\frac12}\,\frac{[j-n+\frac12]^{1/2}}{[2j+1]} \end{pmatrix} |j^+\mu^+n^-\rangle\rangle,$$

$$b_- |j\mu n\rangle\rangle := q^{(\mu+n-\frac12)/2}[j-\mu]^{\frac12} \begin{pmatrix} -q^{-\frac12}\,\frac{[j+n+\frac12]^{1/2}}{[2j+1]} & -q^{j}\,\frac{[j-n+\frac12]^{1/2}}{[2j]\,[2j+1]} \\ 0 & -\frac{[j+n-\frac12]^{1/2}}{[2j]} \end{pmatrix} |j^-\mu^+n^-\rangle\rangle. \tag{2.3}$$

Here $[N] := (q^{-N} - q^N)/(q^{-1} - q)$ is a "q-integer".

The Dirac operator D that was exhibited in [8] is diagonal in the given orthonormal basis of $\mathcal{H}$, and is one of a family of selfadjoint operators of the form

$$D|j\mu n\rangle\rangle = \begin{pmatrix} d^\uparrow j + c^\uparrow & 0 \\ 0 & d^\downarrow j + c^\downarrow \end{pmatrix} |j\mu n\rangle\rangle, \tag{2.4}$$

where $d^\uparrow, d^\downarrow, c^\uparrow, c^\downarrow$ are real numbers not depending on j, μ, n. In order that the sign of D be nontrivial we need to assume $d^\downarrow d^\uparrow < 0$, so we may as well take $d^\uparrow > 0$ and $d^\downarrow < 0$.

Apart from the issue of their signs, the particular constants that appear in (2.4) are fairly immaterial: $c^\uparrow$ and $c^\downarrow$ do not affect the index calculations later on while $d^\uparrow$ and $|d^\downarrow|$ yield scaling factors on some noncommutative integrals. Thus little generality is lost by making the following choice,

$$D|j\mu n\rangle\rangle = \begin{pmatrix} 2j + \frac32 & 0 \\ 0 & -2j - \frac12 \end{pmatrix} |j\mu n\rangle\rangle. \tag{2.5}$$

whose spectrum (with multiplicity!) coincides with that of the classical Dirac operator of the sphere $\mathbb{S}^3$ equipped with the round metric (indeed, the spin geometry of the 3-sphere can now be recovered by taking $q = 1$).

We let $D = F\,|D|$ be the polar decomposition of D where $|D| := (D^2)^{\frac12}$ and $F = \operatorname{Sign} D$. Explicitly, we see that

$$F|j\mu n\rangle\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} |j\mu n\rangle\rangle, \qquad |D|\,|j\mu n\rangle\rangle = \begin{pmatrix} 2j + \frac32 & 0 \\ 0 & 2j + \frac12 \end{pmatrix} |j\mu n\rangle\rangle.$$

Clearly, $P^\uparrow := \frac12(1 + F)$ and $P^\downarrow := \frac12(1 - F) = 1 - P^\uparrow$ are the orthogonal projectors whose range spaces are $\mathcal{H}^\uparrow$ and $\mathcal{H}^\downarrow$, respectively.

Proposition 2.1. *The triple* $(\mathcal{A}(\mathrm{SU}_q(2)), \mathcal{H}, D)$ *is a regular* 3^+*-summable spectral triple.*

Proof. It was already shown in [8] that this spectral triple is 3^+-summable: indeed, this follows easily from the growth of the eigenvalues in (2.5). The remaining issue is its regularity. Recall [1, 7, 9] that this means that the algebra generated by $\mathcal{A}$ and $[D,\mathcal{A}]$ should lie within the smooth domain $\bigcap_{n=0}^{\infty} \operatorname{Dom}\delta^n$ of the operator derivation $\delta(T) := |D|T - T|D|$.

Since $2j + \frac{3}{2} = 2j^+ + \frac{1}{2}$ and $2j + \frac{1}{2} = 2j^- + \frac{3}{2}$ and due to the triangular forms of the matrices in (2.3), the off-diagonal terms vanish in the 2×2-matrix expressions for $\delta(a_+)$ and $\delta(a_-)$. Indeed one finds,

$$\begin{aligned}
\delta(a_+)|j\mu n\rangle\rangle &= \begin{pmatrix} 2j+\frac{5}{2} & 0 \\ 0 & 2j+\frac{3}{2} \end{pmatrix} a_+|j\mu n\rangle\rangle - a_+ \begin{pmatrix} 2j+\frac{3}{2} & 0 \\ 0 & 2j+\frac{1}{2} \end{pmatrix} |j\mu n\rangle\rangle, \\
\delta(a_-)|j\mu n\rangle\rangle &= \begin{pmatrix} 2j+\frac{1}{2} & 0 \\ 0 & 2j-\frac{1}{2} \end{pmatrix} a_-|j\mu n\rangle\rangle - a_- \begin{pmatrix} 2j+\frac{3}{2} & 0 \\ 0 & 2j+\frac{1}{2} \end{pmatrix} |j\mu n\rangle\rangle.
\end{aligned}$$

In both cases we obtain

$$\delta(a_+) = P^\uparrow a_+ P^\uparrow + P^\downarrow a_+ P^\downarrow, \qquad \delta(a_-) = -P^\uparrow a_- P^\uparrow - P^\downarrow a_- P^\downarrow. \tag{2.6}$$

Replacing a by b, the same triangular matrix structure leads to

$$\delta(b_+) = P^\uparrow b_+ P^\uparrow + P^\downarrow b_+ P^\downarrow, \qquad \delta(b_-) = -P^\uparrow b_- P^\uparrow - P^\downarrow b_- P^\downarrow. \tag{2.7}$$

Thus $\delta(\pi(a)) = \delta(a_+) + \delta(a_-)$ is bounded, with $\|\delta(\pi(a))\| \leqslant \|\pi(a)\|$; and likewise for $\pi(b)$. Next, $\delta([D,a_+]) = [D,\delta(a_+)]$, so that

$$\delta([D,a_+])|j\mu n\rangle\rangle = \begin{pmatrix} 2j+\frac{5}{2} & 0 \\ 0 & -2j-\frac{3}{2} \end{pmatrix} \delta(a_+)|j\mu n\rangle\rangle - \delta(a_+) \begin{pmatrix} 2j+\frac{3}{2} & 0 \\ 0 & -2j-\frac{1}{2} \end{pmatrix} |j\mu n\rangle\rangle,$$

since all matrices appearing are diagonal. This, together with the analogous calculation for $\delta([D,a_-])$, shows that

$$\delta([D,a_+]) = P^\uparrow a_+ P^\uparrow - P^\downarrow a_+ P^\downarrow, \qquad \delta([D,a_-]) = P^\uparrow a_- P^\uparrow - P^\downarrow a_- P^\downarrow. \tag{2.8}$$

A similar argument for b gives

$$\delta([D,b_+]) = P^\uparrow b_+ P^\uparrow - P^\downarrow b_+ P^\downarrow, \qquad \delta([D,b_-]) = P^\uparrow b_- P^\uparrow - P^\downarrow b_- P^\downarrow. \tag{2.9}$$

Combining (2.6), (2.8), and the analogous relations with a replaced by b, we see that both $\mathcal{A}$ and $[D,\mathcal{A}]$ lie within $\operatorname{Dom}\delta$. An easy induction shows that they also lie within $\operatorname{Dom}\delta^k$ for $k = 2,3,\dots$. □

This proposition continues to hold if we replace $\mathcal{A}(\mathrm{SU}_q(2))$ by a suitably completed algebra, which is stable under the holomorphic function calculus.

► Let $\Psi^0(\mathcal{A})$ be the algebra generated by $\delta^k(\mathcal{A})$ and $\delta^k([D,\mathcal{A}])$ for all $k \geqslant 0$ (the notation suggests that, in the spirit of [7] one thinks of it as an "algebra of pseudodifferential operators of order 0"). Since, for instance,

$$\begin{aligned}
P^\uparrow \pi(a) P^\uparrow &= \tfrac{1}{2}\delta^2(\pi(a)) + \tfrac{1}{2}\delta([D,\pi(a)]), \\
P^\uparrow a_+ P^\uparrow &= \tfrac{1}{2}P^\uparrow \pi(a) P^\uparrow + \tfrac{1}{2}P^\uparrow \delta(\pi(a)) P^\uparrow,
\end{aligned}$$

we see that $\Psi^0(\mathcal{A})$ is in fact generated by the diagonal-corner operators $P^\uparrow a_\pm P^\uparrow$, $P^\downarrow a_\pm P^\downarrow$, $P^\uparrow b_\pm P^\uparrow$, $P^\downarrow b_\pm P^\downarrow$ together with the other-corner operators $P^\downarrow a_+ P^\uparrow$, $P^\uparrow a_- P^\downarrow$, $P^\downarrow b_+ P^\uparrow$, and $P^\uparrow b_- P^\downarrow$. Following [5], let $\mathcal{B}$ be the algebra generated by all $\delta^n(\mathcal{A})$ for $n \geqslant 0$. It is a subalgebra of $\Psi^0(\mathcal{A})$ and it is generated by the diagonal operators

$$\tilde{a}_\pm := \pm\delta(a_\pm) = P^\uparrow a_\pm P^\uparrow + P^\downarrow a_\pm P^\downarrow, \qquad \tilde{b}_\pm := \pm\delta(b_\pm) = P^\uparrow b_\pm P^\uparrow + P^\downarrow b_\pm P^\downarrow, \tag{2.10}$$

and by the off-diagonal operators $P^\downarrow a_+ P^\uparrow + P^\uparrow a_- P^\downarrow$ and $P^\downarrow b_+ P^\uparrow + P^\uparrow b_- P^\downarrow$.

► For later convenience we shall introduce an approximate representation $\underline{\pi}$ found in [8], which coincides with π up to compact operators. Note first that the off-diagonal coefficients in (2.3) give rise to smoothing operators in $\mathrm{OP}^{-\infty}$ (see Appendix A), due to the terms appearing in their denominators; we can furthermore simplify the diagonal terms.

We set $\underline{\pi}(a) := \underline{a}_+ + \underline{a}_-$ and $\underline{\pi}(b) := \underline{b}_+ + \underline{b}_-$ with the following definitions:

$$\begin{aligned}
\underline{a}_+ \,|j\mu n\rangle\!\rangle &:= \sqrt{1-q^{2j+2\mu+2}} \begin{pmatrix} \sqrt{1-q^{2j+2n+3}} & 0 \\ 0 & \sqrt{1-q^{2j+2n+1}} \end{pmatrix} |j^+\mu^+n^+\rangle\!\rangle, \\
\underline{a}_- \,|j\mu n\rangle\!\rangle &:= q^{2j+\mu+n+\frac12} \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix} |j^-\mu^+n^+\rangle\!\rangle, \\
\underline{b}_+ \,|j\mu n\rangle\!\rangle &:= q^{j+n-\frac12} \sqrt{1-q^{2j+2\mu+2}} \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix} |j^+\mu^+n^-\rangle\!\rangle, \\
\underline{b}_- \,|j\mu n\rangle\!\rangle &:= -q^{j+\mu} \begin{pmatrix} \sqrt{1-q^{2j+2n+1}} & 0 \\ 0 & \sqrt{1-q^{2j+2n-1}} \end{pmatrix} |j^-\mu^+n^-\rangle\!\rangle.
\end{aligned} \tag{2.11}$$

These formulas can be obtained from (2.3) by truncation, using the pair of estimates

$$\begin{aligned}
\big((q^{-1}-q)[n]\big)^{-1} - q^n &= q^{3n} + O(q^{5n}), \\
1 - \sqrt{1-q^\alpha} &\leqslant q^\alpha, \quad \text{for any} \quad \alpha \geqslant 0.
\end{aligned}$$

The operators $\underline{\pi}(x) - \pi(x)$ are given by sequences of rapid decay, and hence are elements in $\mathrm{OP}^{-\infty}$ (as defined in Appendix A). Therefore, we can replace π by $\underline{\pi}$ when dealing with the local cocycle in the local index theorem in the next section.

Remark 1. These operators differ slightly from the approximate representation given in [8]. Using the inequality $1 - \sqrt{1-q^\alpha} \leqslant q^\alpha$, they can be seen to differ from the operators therein by a compact operator in the principal ideal $\mathcal{K}_q$ generated by the operator $L_q\colon |j\mu n\rangle\!\rangle \mapsto q^j |j\mu n\rangle\!\rangle$. Note that $\mathcal{K}_q \subset \mathrm{OP}^{-\infty}$.

Now, observe that

$$\begin{aligned}
[|D|, \underline{\pi}(a)] &= \underline{a}_+ - \underline{a}_-, \qquad & [D, \underline{\pi}(a)] &= F(\underline{a}_+ - \underline{a}_-), \\
[|D|, \underline{\pi}(b)] &= \underline{b}_+ - \underline{b}_-, \qquad & [D, \underline{\pi}(b)] &= F(\underline{b}_+ - \underline{b}_-),
\end{aligned} \tag{2.12}$$

and also that F commutes with $\underline{a}_\pm$ and $\underline{b}_\pm$. The operators $\underline{a}_\pm$ and $\underline{b}_\pm$ have a simpler expression if we use the following relabelling of the orthonormal basis of $\mathcal{H}$,

$$\begin{aligned}
v^j_{xy\uparrow} &:= |j, x-j, y-j-\tfrac12, \uparrow\rangle \quad \text{for} \quad x = 0,\dots,2j;\ y = 0,\dots,2j+1, \\
v^j_{xy\downarrow} &:= |j, x-j, y-j+\tfrac12, \downarrow\rangle \quad \text{for} \quad x = 0,\dots,2j;\ y = 0,\dots,2j-1.
\end{aligned} \tag{2.13}$$

We again employ the pairs of vectors

$$v_{xy}^j := \begin{pmatrix} v_{xy\uparrow}^j \\ v_{xy\downarrow}^j \end{pmatrix},$$

where the lower component is understood to be zero if $y = 2j$ or $2j+1$, or if $j = 0$. The simplification is that on these vector pairs, all the 2×2 matrices in (2.11) become scalar matrices,

$$\begin{aligned}
\underline{a}_+ v_{xy}^j &= \sqrt{1-q^{2x+2}}\sqrt{1-q^{2y+2}}\, v_{x+1,y+1}^{j^+}, \\
\underline{a}_- v_{xy}^j &= q^{x+y+1}\, v_{xy}^{j^-}, \\
\underline{b}_+ v_{xy}^j &= q^y \sqrt{1-q^{2x+2}}\, v_{x+1,y}^{j^+}, \\
\underline{b}_- v_{xy}^j &= -q^x \sqrt{1-q^{2y}}\, v_{x,y-1}^{j^-}.
\end{aligned} \tag{2.14}$$

These formulas coincide with those found in [5, Sec. 6] up to a doubling of the Hilbert space and the change of conventions $a \leftrightarrow a^*$, $b \leftrightarrow -b$. Indeed, since the spin representation is isomorphic to a direct sum of two copies of the regular representation, the formulas in (2.14) exhibit the same phenomenon for the approximate representations.

3 The cosphere bundle

In [5] Connes constructs a "cosphere bundle" using the regular representation of $\mathcal{A}(\mathrm{SU}_q(2))$. In view of (2.14), the same cosphere bundle may be obtained directly from the spin representation by adapting that construction, as we now proceed to do. In what follows, we use the algebra $\mathcal{A} = \mathcal{A}(\mathrm{SU}_q(2))$, but we could as well replace it with its completion $C^\infty(\mathrm{SU}_q(2))$, which is closed under holomorphic functional calculus (see Appendix A).

We recall two well-known infinite dimensional representations $\pi_\pm$ of $\mathcal{A}(\mathrm{SU}_q(2))$ by bounded operators on the Hilbert space $\ell^2(\mathbb{N})$. On the standard orthonormal basis $\{\, \varepsilon_x : x \in \mathbb{N} \,\}$, they are given by

$$\pi_\pm(a)\, \varepsilon_x := \sqrt{1-q^{2x+2}}\, \varepsilon_{x+1}, \qquad \pi_\pm(b)\, \varepsilon_x := \pm q^x\, \varepsilon_x. \tag{3.1}$$

We may identify the Hilbert space $\mathcal{H}$ spanned by all $v_{xy\uparrow}^j$ and $v_{xy\downarrow}^j$ with the subspace $\mathcal{H}'$ of $\ell^2(\mathbb{N})_x \otimes \ell^2(\mathbb{N})_y \otimes \ell^2(\mathbb{Z})_{2j} \otimes \mathbb{C}^2$ determined by the parameter restrictions in (2.13). Thereby, we get the correspondence

$$\begin{aligned}
\underline{a}_+ &\leftrightarrow \pi_+(a) \otimes \pi_-(a) \otimes V \otimes 1_2, \\
\underline{a}_- &\leftrightarrow -q\, \pi_+(b) \otimes \pi_-(b^*) \otimes V^* \otimes 1_2, \\
\underline{b}_+ &\leftrightarrow -\pi_+(a) \otimes \pi_-(b) \otimes V \otimes 1_2, \\
\underline{b}_- &\leftrightarrow -\pi_+(b) \otimes \pi_-(a^*) \otimes V^* \otimes 1_2,
\end{aligned} \tag{3.2}$$

where V is the unilateral shift operator $\varepsilon_{2j} \mapsto \varepsilon_{2j+1}$ in $\ell^2(\mathbb{Z})$. This again, apart from the 2×2 identity matrix 1_2, coincides with the formula (204) in [5], up to the aforementioned exchange of the generators.

The shift V in the action of the operators $\underline{a}_\pm$ and $\underline{b}_\pm$ on $\mathcal{H}$ can be encoded using the $\mathbb{Z}$-grading coming from the one-parameter group of automorphisms $\gamma(t)$ generated by $|D|$,

$$\gamma(t) = \begin{pmatrix} \gamma_{\uparrow\uparrow}(t) & \gamma_{\uparrow\downarrow}(t) \\ \gamma_{\downarrow\uparrow}(t) & \gamma_{\downarrow\downarrow}(t) \end{pmatrix}, \quad \text{where} \quad \begin{cases} \gamma_{\uparrow\uparrow}(t) : P^\uparrow T P^\uparrow \mapsto P^\uparrow e^{it|D|} T e^{-it|D|} P^\uparrow, \\ \gamma_{\uparrow\downarrow}(t) : P^\uparrow T P^\downarrow \mapsto P^\uparrow e^{it|D|} T e^{-it|D|} P^\downarrow, \\ \gamma_{\downarrow\uparrow}(t) : P^\downarrow T P^\uparrow \mapsto P^\downarrow e^{it|D|} T e^{-it|D|} P^\uparrow, \\ \gamma_{\downarrow\downarrow}(t) : P^\downarrow T P^\downarrow \mapsto P^\downarrow e^{it|D|} T e^{-it|D|} P^\downarrow, \end{cases} \tag{3.3}$$

for any operator T on $\mathcal{H}$. On the subalgebra of "diagonal" operators $T = P^\uparrow T P^\uparrow + P^\downarrow T P^\downarrow$, the compression $\gamma_{\uparrow\uparrow} \oplus \gamma_{\downarrow\downarrow}$ detects the shift of j of the restrictions of T to $\mathcal{H}^\uparrow$ and $\mathcal{H}^\downarrow$ respectively. For example, $\gamma_{\uparrow\uparrow}(t) \oplus \gamma_{\downarrow\downarrow}(t) : a_\pm \mapsto e^{\pm it} a_\pm$, so that the $\mathbb{Z}$-grading encodes the correct shifts $j \to j \pm \frac{1}{2}$ in the formulas for $a_\pm$; and likewise for $b_\pm$.

From equation (3.1) it follows that $b - b^* \in \ker \pi_\pm$, and so the representations $\pi_\pm$ are not faithful on $\mathcal{A}(\mathrm{SU}_q(2))$. We define two algebras $\mathcal{A}(D^2_{q\pm})$ to be the corresponding quotients,

$$0 \to \ker \pi_\pm \to \mathcal{A}(\mathrm{SU}_q(2)) \xrightarrow{r_\pm} \mathcal{A}(D^2_{q\pm}) \to 0. \tag{3.4}$$

We elaborate a little on the structure of the algebras $\mathcal{A}(D^2_{q\pm})$. For convenience, we shall omit the quotient maps $r_\pm$ in this discussion. Then $b = b^*$ in $\mathcal{A}(D^2_{q\pm})$, and from the defining relations (2.1) of $\mathcal{A}(\mathrm{SU}_q(2))$, we obtain

$$ba = q\, ab, \qquad a^* b = q\, ba^*, \qquad a^* a + q^2 b^2 = 1, \qquad aa^* + b^2 = 1. \tag{3.5}$$

These algebraic relations define two isomorphic quantum 2-spheres $\mathbb{S}^2_{q+} \simeq \mathbb{S}^2_{q-} =: \mathbb{S}^2_q$ which have a classical subspace $\mathbb{S}^1$ given by the characters $b \mapsto 0$, $a \mapsto \lambda$ with $|\lambda| = 1$. A substitution $q \mapsto q^2$, followed by $b \mapsto q^{-2} b$ shows that $\mathbb{S}^2_q$ is none other than the equatorial Podleś sphere [11]. Thus, the above quotients of $\mathcal{A}(\mathrm{SU}_q(2))$ with respect to $\ker \pi_\pm$ either coincide with $\mathcal{A}(\mathbb{S}^2_q)$ or are quotients of it. Now, from (3.1) one sees that the spectrum of $\pi_\pm(b)$ is either real positive or real negative, depending on the $\pm$ sign. Hence, the algebras $\mathcal{A}(D^2_{q+})$ and $\mathcal{A}(D^2_{q-})$ describe the two hemispheres of $\mathbb{S}^2_q$ and may be thought of as quantum disks, thus justifying the notation $D_{q\pm}$.

There is a symbol map $\sigma \colon \mathcal{A}(D^2_{q\pm}) \to \mathcal{A}(\mathbb{S}^1)$ that maps these "noncommutative disks" to their common boundary $\mathbb{S}^1$, which is the equator of the equatorial Podleś sphere $\mathbb{S}^2_q$. Explicitly, the symbol map is given as a $*$-homomorphism on the generators of $\mathcal{A}(D^2_{q,\pm})$ by

$$\sigma(r_\pm(a)) := u; \qquad \sigma(r_\pm(b)) := 0, \tag{3.6}$$

where u is the unitary generator of $\mathcal{A}(\mathbb{S}^1)$.

Recall the algebra $\mathcal{B}$ defined around (2.10) with generators $\tilde{a}_\pm$, $\tilde{b}_\pm$ and $P^\downarrow a_+ P^\uparrow + P^\uparrow a_- P^\downarrow$, $P^\downarrow b_+ P^\uparrow + P^\uparrow b_- P^\downarrow$. The following result emulates Proposition 4 of [5] and establishes the correspondence (3.2). The results of [8] on the approximate representation are crucial to its proof.

Proposition 3.1. *There is a $*$-homomorphism*

$$\rho : \mathcal{B} \to \mathcal{A}(D^2_{q+}) \otimes \mathcal{A}(D^2_{q-}) \otimes \mathcal{A}(\mathbb{S}^1) \tag{3.7}$$

defined on generators by

$$\rho(\tilde{a}_+) := r_+(a) \otimes r_-(a) \otimes u, \qquad \rho(\tilde{a}_-) := -q\, r_+(b) \otimes r_-(b^*) \otimes u^*,$$
$$\rho(\tilde{b}_+) := -r_+(a) \otimes r_-(b) \otimes u, \qquad \rho(\tilde{b}_-) := -r_+(b) \otimes r_-(a^*) \otimes u^*.$$

while the off-diagonal operators $P^\downarrow a_+ P^\uparrow + P^\uparrow a_- P^\downarrow$ and $P^\downarrow b_+ P^\uparrow + P^\uparrow b_- P^\downarrow$ are declared to lie in the kernel of ρ.

Proof. First note that the j-dependence of the operators in $\mathcal{B}$ is taken care of by the factor u. Thus, it is enough to show that the following prescription,

$$\rho_1(\tilde{a}_+) := \pi_+(a) \otimes \pi_-(a), \qquad \rho_1(\tilde{a}_-) := -q\, \pi_+(b) \otimes \pi_-(b^*),$$
$$\rho_1(\tilde{b}_+) := -\pi_+(a) \otimes \pi_-(b), \qquad \rho_1(\tilde{b}_-) := -\pi_+(b) \otimes \pi_-(a^*),$$

together with $\rho_1(P^\downarrow a_+ P^\uparrow + P^\uparrow a_- P^\downarrow) = \rho_1(P^\downarrow b_+ P^\uparrow + P^\uparrow b_- P^\downarrow) := 0$, defines a $*$-homomorphism $\rho_1 : \mathcal{B} \to \mathcal{A}(D^2_{q+}) \otimes \mathcal{A}(D^2_{q-})$. In the notation, we have replaced the representations $\pi_\pm$ of $\mathcal{A}(\mathrm{SU}_q(2))$ by corresponding faithful representations of $\mathcal{A}(D^2_{q\pm})$ (omitting the maps $r_\pm$).

We define a map $\Pi : \mathcal{H} \to (\ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N})) \otimes \mathbb{C}^2$, which simply forgets the j-index on the basis vectors v^j_{xy}:

$$\Pi : v^j_{xy} = \begin{pmatrix} v^j_{xy\uparrow} \\ v^j_{xy\downarrow} \end{pmatrix} \mapsto \varepsilon_{xy} := \begin{pmatrix} \varepsilon_{xy\uparrow} \\ \varepsilon_{xy\downarrow} \end{pmatrix},$$

where $\varepsilon_{xy\uparrow} := \varepsilon_x \otimes \varepsilon_y$ and $\varepsilon_{xy\downarrow} := \varepsilon_x \otimes \varepsilon_y$ in the two respective copies of $\ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N})$ in its tensor product with $\mathbb{C}^2$.

For any operator T in $\mathcal{B}$, we define the map ρ_1 by

$$\rho_1(T)\varepsilon_{xy} = \lim_{j\to\infty} \Pi(T v^j_{xy}). \tag{3.8}$$

This map is well-defined, since T is a polynomial in the generators of $\mathcal{B}$. Each such generator shifts the indices x, y, j by $\pm\frac{1}{2}$, with a coefficient matrix that can be bounded uniformly in x, y and j (cf. [8]) so that the limit $j \to \infty$ exists.

First of all, it can be directly verified, using estimates given in [8, Sec. 7], that the off-diagonal operators $P^\downarrow a_+ P^\uparrow + P^\uparrow a_- P^\downarrow$ and $P^\downarrow b_+ P^\uparrow + P^\uparrow b_- P^\downarrow$ are in the kernel of ρ_1. Next, the differences between the generators and the approximate generators $\underline{a}_\pm - \tilde{a}_\pm$ (and similarly $\tilde{b}_\pm - \underline{b}_\pm$) lie in the kernel of ρ_1, as well. Hence we can replace $\tilde{a}_\pm$ and $\tilde{b}_\pm$ by $\underline{a}_\pm$ and $\underline{b}_\pm$, respectively.

Since the coefficients in the definition of $\underline{a}_\pm$ and $\underline{b}_\pm$ (equation (2.14)) are j-independent, we conclude that ρ_1 is of the desired form. For example, we compute:

$$\begin{aligned}\rho_1(\tilde{a}_+)\varepsilon_{xy} = \rho_1(\underline{a}_+)\varepsilon_{xy} &= \lim_{j\to\infty} \sqrt{1 - q^{2x+2}}\sqrt{1 - q^{2y+2}}\,\Pi(v^{j^+}_{x+1,y+1}) \\ &= \sqrt{1 - q^{2x+2}}\sqrt{1 - q^{2y+2}}\,\varepsilon_{x+1,y+1} = (\pi_+(a) \otimes \pi_-(a) \otimes 1_2)\varepsilon_{xy}.\end{aligned}$$

Since a product of the operators $\underline{a}_\pm$ and $\underline{b}_\pm$ still does not contain j-dependent coefficients, ρ_1 respects the multiplication in $\mathcal{B}$. By linearity of the limit, ρ_1 is an algebra map. □

Definition 3.2. The *cosphere bundle on* $\mathrm{SU}_q(2)$ is defined as the range of the map ρ in $\mathcal{A}(D^2_{q+}) \otimes A(D^2_{q-}) \otimes \mathcal{A}(\mathbb{S}^1)$ and is denoted by $\mathcal{A}(\mathbb{S}^*_q)$.

Note that $\mathbb{S}^*_q$ coincides with the cosphere bundle defined in [5, 6], where it is regarded as a noncommutative space over which $D^2_{q+} \times D^2_{q-} \times \mathbb{S}^1$ is fibred.

The symbol map ρ rectifies the correspondence (3.2). Denote by Q the orthogonal projector on $\ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{Z}) \otimes \mathbb{C}^2$ with range $\mathcal{H}'$, which is the Hilbert subspace previously identified with $\mathcal{H}$ just before (3.2). Using (3.2) in combination with Proposition 3.1, we conclude that

$$T - Q(\rho(T) \otimes 1_2)Q \in \mathrm{OP}^{-\infty} \quad \text{for all} \quad T \in \mathcal{B}. \tag{3.9}$$

Here, the action of $\rho(T)$ on $\ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{Z})$ is determined by regarding $\ell^2(\mathbb{Z})$ as the Hilbert space of square-summable Fourier series on $\mathbb{S}^1$.

4 The dimension spectrum

We again follow [5] for the computation of the dimension spectrum. We define three linear functionals $\tau_0^\uparrow$, $\tau_0^\downarrow$ and τ_1 on the algebras $\mathcal{A}(D^2_{q\pm})$. Since their definitions for both disks D^2_{q+} and D^2_{q-} are identical, we shall omit the $\pm$ for notational convenience.

For $x \in \mathcal{A}(D^2_q)$ we define,

$$\begin{aligned}
\tau_1(x) &:= \frac{1}{2\pi} \int_{S^1} \sigma(x), \\
\tau_0^\uparrow(x) &:= \lim_{N\to\infty} \mathrm{Tr}_N\, \pi(x) - (N + \tfrac{3}{2})\tau_1(x), \\
\tau_0^\downarrow(x) &:= \lim_{N\to\infty} \mathrm{Tr}_N\, \pi(x) - (N + \tfrac{1}{2})\tau_1(x),
\end{aligned}$$

where σ is the symbol map (3.6), and Tr_N is the truncated trace

$$\mathrm{Tr}_N(T) := \sum_{k=0}^{N} \langle \varepsilon_k \mid T\varepsilon_k \rangle.$$

The definition of the two different maps $\tau_0^\uparrow$ and $\tau_0^\downarrow$ is suggested by the constants $\frac{3}{2}$ and $\frac{1}{2}$ appearing in our choice of the Dirac operator; it will simplify some residue formulas later on. We find that

$$\begin{aligned}
\mathrm{Tr}_N(\pi(a)) &= (N + \tfrac{3}{2})\tau_1(a) + \tau_0^\uparrow(a) + O(N^{-k}) \\
&= (N + \tfrac{1}{2})\tau_1(a) + \tau_0^\downarrow(a) + O(N^{-k}) \quad \text{for all} \quad k > 0.
\end{aligned}$$

Let us denote by r the restriction homomorphism from $\mathcal{A}(D^2_{q+}) \otimes A(D^2_{q-}) \otimes \mathcal{A}(\mathbb{S}^1)$ onto the first two legs of the tensor product. In particular, we will use it as a map

$$r : \mathcal{A}(\mathbb{S}^*_q) \to \mathcal{A}(D^2_{q+}) \otimes A(D^2_{q-}).$$

In the following, we adopt the notation [7]:

$$⨍ T := \mathrm{Res}_{z=0}\, \mathrm{Tr}\, T|D|^{-z}.$$

Theorem 4.1. *The dimension spectrum of the spectral triple* $(\mathcal{A}(\mathrm{SU}_q(2)), \mathcal{H}, D)$ *is simple and given by* $\{1,2,3\}$*; the corresponding residues are*

$$\begin{aligned}
⨍ T|D|^{-3} &= 2(\tau_1 \otimes \tau_1)\big(r\rho(T)^0\big), \\
⨍ T|D|^{-2} &= \big(\tau_1 \otimes (\tau_0^{\uparrow} + \tau_0^{\downarrow}) + (\tau_0^{\uparrow} + \tau_0^{\downarrow}) \otimes \tau_1\big)\big(r\rho(T)^0\big), \\
⨍ T|D|^{-1} &= (\tau_0^{\uparrow} \otimes \tau_0^{\downarrow} + \tau_0^{\downarrow} \otimes \tau_0^{\uparrow})\big(r\rho(T)^0\big),
\end{aligned}$$

with $T \in \Psi^0(\mathcal{A})$.

Proof. If we identify $\mathcal{H}' \subset \ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{Z}) \otimes \mathbb{C}^2$ with $\mathcal{H}$ as above, the one-parameter group of automorphisms $\gamma(t)$ induces a $\mathbb{Z}$-grading on $\mathcal{A}(\mathbb{S}_q^*)$, in its representation on $\mathcal{H}'$. We denote by $\rho(T)^0$ the degree-zero part of the diagonal operator $\rho(T)$, for $T \in \mathcal{B}$. For the calculation of the dimension spectrum we need to find the poles of the zeta function $\zeta_T(z) := \mathrm{Tr}(T|D|^{-z})$ for all $T \in \Psi^0(\mathcal{A})$. From our discussion of the generators of $\Psi^0(\mathcal{A})$, we see that we only need to adjoin $P^{\uparrow}\mathcal{B}$ to $\mathcal{B}$.

In the zeta function $\zeta_T(z)$ for $T \in \mathcal{B}$, we can replace T by $Q(\rho(T) \otimes 1_2)Q$ since their difference is a smoothing operator by (3.9). The operator $Q(\rho(T) \otimes 1_2)Q$ commutes with the projector $P^{\uparrow}$ so we can first calculate

$$\begin{aligned}
\mathrm{Tr}(P^{\uparrow}Q(\rho(T) \otimes 1_2)Q\,|D|^{-z}) &= \sum_{2j=0}^{\infty} (2j + \tfrac{3}{2})^{-z} (\mathrm{Tr}_{2j} \otimes \mathrm{Tr}_{2j+1})(r\rho(T)^0) \\
&= (\tau_1 \otimes \tau_1)(r\rho(T)^0)\,\zeta(z-2) + (\tau_1 \otimes \tau_0^{\downarrow} + \tau_0^{\uparrow} \otimes \tau_1)(r\rho(T)^0)\,\zeta(z-1) \\
&\quad + (\tau_0^{\uparrow} \otimes \tau_0^{\downarrow})(r\rho(T)^0)\,\zeta(z) + f_{\uparrow}(z),
\end{aligned} \tag{4.1}$$

where $f_{\uparrow}(z)$ is holomorphic in $z \in \mathbb{C}$. Similarly,

$$\begin{aligned}
\mathrm{Tr}(P^{\downarrow}Q(\rho(T) \otimes 1_2)Q\,|D|^{-z}) &= \sum_{2j=0}^{\infty} (2j + \tfrac{3}{2})^{-z} (\mathrm{Tr}_{2j+1} \otimes \mathrm{Tr}_{2j})(r\rho(T)^0) \\
&= (\tau_1 \otimes \tau_1)(r\rho(T)^0)\,\zeta(z-2) + (\tau_1 \otimes \tau_0^{\uparrow} + \tau_0^{\downarrow} \otimes \tau_1)(r\rho(T)^0)\,\zeta(z-1) \\
&\quad + (\tau_0^{\downarrow} \otimes \tau_0^{\uparrow})(r\rho(T)^0)\,\zeta(z) + f_{\downarrow}(z),
\end{aligned} \tag{4.2}$$

where $f_{\downarrow}(z)$ is holomorphic in z. Since $\zeta(z)$ has a simple pole at $z = 1$, we see that the zeta function ζ_T has simple poles at 1, 2 and 3. □

From the above proof, we derive the following formulas which will be used later on:

$$\begin{aligned}
⨍ P^{\uparrow}T|D|^{-3} &= (\tau_1 \otimes \tau_1)\big(r\rho(T)^0\big), \\
⨍ P^{\uparrow}T|D|^{-2} &= \big(\tau_1 \otimes \tau_0^{\downarrow} + \tau_0^{\uparrow} \otimes \tau_1\big)\big(r\rho(T)^0\big), \\
⨍ P^{\uparrow}T|D|^{-1} &= (\tau_0^{\uparrow} \otimes \tau_0^{\downarrow})\big(r\rho(T)^0\big),
\end{aligned} \tag{4.3}$$

with T any element in $\Psi^0(\mathcal{A})$.

5 Local index formula ($d = 3$)

We begin by discussing the local cyclic cocycles giving the local index formula, in the general case when the spectral triple $(\mathcal{A}, \mathcal{H}, D)$ has simple discrete dimension spectrum not containing 0 and bounded above by 3.

Let us recall that with a general (odd) spectral triple $(\mathcal{A}, \mathcal{H}, D)$ there comes a Fredholm index of the operator D as an additive map $\varphi : K_1(\mathcal{A}) \to \mathbb{Z}$ defined as follows. If $F = \operatorname{Sign} D$ and P is the projector $P = \frac{1}{2}(1 + F)$, then

$$\varphi([u]) = \operatorname{Index}(PuP), \tag{5.1}$$

with $u \in \operatorname{Mat}_r(\mathcal{A})$ a unitary representative of the K_1 class (the operator PuP is automatically Fredholm). The above map is computed by pairing $K_1(\mathcal{A})$ with "nonlocal" cyclic cocycles χ_n given in terms of the operator F and of the form

$$\chi_n(a_0, \dots, a_n) = \lambda_n \operatorname{Tr}(a_0 [F, a_1] \cdots [F, a_n]), \quad \text{for all} \quad a_j \in \mathcal{A}, \tag{5.2}$$

where λ_n is a suitable normalization constant. The choice of the integer n is determined by the degree of summability of the Fredholm module $(\mathcal{H}, F)$ over $\mathcal{A}$; any such module is declared to be p-summable if the commutator $[F, a]$ is an element in the p-th Schatten ideal $\mathcal{L}^p(\mathcal{H})$, for any $a \in A$. The minimal n in (5.2) needs to be taken such that $n \geqslant p$.

On the other hand, the Connes–Moscovici local index theorem [7] expresses the index map in terms of a local cocycle ϕ_{odd} in the (b, B) bicomplex of $\mathcal{A}$ which is a local representative of the cyclic cohomology class of χ_n (the cyclic cohomology Chern character). The cocycle ϕ_{odd} is given in terms of the operator D and is made of a finite number of terms $\phi_{\text{odd}} = (\phi_1, \phi_3, \dots)$; the pairing of the cyclic cohomology class $[\phi_{\text{odd}}] \in HC^{\text{odd}}(\mathcal{A})$ with $K_1(\mathcal{A})$ gives the Fredholm index (5.1) of D with coefficients in $K_1(\mathcal{A})$. The components of the cyclic cocycle ϕ_{odd} are explicitly given in [7]; we shall presently give them for our case.

We know from Proposition 2.1 that our spectral triple $(\mathcal{A}, \mathcal{H}, D)$ with $\mathcal{A} = \mathcal{A}(\mathrm{SU}_q(2))$ has metric dimension equal to 3. As for the corresponding Fredholm module $(\mathcal{H}, F)$ over $\mathcal{A} = \mathcal{A}(\mathrm{SU}_q(2))$, it is 1-summable since all commutators $[F, \pi(x)]$, with $x \in \mathcal{A}$, are off-diagonal operators given by sequences of rapid decay. Hence each $[F, \pi(x)]$ is trace-class and we need only the first Chern character $\chi_1(a_0, a_1) = \operatorname{Tr}(a_0 [F, a_1])$, with $a_1, a_2 \in \mathcal{A}$ (we shall omit discussing the normalization constant for the time being and come back to it in the next section). An explicit expression for this cyclic cocycle on the PBW-basis of $\mathrm{SU}_q(2)$ was obtained in [10].

The local cocycle has two components, $\phi_{\text{odd}} = (\phi_1, \phi_3)$, the cocycle condition $(b + B)\phi_{\text{odd}} = 0$ reading $B\phi_1 = 0$, $b\phi_1 + B\phi_3 = 0$, $b\phi_3 = 0$ (see Appendix A); it is explicitly given by

$$\begin{aligned}
\phi_1(a_0, a_1) &:= \fint a_0 [D, a_1] |D|^{-1} - \frac{1}{4} \fint a_0 \nabla([D, a_1]) |D|^{-3} + \frac{1}{8} \fint a_0 \nabla^2([D, a_1]) |D|^{-5}, \\
\phi_3(a_0, a_1, a_2, a_3) &:= \frac{1}{12} \fint a_0 [D, a_1] [D, a_2] [D, a_3] |D|^{-3},
\end{aligned}$$

where $\nabla(T) := [D^2, T]$ for any operator T on $\mathcal{H}$. Under the assumption that $[F, a]$ is traceclass for each $a \in \mathcal{A}$, these expressions can be rewritten as follows:

$$\begin{aligned}
\phi_1(a_0, a_1) &= \fint a_0\, \delta(a_1) F|D|^{-1} - \frac{1}{2} \fint a_0\, \delta^2(a_1) F|D|^{-2} + \frac{1}{4} \fint a_0\, \delta^3(a_1) F|D|^{-3}, \\
\phi_3(a_0, a_1, a_2, a_3) &= \frac{1}{12} \fint a_0\, \delta(a_1)\, \delta(a_2)\, \delta(a_3) F|D|^{-3}.
\end{aligned} \tag{5.3}$$

We now quote Proposition 2 of [5], referring to that paper for its proof.

Proposition 5.1. *Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple with discrete simple dimension spectrum not containing* 0 *and bounded above by* 3. *If $[F, a]$ is trace-class for all $a \in \mathcal{A}$, then the Chern character χ_1 is equal to $\phi_{\mathrm{odd}} - (b + B)\phi_{\mathrm{ev}}$ where the cochain $\phi_{\mathrm{ev}} = (\phi_0, \phi_2)$ is given by*

$$\phi_0(a) := \operatorname{Tr}(Fa\,|D|^{-z})\big|_{z=0},$$
$$\phi_2(a_0, a_1, a_2) := \frac{1}{24} \fint a_0\, \delta(a_1)\, \delta^2(a_2) F|D|^{-3}.$$

The absence of 0 in the dimension spectrum is needed for the definition of ϕ_0. The cochain $\phi_{\mathrm{ev}} = (\phi_0, \phi_2)$ was named η-cochain in [5]. In components, the equivalence of the characters means that

$$\phi_1 = \chi_1 + b\phi_0 + B\phi_2\,, \qquad \phi_3 = b\phi_2\,.$$

The following general result, in combination with the above proposition, shows that χ_1 can be given (up to coboundaries) in terms of one single (b, B)-cocycle ψ_1.

Proposition 5.2. *Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple with discrete simple dimension spectrum not containing* 0 *and bounded above by* 3. *Assume that $[F, a]$ is trace class for all $a \in \mathcal{A}$, and set $P := \frac{1}{2}(1 + F)$. Then, the local Chern character ϕ_{odd} is equal to $\psi_1 - (b + B)\phi'_{\mathrm{ev}}$, where*

$$\psi_1(a_0, a_1) := 2 \fint a_0\, \delta(a_1) P|D|^{-1} - \fint a_0\, \delta^2(a_1) P|D|^{-2} + \frac{2}{3} \fint a_0\, \delta^3(a_1) P|D|^{-3},$$

and $\phi'_{\mathrm{ev}} = (\phi'_0, \phi'_2)$ is given by

$$\phi'_0(a) := \operatorname{Tr}(a\,|D|^{-z})\big|_{z=0},$$
$$\phi'_2(a_0, a_1, a_2) := -\frac{1}{24} \fint a_0\, \delta(a_1)\, \delta^2(a_2) F|D|^{-3}.$$

Proof. One needs to verify the following equalities between cochains in the (b, B) bicomplex:

$$\phi_1 + b\phi'_0 + B\phi'_2 = \psi_1, \qquad \phi_3 + b\phi'_2 = 0.$$

The second equality follows from a direct computation of $b\phi'_2$ and comparing with equation (5.3). Note that this identity proves that ψ_1 is indeed a cyclic cocycle. One also shows that

$$B\phi'_2(a_0, a_1) = \frac{1}{12} \fint a_0\, \delta^3(a_1) F|D|^{-3}.$$

Then, using the asymptotic expansion [7]:

$$|D|^{-z} a \sim \sum_{k \geqslant 0} \binom{-z}{k} \delta^k(a)\, |D|^{-z-k}$$

modulo very low powers of $|D|$, one computes

$$b\phi'_0(a_0, a_1) = \fint a_0\, \delta(a_1)|D|^{-1} - \frac{1}{2} \fint a_0\, \delta^2(a_1)|D|^{-2} + \frac{1}{3} \fint a_0\, \delta^3(a_1)|D|^{-3},$$

and it is now immediate that $\phi_1 + b\phi'_0 + B\phi'_2$ gives the cyclic cocycle ψ_1. $\square$

Remark 2. The term involving $P|D|^{-3}$ would vanish if the latter were traceclass, which is the case in [5] (this is the statement that the metric dimension of the projector P is 2).

Combining these two propositions, it follows that the cyclic 1-cocycles χ_1 and ψ_1 are related as:

$$\chi_1 = \psi_1 - b\beta, \tag{5.4}$$

where $\beta(a) = 2\,\mathrm{Tr}(Pa\,|D|^{-z})\big|_{z=0}$.

6 The pairing between HC^1 and K_1

In this section, we shall calculate the value of the index map (5.1) when U is the unitary operator representing the generator of $K_1(\mathcal{A}(\mathrm{SU}_q(2)))$,

$$\varphi([U]) = \mathrm{Index}(PUP) := \dim\ker PUP - \dim\ker PU^*P,$$

with

$$U = \begin{pmatrix} a & b \\ -qb^* & a^* \end{pmatrix}, \tag{6.1}$$

acting on the doubled Hilbert space $\mathcal{H} \otimes \mathbb{C}^2$ via the representation $\pi \otimes 1_2$. The projector P was denoted $P^{\uparrow}$ in Section 2. One expects this index to be nonzero, since the K-homology class of $(\mathcal{A}, \mathcal{H}, D)$ is non-trivial. This has been remarked also in [3], where our spectral triple is decomposed in terms of the spectral triple constructed in [2].

We first compute the above index directly, which is possible due to the simple nature of this particular example. A short computation shows that the kernel of the operator PU^*P is trivial, whereas the kernel of PUP contains only elements proportional to the vector

$$\begin{pmatrix} |0, 0, -\frac{1}{2}, \uparrow\rangle \\ -q^{-1}|0, 0, \frac{1}{2}, \uparrow\rangle \end{pmatrix},$$

leading to $\varphi([U]) = \mathrm{Index}(PUP) = 1$.

► Recall that for $\mathcal{A} = \mathcal{A}(\mathrm{SU}_q(2))$, our Fredholm module $(\mathcal{H}, F)$ over $\mathcal{A}(\mathrm{SU}_q(2))$ is 1-summable. From the previous section we know that $\mathrm{Index}(PUP)$ can be computed using the local cyclic cocycle ψ_1 – see Equation (5.4). To prepare for this index computation via ψ_1, we recall the following lemma [4, IV.1.γ], which fixes the normalization constant in front of χ_1. For completeness we recall the proof.

Lemma 6.1. *Let $(\mathcal{H}, F)$ be a 1-summable Fredholm module over $\mathcal{A}$ with $P = \frac{1}{2}(1 + F)$; let $u \in \mathrm{Mat}_r(\mathcal{A})$ be unitary with a suitable r. Then PuP is a Fredholm operator on $P\mathcal{H}$ and*

$$\mathrm{Index}(PuP) = -\tfrac{1}{2}\,\mathrm{Tr}(u^*[F, u]) = -\tfrac{1}{2}\chi_1(u^*, u).$$

Proof. We claim that Pu^*P is a parametrix for PuP, that is, an inverse modulo compact operators on $P\mathcal{H}$. Indeed, since $P - u^*Pu = -\frac{1}{2}u^*\,[F, u]$ is traceclass by 1-summability, by composing it from both sides with P it follows that $P - Pu^*PuP$ is traceclass. Therefore,

$$\mathrm{Index}(PuP) = \mathrm{Tr}(P - Pu^*PuP) - \mathrm{Tr}(P - PuPu^*P), \tag{6.2}$$

and the identities $P - Pu^*PuP = -\frac{1}{2}Pu^*\,[F, u]P$ and $[F, u]\,u^* + u\,[F, u^*] = 0$, together with $[F, [F, u]]_+ = 0$, imply the statement. □

Thus, the index of PUP, for the U of (6.1) is given, up to an overall $-\frac{1}{2}$ factor, by

$$\psi_1(U^{-1}, U) = 2 \fint U^*_{kl}\, \delta(U_{lk}) P|D|^{-1} - \fint U^*_{kl}\, \delta^2(U_{lk}) P|D|^{-2} + \frac{2}{3} \fint U^*_{kl}\, \delta^3(U_{lk}) P|D|^{-3},$$

with summation over $k, l = 0, 1$ understood. We compute this expression using equation (4.3). First note that since the entries of U are generators of $\mathcal{A}(\mathrm{SU}_q(2))$, we see from (2.6) and (2.7) that $\rho(\delta^2(U_{kl})) = \rho(U_{kl})$, a relation that simplifies the above formula. We compute the degree 0 part of $\rho(U^*_{kl}\, \delta(U_{lk}))$ with respect to the grading coming from $\gamma(t)$ – the only part that contributes to the trace – using the algebra relations of $\mathcal{A}(D^2_{q\pm})$,

$$\rho(U^*_{kl}\, \delta(U_{lk}))^0 = 2(1 - q^2)\, 1 \otimes r_-(b)^2.$$

Using the basic equalities

$$\tau_1(1) = 1, \quad \tau_1(r_\pm(b)^n) = 0, \quad \tau_0^\uparrow(1) = -\tau_0^\downarrow(1) = -\tfrac{1}{2}, \quad \tau_0^\uparrow(r_\pm(b)^n) = \tau_0^\downarrow(r_\pm(b)^n) = \frac{(\pm 1)^n}{1 - q^n},$$

we find that

$$\psi_1(U^{-1}, U) = 2(1 - q^2)(2\tau_0^\uparrow \otimes \tau_0^\downarrow + \frac{2}{3}\tau_1 \otimes \tau_1)\big(1 \otimes r_-(b)^2\big) - (\tau_1 \otimes \tau_0^\downarrow + \tau_0^\uparrow \otimes \tau_1)\big(1 \otimes 1\big) = -2.$$

Taking the proper coefficients, we finally obtain

$$\mathrm{Index}(PUP) = -\tfrac{1}{2}\psi_1(U^{-1}, U) = 1.$$

A Pseudodifferential calculus and cyclic cohomology

Recall [1, 7, 9] that a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is *regular* (or *smooth*, or QC^∞) if the algebra generated by $\mathcal{A}$ and $[D, \mathcal{A}]$ lies within the smooth domain $\bigcap_{n=0}^\infty \mathrm{Dom}\, \delta^n$ of the operator derivation $\delta(T) := |D|T - T|D|$. This condition permits to introduce the analogue of Sobolev spaces $\mathcal{H}^s := \mathrm{Dom}(1 + D^2)^{s/2}$ for $s \in \mathbb{R}$. Let $\mathcal{H}^\infty := \bigcap_{s \geqslant 0} H^s$, which is a core for $|D|$. Then $T\colon \mathcal{H}^\infty \to \mathcal{H}^\infty$ has *analytic order* $\leqslant k$ if T extends to a bounded operator from $\mathcal{H}^{k+s}$ to $\mathcal{H}^s$ for all $s \geqslant 0$. It turns out that $\mathcal{A}(\mathcal{H}^\infty) \subset \mathcal{H}^\infty$.

Assume that $|D|$ is invertible – which is a generic case of the D used in this paper (for a careful treatment of the noninvertible case, see [1]). The space OP^α of *operators of order* $\leqslant \alpha$ consists of those $T\colon \mathcal{H}^\infty \to \mathcal{H}^\infty$ such that

$$|D|^{-\alpha} T \in \bigcap_{n=1}^\infty \mathrm{Dom}\, \delta^n.$$

(Operators of order α have analytic order α). In particular, $\mathrm{OP}^0 = \bigcap_{n=1}^\infty \mathrm{Dom}\, \delta^n$, the algebra of operators of order $\leqslant 0$, includes $A \cup [D, \mathcal{A}]$ and their iterated commutators with $|D|$. Moreover, $[D^2, \mathrm{OP}^\alpha] \subset \mathrm{OP}^{\alpha+1}$ and $\mathrm{OP}^{-\infty} := \bigcap_{\alpha \leqslant 0} \mathrm{OP}^\alpha$ is a two-sided ideal in OP^0.

The algebra structure can be read off in terms of an *asymptotic expansion*: $T \sim \sum_{j=0}^\infty T_j$ whenever T and each T_j are operators from $\mathcal{H}^\infty$ to $\mathcal{H}^\infty$; and for each $m \in \mathbb{Z}$, there exists N such that for all

$M > N$, the operator $T - \sum_{j=1}^{M} T_j$ has analytic order $\leqslant m$. For instance, for complex powers of $|D|$ (defined by the Cauchy formula) there is a binomial expansion:

$$[|D|^z, T] \sim \sum_{k=1}^{\infty} \binom{z}{k} \delta^k(T)\, |D|^{z-k}.$$

Thus far, we have employed finitely generated algebras $\mathcal{A}(X)$, where $X = \mathrm{SU}_q(2)$, $D^2_{q\pm}$, $\mathbb{S}^1$ or $\mathbb{S}^2_q$. In each case, we can enlarge them to algebras $C^\infty(X)$ by replacing polynomials in the generators (given in a prescribed order) by series with coefficients of rapid decay: this is clear when $X = \mathbb{S}^1$, where smooth functions have rapidly decaying Fourier series. Using the symbol maps (3.4), (3.6) and (3.7) together with Lemma 2 of [6], we can check that each such $C^\infty(X)$ is closed under holomorphic functional calculus. The foregoing results apply, mutatis mutandis, to the regular spectral triple $(C^\infty(\mathrm{SU}_q(2)), \mathcal{H}, D)$.

► For convenience, we also summarize here the cyclic cohomology of the algebra $\mathcal{A}(\mathrm{SU}_q(2))$. A cyclic n-cochain on an algebra $\mathcal{A}$ is an element $\varphi \in C^n_\lambda(\mathcal{A})$, the collection of $(n+1)$-linear functionals on $\mathcal{A}$ which in addition are cyclic, $\lambda\varphi = \varphi$, with

$$\lambda\varphi(a_0, a_1, \dots, a_n) = (-1)^n \varphi(a_n, a_0, \dots, a_{n-1}).$$

There is a cochain complex $(C^\bullet_\lambda(\mathcal{A}) = \bigoplus_n C^n_\lambda(\mathcal{A}), b)$ with (Hochschild) coboundary operator $b\colon C^n(\mathcal{A}) \to C^{n+1}(\mathcal{A})$ defined by

$$b\varphi(a_0, a_1, \dots, a_{n+1}) := \sum_{j=0}^{n} (-1)^j \varphi(a_0, \dots, a_j a_{j+1}, \dots, a_{n+1}) + (-1)^{n+1} \varphi(a_{n+1}a_0, a_1, \dots, a_n).$$

The cyclic cohomology $HC^\bullet(\mathcal{A})$ of the algebra $\mathcal{A}$ is the cohomology of this complex,

$$HC^n(\mathcal{A}) := H^n(C^\bullet_\lambda(\mathcal{A}), b).$$

Equivalently, $HC^\bullet(\mathcal{A})$ can be described [4, 9] by using the second filtration of a (b, B) bicomplex of arbitrary (i.e., noncyclic) cochains on $\mathcal{A}$. Here the operator B decreases the degree $B\colon C^n(\mathcal{A}) \to C^{n-1}(\mathcal{A})$, and is defined as $B = NB_0$, with

$$(B_0\varphi)(a_0, \dots, a_{n-1}) := \varphi(1, a_0, \dots, a_{n-1}) - (-1)^n \varphi(a_0, \dots, a_{n-1}, 1)$$
$$(N\psi)(a_0, \dots, a_{n-1}) := \sum_{j=0}^{n-1} (-1)^{(n-1)j} \psi(a_j, \dots, a_{n-1}, a_0, \dots, a_{j-1}).$$

It is straightforward to check that $B^2 = 0$ and that $bB + Bb = 0$; thus $(b+B)^2 = 0$. By putting together these two operators, one gets a bicomplex $(C^\bullet(\mathcal{A}), b, B)$ with $C^{p-q}(\mathcal{A})$ in bidegree (p, q). To a cyclic n-cocycle one associates the (b, B) cocycle φ, $(b+B)\varphi = 0$, having only one nonvanishing component $\varphi_{n,0}$ given by $\varphi_{n,0} := (-1)^{\lfloor n/2 \rfloor}\psi$.

► The cyclic cohomology of the algebra $\mathcal{A}(\mathrm{SU}_q(2))$ was computed in [10]. The even components vanish while the odd ones were found to be one-dimensional and generated by the cyclic 1-cocycle $\tau_{\mathrm{odd}} \in HC^1(\mathcal{A}(\mathrm{SU}_q(2)))$ which was obtained as a character of a 1-summable Fredholm module,

$$\tau_{\mathrm{odd}}(a^l b^m (b^*)^n, a^{l'} b^{m'} (b^*)^{n'}) = (n-m) \frac{q^{l(m'+n')} \prod_{i=1}^{l} (1 - q^{2i})}{\prod_{i=0}^{l} (1 - q^{2i+2n+2n'})}\, \delta_{n+n', m+m'}\, \delta_{l,-l'}$$

where we use the notation $a^{-l} = (a^*)^l$ for $l > 0$. Since $HC^1(\mathcal{A}(\mathrm{SU}_q(2)))$ is one-dimensional, the characters of the 1-summable Fredholm modules found in [5] and in this paper, are all cohomologous to this cyclic cocycle.

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References

[1] A. L. Carey, J. Phillips, A. Rennie and F. A. Sukochev, "The Hochschild class of the Chern character for semifinite spectral triples", J. Funct. Anal. **213** (2004), 111–153.

[2] P. S. Chakraborty, A. Pal, "Equivariant spectral triples on the quantum SU(2) group", K-Theory **28** (2003), 107–126.

[3] P. S. Chakraborty, A. Pal, "On equivariant Dirac operators for $\mathrm{SU}_q(2)$", Proc. Indian Acad. Sci. (Math. Sci.) **116** (2006), 531–541.

[4] A. Connes, *Noncommutative Geometry*, Academic Press, London and San Diego, 1994.

[5] A. Connes, "Cyclic cohomology, quantum group symmetries and the local index formula for $\mathrm{SU}_q(2)$", J. Inst. Math. Jussieu **3** (2004), 17–68.

[6] A. Connes, "Cyclic cohomology, noncommutative geometry and quantum group symmetries", in *Noncommutative Geometry*, S. Doplicher and R. Longo, eds., Lecture Notes in Mathematics **1831**, Springer, Berlin, 2004.

[7] A. Connes and H. Moscovici, "The local index formula in noncommutative geometry", Geom. Funct. Anal. **5** (1995), 174–243.

[8] L. Dąbrowski, G. Landi, A. Sitarz, W. van Suijlekom and J. C. Várilly. "The Dirac operator on $\mathrm{SU}_q(2)$", Commun. Math. Phys. **259** (2005), 729–759.

[9] J. M. Gracia-Bondía, J. C. Várilly and H. Figueroa, *Elements of Noncommutative Geometry*, Birkhäuser, Boston, 2001.

[10] T. Masuda, Y. Nakagami and J. Watanabe, "Noncommutative differential geometry on the quantum SU(2). I: An algebraic viewpoint", K-Theory **4** (1990), 157–180.

[11] P. Podleś, "Quantum spheres", Lett. Math. Phys. **14** (1987), 521–531.