

Combinatorics of renormalization as matrix calculus

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Abstract

We give a simple presentation of the combinatorics of renormalization in perturbative quantum field theory in terms of triangular matrices. The prescription, that may be of calculational value, is derived from first principles, to wit, the “Birkhoff decomposition” in the Hopf-algebraic description of renormalization by Connes and Kreimer.

1 Introduction

A Hopf algebra structure underlying the combinatorics of perturbative renormalization was recognized by Kreimer [1]. Some have worried about the practical usefulness of his insight for organizing everyday computations in quantum field theories. A partial answer to this legitimate question is given in [2], for instance. This paper presents another partial answer, in a different vein. We show that the Feynman rules collectively possess a triangular matrix representation, such that the renormalization map becomes a matrix operation. This infinite matrix can be truncated almost *ad libitum*.

The procedure is general, largely independent of the renormalization scheme (although we illustrate everything with the MS-scheme in dimensional regularization), and essentially independent of the particular field theory model one works with. The latter enters just in filling up the matrix entries, generally entailing further simplification. Only knowledge of linear algebra and quantum fields are required; no Hopf algebra is (openly) used. More detail is found in [3].

First we recall the algebraic behaviour of the subtraction map K : the whole paper turns around its Rota–Baxter property. In Section 3, K is lifted to the matrix level and the computational recipe for the matrix counterterm and matrix renormalization maps is found. In the following section we verify that this reproduces the diagrammatic Bogoliubov operation; we take examples from the ϕ_4^4 model, and compare with the tables in [4]. In Section 5 we rework the matrix representation using the map K_+ that picks up the finite parts. The next two sections contain mathematical summaries. In Section 7 we sketch our derivation of the matrix representation. Finally we examine the outlook.

2 The subtraction map as a Rota–Baxter operator

Consider Laurent series

$$S(\varepsilon) = \frac{a_{-n}}{\varepsilon^n} + \frac{a_{-n+1}}{\varepsilon^{n-1}} + \cdots + \frac{a_{-1}}{\varepsilon} + a_0 + a_1\varepsilon + \cdots . \quad (1)$$

With the ordinary multiplication, they form a commutative algebra V with unit. Consider further the operation K which picks out the pure pole part

$$K[S](\varepsilon) = \frac{a_{-n}}{\varepsilon^n} + \frac{a_{-n+1}}{\varepsilon^{n-1}} + \cdots + \frac{a_{-1}}{\varepsilon},$$

and the operation $K_+ := \text{id} - K$ keeping the finite part,

$$K_+[S](\varepsilon) = a_0 + a_1\varepsilon + \cdots .$$

The projector condition $K^2 = K$ ensures that the intersection between $K(V)$ and $K_+(V)$ is zero. The product of two elements of $K(V)$ remains in $K(V)$ – and likewise for $K_+(V)$. The key property

$$K[S_1]K[S_2] = K[K[S_1]S_2 + S_1K[S_2] - S_1S_2], \quad (2)$$

is easy to check – see Section 6. It makes K a *Rota–Baxter* operator [5]; also K_+ is a *Rota–Baxter* operator. All this applies in particular to series corresponding to dimensionally regularized integrals in the MS-scheme subtraction. Our arguments are purely combinatorial, so we need not worry about the precise form of the a_i coefficients. We adopt for the subtraction operator K the notation of [6], followed in [4]; sometimes we write K_- for clarity.

3 Setting up the recipe

In this section we suppose given the pair (V, K) of a commutative algebra with unit and a *Rota–Baxter* projector, first as an abstract framework; we always have in mind the algebra of unrenormalized Feynman amplitudes of the form (1). Consider upper triangular matrices of finite size with entries in V , of two types: nilpotent, i.e., with 0's on the diagonal, and unipotent, i.e., with 1's on the diagonal:

$$Z = \begin{pmatrix} 0 & * & \dots & \dots & * \\ 0 & 0 & \ddots & Z_{ij} & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & * \\ 0 & \dots & \dots & 0 & 0 \end{pmatrix}; \quad \varphi = \begin{pmatrix} 1 & * & \dots & \dots & * \\ 0 & 1 & \ddots & \varphi_{ij} & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & * \\ 0 & \dots & \dots & 0 & 1 \end{pmatrix}.$$

We define *Rota–Baxter* operations $\mathcal{K}_- = \mathcal{K}$ and \mathcal{K}_+ , on algebras $M^{\text{up}}(V)$ of upper triangular matrices with scalar diagonals and with entries in V , by extending the maps K and K_+ componentwise,

$$(\mathcal{K}[\varphi])_{ij} := K[\varphi_{ij}], \quad (\mathcal{K}_+[\varphi])_{ij} := K_+[\varphi_{ij}].$$

Verification for $\mathcal{K}, \mathcal{K}_+$ of the analogue of (2) is immediate; but the algebras $M^{\text{up}}(V)$ are no longer commutative. We seek to factorize an arbitrary unipotent element φ in the form

$$\varphi = \varphi_+ \varphi_-^{-1}, \quad (3)$$

where the factors $\varphi_- \in 1 + \mathcal{K}[M^{\text{up}}(V)]$, $\varphi_+ \in \mathcal{K}_+[M^{\text{up}}(V)]$ are also unipotent; note that they are unique. This can be called a matrix Birkhoff decomposition.

If $\mathcal{K}[\log \varphi]$ and $\mathcal{K}_+[\log \varphi]$ happened to commute, it would be enough to choose $\varphi_+ = e^{\mathcal{K}_+[\log \varphi]}$ and $\varphi_- = e^{-\mathcal{K}[\log \varphi]}$. In general, that is not so; but we are able to compensate for the lack of commutativity between the images of \mathcal{K}_- and \mathcal{K}_+ . For that, consider the equations

$$\varphi_- = 1 - \mathcal{K}_-[(\varphi - 1)\varphi_-] \quad \text{and} \quad \varphi_+^{-1} = 1 - \mathcal{K}_+[\varphi_+^{-1}(\varphi - 1)],$$

respectively solved by

$$\varphi_- = 1 - \mathcal{K}_-[\varphi - 1] + \mathcal{K}_-[(\varphi - 1)\mathcal{K}_-[\varphi - 1]] - \dots; \quad (4)$$

$$\varphi_+^{-1} = 1 - \mathcal{K}_+[\varphi - 1] + \mathcal{K}_+[\mathcal{K}_+\varphi - 1] - \dots. \quad (5)$$

Both series terminate. Atkinson's theorem [7] asserts that these matrices φ_- , φ_+^{-1} verify (3). The proof runs as follows:

$$\begin{aligned} \varphi_+^{-1}\varphi_- &= 1 - \mathcal{K}_+[\varphi_+^{-1}(\varphi - 1)] - \mathcal{K}_-[(\varphi - 1)\varphi_-] + \mathcal{K}_+[\varphi_+^{-1}(\varphi - 1)]\mathcal{K}_-[(\varphi - 1)\varphi_-] \\ &= 1 - \varphi_+^{-1}(\varphi - 1)\varphi_-, \end{aligned}$$

after some work with the Rota–Baxter property.

The matrix φ_+ is what we are really after. It can obviously be obtained as $\varphi\varphi_-$, or by inverting (5), using the geometric series formula

$$\varphi_+ = 1 - (\varphi_+^{-1} - 1) + (\varphi_+^{-1} - 1)^2 - \dots.$$

A better course is perhaps to observe that, by the same token as in (4), we obtain

$$\varphi_+ = 1 - \mathcal{K}_+[(\varphi^{-1} - 1)\varphi_+].$$

Thus, the respective formulas for the components of φ_- and φ_+ are

$$\begin{aligned} \varphi_{-ij} &= -K_-[\varphi_{ij}] + \sum_{k=1}^{j-i-1} \sum_{i < l_1 < l_2 < \dots < l_k < j} (-)^{k-1} K_-[\varphi_{il_1} K_-[\varphi_{l_1 l_2} \dots K_-[\varphi_{l_k j}] \dots]], \\ \varphi_{+ij} &= -K_+[\varphi_{ij}^{-1}] + \sum_{k=1}^{j-i-1} \sum_{i < l_1 < l_2 < \dots < l_k < j} (-)^{k-1} K_+[\varphi_{il_1}^{-1} K_+[\varphi_{l_1 l_2}^{-1} \dots K_+[\varphi_{l_k j}^{-1}] \dots]]. \end{aligned} \quad (6)$$

These similar formulas are our workhorses; with the appropriate definition of φ , the matrix φ_- will be seen to contain all the information on counterterms in renormalization; and φ_+ on the renormalized quantities.

4 Making the recipe work

Now we make explicit how the Feynman rules specify such operators φ . Recall that if $\Gamma_i \subseteq \Gamma_j$ is a superficially divergent subgraph of Γ_j , the *cograph* Γ_j/Γ_i is obtained by shrinking Γ_i to a vertex within Γ_j . We only consider subgraphs that are generalized vertices [6]. Chosen an n -point function,

the spaces of vectors on which the matrices act are spanned by the corresponding (superficially divergent, *connected*, amputated) Feynman graphs. We may use the familiar bra-ket notation to denote the diagrams as vectors. A basis $|\Gamma_1\rangle, |\Gamma_2\rangle, |\Gamma_3\rangle, \dots$ for such a space can be ordered in many ways, the only conditions being that $|\Gamma_1\rangle = |\emptyset\rangle$ – the empty diagram – and that each cograph of any Γ_l occurs in the basis as some Γ_m with $m < l$. It is then convenient to order the basis by number of loops (or vertices, if we work on coordinate space); but the order within a given loop-number sector is immaterial. Once the external structure and the basis are fixed, we fill up the entries of a matrix by the rule:

$$\text{for } i \neq j, \quad \varphi_{ij} = \sum_{\Gamma'} (\text{unrenormalized}) \text{ amplitude of } \Gamma' \quad \text{if } \Gamma_i \simeq \Gamma_j/\Gamma',$$

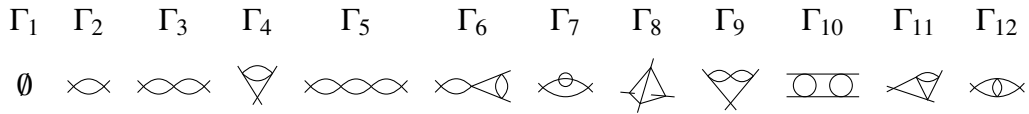
otherwise $\varphi_{ij} = 0$. This entails triangularity, since $\varphi_{ij} = 0$ if $i > j$. We set $\varphi_{ii} = 1$ for all i . Note that Γ' need not belong to the basis list (it might be disconnected, for one thing). Let $\tilde{\varphi}(\Gamma')$ be the unrenormalized amplitude of Γ' . We just said that the coefficient of $|\Gamma_i\rangle$ in $\varphi(|\Gamma_j\rangle)$ is $\sum_{\Gamma'} \tilde{\varphi}(\Gamma')$ for $\Gamma_i \simeq \Gamma_j/\Gamma'$. The notation is appropriate because $\tilde{\varphi}$ is the abstract object represented by the matrix φ (see Section 7); it has the property that

$$\tilde{\varphi}(\Gamma_i \Gamma_j) := \tilde{\varphi}(\Gamma_i \cup \Gamma_j) = \tilde{\varphi}(\Gamma_i) \tilde{\varphi}(\Gamma_j).$$

This property is shared by the elements of φ_- and φ_+ . As a bonus, it allows to simplify the notation later on, by simply omitting $\tilde{\varphi}$.

When we truncate the matrices we are not obliged to include all the diagrams belonging to the higher sector – and we can also choose, for whatever purpose, particular classes of diagrams, subject to the aforementioned two conditions. With that, the first row of φ is given by $1, \tilde{\varphi}(\Gamma_2), \tilde{\varphi}(\Gamma_3), \dots$, the unrenormalized amplitudes of all the diagrams; and the analogously defined first row $(1, \tilde{\varphi}_-(\Gamma_2), \tilde{\varphi}_-(\Gamma_3), \dots)$ of φ_- will yield all the counterterms of the theory!

We take as a simple *example* the space of graphs relevant to the 4-point function for the ϕ_4^4 model, truncated to the 12 tadpole-free diagrams (including the empty one) up to three loops. We adopt the order of [4] for the basis, as follows:



Notice that $\Gamma_2 = \Gamma_7/\Gamma'$ where the sunset diagram $\Gamma' = \bigoplus$ does not appear in the basis list. We find that φ is equal to

$$\begin{pmatrix} 1 & \tilde{\varphi}(\Gamma_2) & \tilde{\varphi}(\Gamma_3) & \tilde{\varphi}(\Gamma_4) & \tilde{\varphi}(\Gamma_5) & \tilde{\varphi}(\Gamma_6) & \tilde{\varphi}(\Gamma_7) & \tilde{\varphi}(\Gamma_8) & \tilde{\varphi}(\Gamma_9) & \tilde{\varphi}(\Gamma_{10}) & \tilde{\varphi}(\Gamma_{11}) & \tilde{\varphi}(\Gamma_{12}) \\ & 1 & 2\tilde{\varphi}(\Gamma_2) & \tilde{\varphi}(\Gamma_2) & 2\tilde{\varphi}(\Gamma_3) + \tilde{\varphi}^2(\Gamma_2) & \tilde{\varphi}(\Gamma_4) + \tilde{\varphi}^2(\Gamma_2) & \tilde{\varphi}(\Gamma') & 0 & \tilde{\varphi}(\Gamma_3) & \tilde{\varphi}^2(\Gamma_2) & \tilde{\varphi}(\Gamma_4) & 2\tilde{\varphi}(\Gamma_4) \\ & & 1 & 0 & 3\tilde{\varphi}(\Gamma_2) & \tilde{\varphi}(\Gamma_2) & 0 & 0 & 0 & 0 & 0 & \tilde{\varphi}(\Gamma_2) \\ & & & 1 & 0 & \tilde{\varphi}(\Gamma_2) & 0 & 0 & 2\tilde{\varphi}(\Gamma_2) & 2\tilde{\varphi}(\Gamma_2) & \tilde{\varphi}(\Gamma_2) & 0 \\ & & & & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & & & 1 & 0 & 0 & 0 & 0 & 0 \\ & & & & & & & 1 & 0 & 0 & 0 & 0 \\ & & & & & & & & 1 & 0 & 0 & 0 \\ & & & & & & & & & 1 & 0 & 0 \\ & & & & & & & & & & 1 & 0 \\ & & & & & & & & & & & 1 \end{pmatrix}$$

The abundance of zeros is welcome in the calculation: with this concrete form of φ the series (4) stops after three iterations. Omitting $\tilde{\varphi}$ from the notation as advertised, one reads off from (6) the first row of φ_- , containing the counterterms for the eleven nontrivial diagrams, that we write as the column matrix φ_{-j1}^T in the following display. These expressions for $\tilde{\varphi}_-(\Gamma_j)$ coincide with those listed in the tables in [4], where they are denoted $-K\bar{R}(\Gamma_j)$. But note that here the Bogoliubov preparation map \bar{R} does not appear explicitly; our result, with one exception, is not recursively presented, and, with the help of some symbolic programming, can be obtained at one stroke. It is clear that the method will jointly handle large numbers of multi-loop diagrams with ease. (Of course, we are not claiming that it is always quicker than the standard procedures.)

$$\begin{pmatrix} 1 \\ \tilde{\varphi}_-(\text{fish}) \\ \tilde{\varphi}_-(\text{sun}) \\ \tilde{\varphi}_-(\text{triangle}) \\ \tilde{\varphi}_-(\text{box}) \\ \tilde{\varphi}_-(\text{figure-eight}) \\ \tilde{\varphi}_-(\text{pentagon}) \\ \tilde{\varphi}_-(\text{triangle with dot}) \\ \tilde{\varphi}_-(\text{triangle with dot}) \\ \tilde{\varphi}_-(\text{triangle with dot}) \\ \tilde{\varphi}_-(\text{triangle with dot}) \\ \tilde{\varphi}_-(\text{triangle with dot}) \\ \tilde{\varphi}_-(\text{triangle with dot}) \\ \tilde{\varphi}_-(\text{triangle with dot}) \end{pmatrix} = \begin{pmatrix} 1 \\ -K[\text{fish}] \\ K[\text{fish}]K[\text{fish}] \\ -K[\text{triangle}] + K[\text{fish}K[\text{fish}]] \\ -K[\text{fish}]K[\text{fish}]K[\text{fish}] \\ -K[\text{fish}]\tilde{\varphi}_-(\text{triangle}) \\ -K[\text{figure-eight}] + K[\text{fish}K[\text{circle}]] \\ -K[\text{pentagon}] \\ -K[\text{triangle with dot}] + 2K[\text{triangle with dot}K[\text{fish}]] - K[\text{fish}(K[\text{fish}])^2] \\ -K[\text{triangle with dot}] + 2K[\text{triangle with dot}K[\text{fish}]] - K[\text{fish}(K[\text{fish}])^2] \\ -K[\text{triangle with dot}] + K[\text{fish}]K[\text{triangle with dot}] + K[\text{fish}K[\text{triangle with dot}]] - [K[\text{fish}]]^{\{3\}} \\ -K[\text{figure-eight}] + K[\text{fish}^2K[\text{fish}]] + 2K[\text{fish}K[\text{triangle with dot}]] - 2[K[\text{fish}]]^{\{3\}} \end{pmatrix}.$$

Above we wrote $[K[\text{fish}]]^{\{3\}} := K[\text{fish}K[\text{fish}K[\text{fish}]]]$. The graphs $\Gamma_3, \Gamma_5, \Gamma_6$ are cutvertex, for which $\tilde{\varphi}$ and the renormalization map are known to factorize. For them (6) *prima facie* gives a more complicated expression, that can be reduced to the expressions shown by some Rota–Baxter gymnastics. We give the example of $\tilde{\varphi}_-(\text{figure-eight})$. From our matrix operations:

$$\begin{aligned} \tilde{\varphi}_-(\text{figure-eight}) &= -K[\text{figure-eight}] + K[\text{triangle with dot}K[\text{fish}]] + K[\text{fish}K[\text{triangle with dot}]] \\ &\quad + K[\text{fish}K[\text{fish}]] + K[\text{fish}K[\text{fish}^2]] - 3[K[\text{fish}]]^{\{3\}}. \end{aligned}$$

Since

$$K[\text{figure-eight}] = K[\text{fish}K[\text{triangle with dot}]] = -K[\text{fish}]K[\text{triangle with dot}] + K[K[\text{fish}]K[\text{triangle with dot}]] + K[K[\text{triangle with dot}]\text{fish}],$$

we get

$$\begin{aligned} \tilde{\varphi}_-(\text{figure-eight}) &= K[\text{fish}]K[\text{triangle with dot}] + K[\text{fish}^2K[\text{fish}]] + K[\text{fish}K[\text{fish}^2]] - 3[K[\text{fish}]]^{\{3\}} \\ &= K[\text{fish}]K[\text{triangle with dot}] + K[\text{fish}^3] + K[\text{fish}^2]K[\text{fish}] - 3[K[\text{fish}]]^{\{3\}}. \end{aligned}$$

To continue, we invoke the classical Bohnenblust–Spitzer identity [8]:

$$n! [K[A]]^{\{n\}} := n! \underbrace{K[AK[A \dots K[A] \dots]]}_{n \text{ times}} = \sum_{P \in \Pi_n} \prod_{p \in P} (|p| - 1)! K(A^{|p|}),$$

which is itself derivable from the Rota–Baxter identity; here Π_n is the set of partitions P of the set $\{1, \dots, n\}$. For the present case,

$$6[K[\infty]]^{\{3\}} = K[\infty] K[\infty] K[\infty] + 3K[\infty^2] K[\infty] + 2K[\infty^3],$$

implying that

$$\begin{aligned} \tilde{\varphi}_-(\infty \circlearrowleft) &= K[\infty] K[\infty] - \frac{1}{2} K[\infty] K[\infty] K[\infty] - \frac{1}{2} K[\infty] K[\infty^2] = K[\infty] K[\infty] \\ &\quad - K[\infty] K[\infty K[\infty]] = (-K[\infty]) (-K[\infty] + K[\infty K[\infty]]) = \tilde{\varphi}_-(\infty) \tilde{\varphi}_-(\infty). \end{aligned}$$

According to the theory underlying the matrix representation (Section 7), the factorization property of $\tilde{\varphi}_-$ is automatic: we did not *prove* anything, just performed an internal check.

5 The matrix representation in terms of K_+

Renormalization theory is usually formulated in terms of subtractions [4, 6, 9]. For good reasons: for instance, in the MS-scheme the counterterms are local, independent of the mass and the renormalization scale; this helps to establish renormalization group equations. Also, the derivation of the whole procedure is simpler in terms of subtractions. However, the understanding of renormalization as an approximation process, rather than a cancellation of infinities, is thereby lost. Recently 't Hooft expressed the *desideratum* of a renormalization scheme exclusively containing dressed vertices [10]. We take here a step in this direction by rephrasing the renormalization of (regularized) Feynman graphs with subdivergences in terms of K_+ . The magic of the triangular matrix representation implies that the matrix φ_+ must give, graph by graph, the completely renormalized expressions $(\text{id} - K)\bar{R}(\Gamma) = K_+\bar{R}(\Gamma)$. Inspection of the equations in Section 3, on the other hand, shows that the same calculation method for φ_- in terms of K_- yields $(1, \tilde{\varphi}_+(\Gamma_2), \tilde{\varphi}_+(\Gamma_3), \dots)$ in terms of K_+ , provided one starts by inverting the matrix φ . This is a small price to pay, and again a bit of symbolic programming goes a long way. We illustrate the procedure with the same model example:

$$\begin{pmatrix} \tilde{\varphi}_+(\infty\infty) \\ \tilde{\varphi}_+(\nabla) \\ \tilde{\varphi}_+(\infty\infty\infty) \\ \tilde{\varphi}_+(\infty\infty) \\ \tilde{\varphi}_+(\infty) \\ \tilde{\varphi}_+(\nabla) \\ \tilde{\varphi}_+(\infty\infty) \\ \tilde{\varphi}_+(\infty) \\ \tilde{\varphi}_+(\infty) \end{pmatrix} = \begin{pmatrix} K_+[\infty\infty] K_+[\infty\infty] \\ K_+[\nabla] + K_+[\infty] K_+[\infty] - K_+[\infty K_+[\infty]] \\ K_+[\infty] K_+[\infty] K_+[\infty] \\ K_+[\infty] \left(K_+[\nabla] + K_+[\infty] K_+[\infty] - K_+[\infty K_+[\infty]] \right) \\ K_+[\infty] + K_+[\infty] K_+[\ominus] - K_+[\ominus K_+[\infty]] \\ \left\{ \begin{array}{l} K_+[\nabla] + 2K_+[\nabla] K_+[\infty] - K_+[\infty] K_+[\infty^2] \\ - 2K_+[\infty K_+[\nabla]] - K_+[\infty^2 K_+[\infty]] + 2[K_+[\infty]]^{\{3\}} \end{array} \right\} \\ \left\{ \begin{array}{l} K_+[\infty\infty] + 2K_+[\nabla] K_+[\infty] - K_+[\infty] K_+[\infty^2] \\ - 2K_+[\infty K_+[\nabla]] - K_+[\infty^2 K_+[\infty]] + 2[K_+[\infty]]^{\{3\}} \end{array} \right\} \\ \left\{ \begin{array}{l} K_+[\nabla] + K_+[\nabla] K_+[\infty] - K_+[\infty] K_+[\infty^2] \\ - K_+[\nabla\infty] + [K_+[\infty]]^{\{3\}} \end{array} \right\} \\ \left\{ \begin{array}{l} K_+[\infty] + 2K_+[\nabla] K_+[\infty] - K_+[\infty] K_+[\infty^2] \\ - 2K_+[\nabla K_+[\infty]] + K_+[\infty(K_+[\infty])^2] \end{array} \right\} \end{pmatrix}.$$

The simpler cases have been omitted. The reader is reminded that $K_+[\infty]$ actually means $K_+[\tilde{\varphi}(\infty)]$, and so on.

6 More on Rota–Baxter operators

A few extra comments on the Rota–Baxter property of K_- and K_+ are in order. In the work by Kreimer, the Rota–Baxter property appears for the first time in [11], under the name “multiplicativity constraint”. The maps K_- , K_+ may be regarded as generalized integrals. Indeed, let us insert a parameter θ (a Rota–Baxter *weight*) before the last term of (2):

$$K[S_1] K[S_2] = K[K[S_1]S_2 + S_1K[S_2] - \theta S_1S_2].$$

The case $\theta = 0$ corresponds to a property of the integral $I[f](x) := \int_0^x f(t) dt$, to wit,

$$I[f_1] I[f_2] = I[I[f_1]f_2 + f_1I[f_2]], \quad (7)$$

which is just integration by parts. For g fixed, the solution of the equation $f = 1 - I[gf]$ with I satisfying (7) is given by

$$f = 1 - I[g] + I[gI[g]] - \dots = e^{-I[g]},$$

which follows from (7) for $f_1 = f_2 = f$, and illustrates our approach in this paper.

Although (2) is elementary, we wish to prove it here. This is warranted because the Rota–Baxter property is persistently ignored in field theory treatises; and this neglect is not without consequences. For instance, in Section 5.3.3 of the standard text [9], we find a tortured argument to try to prove

$\tilde{\varphi}_-(\Gamma_i \cup \Gamma_j) = \tilde{\varphi}_-(\Gamma_i)\tilde{\varphi}_-(\Gamma_j)$, in which the intermediate formulas (5.3.15) and (5.3.16) are plain wrong. To see why (2) holds, notice that

$$\begin{aligned} K[S_1]S_2 + S_1K[S_2] - S_1S_2 &= K[S_1](K[S_2] + K_+[S_2]) - (K[S_1] + K_+[S_1])K_+[S_2] \\ &= K[S_1]K[S_2] - K_+[S_1]K_+[S_2], \end{aligned}$$

and applying K to this equality kills the term $K_+[S_1]K_+[S_2]$, leaving $K[S_1]K[S_2]$ unchanged.

7 The rationale for the matrix representation

The not so mathematically-minded might wish to skip this section. In the formalism Kreimer developed jointly with Connes [12, 13], Feynman diagrams are organized in a Hopf algebra $\mathcal{H}_{\mathcal{F}}$ of graphs; Feynman rules are understood as linear and multiplicative maps $\tilde{\varphi}$ of $\mathcal{H}_{\mathcal{F}}$ into an algebra V (commutative, with unit) of quantum amplitudes; and the disentangling of subdivergences is formulated as a factorization problem (Birkhoff decomposition). The original version had a strong geometrical flavour, but its supporting algebraic frame has emerged since then [5].

The space $\mathcal{H}_{\mathcal{F}}$ is the algebra of polynomials with connected Feynman graphs as indeterminates, multiplication being given by simple juxtaposition of graphs. Connes and Kreimer introduced on $\mathcal{H}_{\mathcal{F}}$ a coproduct $\Delta : \mathcal{H}_{\mathcal{F}} \rightarrow \mathcal{H}_{\mathcal{F}} \otimes \mathcal{H}_{\mathcal{F}}$, serving to encode the superficially divergent subgraphs, by setting $\Delta(\Gamma) := \sum_{\Gamma'} \Gamma' \otimes \Gamma/\Gamma'$, in the notation of Section 4. For the coproducts of a Hopf algebra \mathcal{H} one writes $\Delta a = \sum a_{(1)} \otimes a_{(2)}$, for $a \in \mathcal{H}$.

Let (V, K) be a commutative Rota–Baxter algebra, and consider $\text{Hom}(\mathcal{H}, V)$, the space of linear maps from \mathcal{H} to V ; this is an *algebra* with the convolution operation, given by $f \star g = \sum f(a_{(1)})g(a_{(2)})$, for $a \in \mathcal{H}$. In our case the multiplicative (that is, product-respecting) elements of $\text{Hom}(\mathcal{H}_{\mathcal{F}}, V)$, with V the algebra of Feynman amplitudes, are of particular interest. Clearly they are determined by their action on the subspace \mathcal{F} of connected graphs. We construct a representation Ψ of $\text{Hom}(\mathcal{H}_{\mathcal{F}}, V)$ by infinite triangular matrices with entries in V by taking the composition

$$\Psi[f] : V \otimes \mathcal{F} \xrightarrow{\text{id}_V \otimes \Delta} V \otimes \mathcal{H}_{\mathcal{F}} \otimes \mathcal{F} \xrightarrow{\text{id}_V \otimes f \otimes \text{id}_{\mathcal{F}}} V \otimes V \otimes \mathcal{F} \xrightarrow{m_V \otimes \text{id}_{\mathcal{F}}} V \otimes \mathcal{F}, \quad (8)$$

where m_V is just multiplication on V . The plot works because the external structure of the cographs Γ/Γ' is the same as that of Γ , so Δ actually sends \mathcal{F} into $\mathcal{H}_{\mathcal{F}} \otimes \mathcal{F}$. Thus for any $f \in \text{Hom}(\mathcal{H}_{\mathcal{F}}, V)$ a connected graph is sent by $\Psi[f]$ into a linear combination of connected graphs with coefficients in V , corresponding to the same n -point function. In fact, Ψ is an antirepresentation, since $\Psi[f \star g] = \Psi[g]\Psi[f]$.

With the operator K given by $K[f](a) := K[f(a)]$, the space $\text{Hom}(\mathcal{H}_{\mathcal{F}}, V)$ becomes a (noncommutative) Rota–Baxter algebra; then $\Psi[K[f]] = \mathcal{K}[\Psi[f]]$, with \mathcal{K} the known matrix Rota–Baxter map. Let finally $\tilde{\varphi} \in \text{Hom}(\mathcal{H}_{\mathcal{F}}, V)$ be the Feynman rule, which is multiplicative, and denote

$$\varphi := \Psi[\tilde{\varphi}].$$

This will be a unipotent matrix. We have at last reproduced the setting of this paper. The matrix decomposition (3) is a consequence of Connes' and Kreimer's algebraic Birkhoff decomposition $\tilde{\varphi} = \tilde{\varphi}_-^{-1} \star \tilde{\varphi}_+$, where the two factors are multiplicative as well [12]. Proofs and details are found in [3].

8 Conclusion

Inspired by the Connes–Kreimer Hopf algebra formalism, we have exhibited the combinatorics of renormalization as a collective process, mechanized by means of simple matrix calculus. Our approach neatly resolves the tension between the “additive” and the “multiplicative” sides of renormalization: the recursive diagrammatic subtraction of subdivergences is the outcome of a multiplicative process (this is not quite a trivial remark: a direct check of the relation $\tilde{\varphi}_+(\Gamma) = K_+\overline{R}(\Gamma)$ if and only if $\tilde{\varphi}_-(\Gamma) = -K_-\overline{R}(\Gamma)$ involves somewhat messy calculations). As a consequence, the renormalization of the Lagrangian’s parameters by counterterms takes place by composition of series; the latter has been known since 1855 to have a triangular matrix representation [14]. All this is more or less clear from the analysis in [6, 13, 15]; but probably deserves further elucidation.

Also, we have rewritten the renormalization map in terms of the projection K_+ on the finite part. In dimensional regularization this prescription falls short of ’t Hooft’s desideratum [10], as some of the terms in φ_+ contain coefficients of the pole parts; this objection, nevertheless, loses force in regularization-free schemes like BPHZ and Epstein–Glaser renormalization, that also possess a Rota–Baxter property [16].

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References

- [1] D. Kreimer, “On the Hopf algebra structure of perturbative quantum field theories”, *Adv. Theor. Math. Phys.* **2** (1998), 303–334.
- [2] D. J. Broadhurst and D. Kreimer, “Combinatoric explosion of renormalization tamed by Hopf algebra: 30-loop Padé–Borel resummation”, *Phys. Lett. B* **475** (2000), 63–70.
- [3] K. Ebrahimi-Fard and L. Guo, “Matrix representation of renormalization in perturbative quantum field theory”, hep-th/0508155, Bonn, 2005.
- [4] H. Kleinert and V. Schulte-Frohlinde, *Critical Properties of ϕ^4 -Theories*, World Scientific, Singapore, 2001.
- [5] K. Ebrahimi-Fard, L. Guo and D. Kreimer, “Spitzer’s identity and the algebraic Birkhoff decomposition in pQFT”, *J. Phys. A* **37** (2004), 11037–11052.
- [6] W. E. Caswell and A. D. Kennedy, “Simple approach to renormalization theory”, *Phys. Rev. D* **25** (1982), 392–408.
- [7] F. V. Atkinson, “Some aspects of Baxter’s functional equation”, *J. Math. Anal. Appl.* **7** (1963), 1–30.
- [8] K. Ebrahimi-Fard and L. Guo, “The Spitzer identity”, preprint, Bonn, 2005.
- [9] J. C. Collins, *Renormalization*, Cambridge University Press, Cambridge, 1984.
- [10] G. ’t Hooft, “Renormalization without infinities”, *Int. J. Mod. Phys. A* **20** (2005), 1336–1345.

- [11] D. Kreimer, “Chen’s iterated integral represents the operator product expansion”, *Adv. Theor. Math. Phys.* **3** (1999), 627–670.
- [12] A. Connes and D. Kreimer, “Renormalization in quantum field theory and the Riemann–Hilbert problem I: the Hopf algebra structure of graphs and the main theorem”, *Commun. Math. Phys.* **210** (2000), 249–273.
- [13] A. Connes and D. Kreimer, “Renormalization in quantum field theory and the Riemann–Hilbert problem II: the β -function, diffeomorphisms and the renormalization group”, *Commun. Math. Phys.* **216** (2001), 215–241.
- [14] H. Figueroa, J. M. Gracia-Bondía and J. C. Várilly, “Faà di Bruno Hopf algebras”, *Rev. Col. Mat.* **56** (2022), 1–12.
- [15] G. B. Pivovarov and F. V. Tkachov, “Euclidean asymptotic expansions of Green functions of quantum fields II. Combinatorics of the asymptotic operation”, *Int. J. Mod. Phys. A* **8** (1993), 2241–2286.
- [16] K. Ebrahimi-Fard, J. M. Gracia-Bondía and L. Guo, forthcoming.