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# Numerical simulations of mountain wave generation past an isolated obstacle

JORGE A. GUTIÉRREZ[1]

Laboratorio de Investigaciones Atmosféricas y Planetarias y Centro de Investigaciones Geofísicas
Universidad de Costa Rica, San José, Costa Rica.

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**ABSTRACT**

Numerical experiments concerning high Froude number flow without background rotation past an isolated three-dimensional obstacle are performed and comparisons with the results obtained and R. Smith´s linear theory are made. Linear and non-linear effects are investigated.

## 1. Introduction

Theories on the dynamics of orographic flows were first developed in the 1940's and 1950's. From then until recently the field has been divided according to scale. Early work on mesoscale mountain airflow was focused on an effort to understand the role of internal gravity waves in phenomena that could be observed in wave clouds, strong downslope winds, clear-air turbulence (CAT) and glider ascents, see for example Queney (1948).

Theories and models of mountain flows are now addressing a wider range of physical dimensions and scientists are developing ways to understand the connections between differing scales, Smith (1979, 1989), Blumen (1990), Baines (1995).

The interaction of rapidly moving flow with an obstacle may generate gravity waves. These waves may propagate upwards and break when reaching the stratosphere generating clear air turbulence.

Under certain circumstances these waves cannot propagate in the vertical and are advected downstream carrying with them great amounts of energy that generate very strong winds on the lee of the orography. These winds can produce structural damage on buildings located downstream of the mountain.

In this paper the behaviour of a continuously stratified flow past an isolated mountain is investigated by means of computer simulations performed in a *Sun* workstation at the Laboratory for Atmospheric and Planetary Research of the University of Costa Rica. The choice of flow parameters is such as to ensure the flow regime is within the range covered by Smith's linear theory. The model used is a non-linear, non-hydrostatic model developed at the University of Reading by P. Miranda. In the linear regime the flow is expected to approach the behaviour predicted by linear theory, deviations from this behaviour will be analised and discussed.

A linear theory of stratified hydrostatic flow past an isolated mountain was developed by R. Smith in 1980. His work complements and extends that of Wurtele (1957) and Crapper (1959, 1962) which are an extension of the linear theory of Lyra and Queney with constant background flow speed $u$ and background stratification $N$ to three dimensions. Smith´s linear theory was put forth in an attempt to investigate the nature of the wave´s hydrostatic part and also the nature of the flow near the mountain.

Smith (1980) found that the small-amplitude theory describes how the flow tends to be diverted around the topography. An important parameter in describing the behaviour of the flow is the Froude number, which, throughout this paper will be chosen to be $Fr = u/Nh$. In this definition $u$ represents the background speed of the flow, $N$ is the background stratification of the flow and $h$ is the height of the obstacle.

[1] Corresponding autor address: Dr Jorge A Gutierrérrez,
Laboratorio de Investigaciones Atmosféricas y Planetarias y
Centro de Investigaciones Geofísicas
Universidad de Costa Rica, E-mail: jorgegc@cariari.ucr.ac.cr

Given that in small-amplitude theory it is assumed that $h$ is small the Froude number of the flow investigated will be large, this means the speed of propagation of the gravity waves moving horizontally upstream is far smaller than the background speed of the flow.

In the numerical experiment discussed in this paper the flow parameters were chosen to be such that $h = 10$ m, $u = 20ms^{-1}$, $N = 10^{-2}s^{-1}$, thus the Froude number is equal to 200 which corresponds to a linear regime.

## 2. The numerical model

The numerical model used is a three-dimensional, non-hydrostatic, non-linear, dry numerical model developed at the University of Reading by P. Miranda. The model has been described in various papers, amongst them Miranda and James (1992) and Gutiérrez (1997).

The model uses second-order centred finite differences. The variables are located on a staggered grid (Arakawa C-grid). The time differencing also uses a second-order centred scheme. The model is written for constant grid spacing in each of the two horizontal directions, although $\Delta x$ may be different from $\Delta y$. The vertical grid spacing is treated as a variable. The finite-difference equations are solved only for the interior points. Values of the variables at the boundaries or just outside them (needed due to the staggering of the grid) are obtained from the boundary conditions.

The boundary conditions imposed at the lateral boundaries and at the top of the model domain are intended to minimise wave reflections at those boundaries. The elimination of the wave reflections from the lateral boundaries or from the top of the model is very important in order to be able to isolate and investigate the interaction of continuously stratified flow with isolated orography.

On the top of the model this is done by the inclusion of an absorption layer. Lateral boundary conditions are given by an extrapolation technique known as a radiative boundary condition.

Bottom boundary conditions for the horizontal wind field are free-slip conditions, $\partial u/\partial s = \partial v/\partial s = 0$. For the potential temperature the condition, $\partial\theta'/\partial s = 0$ is applied.

For each experiment, the data input to the model consists of the reference state potential temperature $\theta_s(z)$ and the wind field $(u_s, v_s)$, the reference pressure at $z = 0$ and the reference pressure at the top of the model. The surface pressure and the actual distribution of $\theta_s(x, y, z)$ are obtained iteratively, in a procedure that ensures the existence of hydrostatic equilibrium.

## 3. Smith´s linear theory

Smith (1980) investigated the steady flow of a vertically infinite, continuously stratified Boussinesq fluid, over topography with small amplitude. In what follows the topography will be described by $z = h(x, y)$. Let denote the perturbations to the background wind, pressure, and density fields by the symbols, u´, p´, $\rho$´. After linearising the momentum and continuity equations Smith found the following set of equations:

$$\rho_o u \frac{\partial u'}{\partial x} = -\frac{\partial p'}{\partial x} \qquad (1)$$

$$\rho_o u \frac{\partial v'}{\partial x} = -\frac{\partial p'}{\partial y} \qquad (2)$$

$$\rho_o u \frac{\partial w'}{\partial x} = -\frac{\partial p'}{\partial z} - \rho' g \qquad (3)$$

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0 \qquad (4)$$

$$\rho' = -\frac{d\rho}{dz}\eta \qquad (5)$$

where x,y,z are the downstream, cross-stream and vertical coordinates; u´, v´, w´, $\rho'$, p´, $\eta$ are the perturbation velocity components, the density perturbation, pressure perturbation and vertical displacement, $\rho_o, u, d\rho/dz$, are the background mean density, wind speed and vertical density gradient.

Using the kinematic condition for steady flow and taking u as a constant Smith was able to reduce the system of equations (1), (2), (3), (4), (5) to a single equation for $\eta(x, y, z)$, the vertical displacement for a fluid parcel or a density surface about its undisturbed level.

$$\frac{\partial^2}{\partial x^2}\left(\nabla^2\eta\right) + \frac{N^2}{u^2}\nabla_H^2\eta = 0 \qquad (6)$$

where $N^2 = -\frac{g}{\rho_o}\frac{d\rho}{dz}$ (7),

is a measure of the stratification of the flow. In order to find a solution to (6) Smith represented $\eta$ as a double Fourier integral,

$$\eta(x, y, z) = \iint \hat{\eta}(k, l, z)\, e^{i(kx+ly)}\, dk\, dl \qquad (8)$$

substituting (8) onto (6) the following result can be obtained,

$$\frac{\partial^2 \hat{\eta}}{\partial z^2} + m^2 \eta = 0 \qquad (9)$$

where,

$$m^2 = \frac{k^2 + l^2}{k^2}\left(\frac{N^2}{u^2} - k^2\right) \qquad (10)$$

If $N^2$ is taken as a constant the solution to (9) is found to be

$$\hat{\eta}(k,l,z) = \hat{\eta}(k,l,0)\, e^{im(k,l)z} \qquad (11)$$

When $k^2 > N^2/u^2$, the positive imaginary root of (10) must be selected to eliminate the non-physical growth of the disturbance amplitude with height. In the case of $k^2 < N^2/u^2$ the sign of m needs to be chosen to be the same as the sign of k, in order to satisfy the radiation condition aloft, that is, for westerly flow the phase-lines need to be tilted towards the west. The orography used by Smith was

$$h(x,y) = \frac{h}{\left(\frac{r^2}{a^2} + 1\right)^{3/2}}; \quad \text{where} \quad r = \left(x^2 + y^2\right)^{1/2} \qquad (12)$$

the same type of orography was chosen by Crapper (1959) because of its simple Fourier transform which is:

$\hat{h}(k,l) = \frac{1}{2\pi} h a^2 e^{-a\kappa}$, where $\kappa = \left(k^2 + l^2\right)^{1/2}$ is the module of the horizontal wavenumber vector. One can further simplify the problem by using the hydrostatic approximation in which the vertical acceleration of air parcels is taken to be equal to zero. I

The hydrostatic assumption reduces (10) to

$$m = \frac{N}{u}\frac{\left(k^2 + l^2\right)^{1/2}}{k} \qquad (13)$$

It is found that this approximation is valid for Fourier components with k such that $|k| < N/u$ and is valid for the entire flow field if the mountain is broad enough that it creates only small k components. This is valid when

$$Na/u >> 1 \qquad (14).$$

This last expression can be rewritten as $\frac{a}{u/N} >> 1$ which implies that the horizontal scale of the mountain, a, must be much larger than $u/N$ ; the distance travelled by an air parcel during the time needed to produce a buoyancy oscillation.

The choice of common atmospheric values of $u = 10ms^{-1}$, $N = 0.01s^{-1}$ Smith found that condition (13) is satisfied when the horizontal scale is much greater than 1 km; say, between 5 km to 50 km. No investigation on the effect of background rotation on the flow is shown in this paper, nevertheless it is expected that for flow past mountains broader than 50 km subject to the effect of background rotation will suffer the influence of the Coriolis force.

**Low level flow- Linear theory**

It can be seen, Smith (1980), that in hydrostatic flow, the horizontal structure scales on "a" whereas the vertical structure scales on the length $u/N$ . Smith found that close to the ground the pattern of vertical displacement resembles the surface topography. This is needed in order to satisfy the linear lower boundary condition,

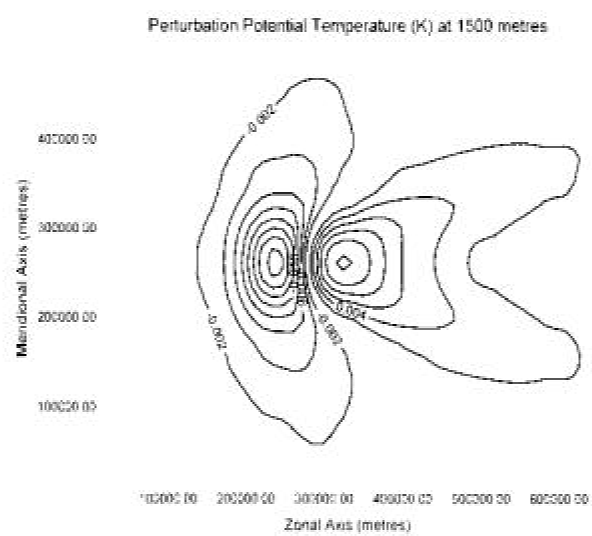


Figure 1: $\theta'$ at 900 metres above level ground. Integration time 20 hours. Fr=200.

$$\eta(x, y, z = 0) = h(x, y) \qquad (15)$$

as one moves upward, there is development of a region of downward displacement over the lee slope of the mountain. Higher up, the region of down motion divides itself and becomes wider forming a U-shaped area.

**Low level flow – Numerical simulations**

The orography used in the calculations shown in this paper corresponds to a bell shaped mountain like that used by Smith in his calculations dealing with linear theory.

The results of a 20-hour integration are shown in figure 1, which shows the $\theta'$ (perturbation potential temperature) field at a height of 0.9 km, that is, close to the orography. This result indicates that when close to the mountain the perturbation to the flow is such that there is a region of negative $\theta'$ located on the upstream side of the orography where air is moving upwards and a region of positive $\theta'$ of descending air located on the lee of the orography. It is important to point out here that the model used is a non-linear numerical model. But given that the flow investigated has a high Froude number the non-linear model is able to simulate linear flow behaviour such as that predicted by Smith´s linear theory.

Figure 2 shows the structure of the flow at higher altitude. This figure shows the $\theta'$ field at 1.5 km above level ground. It can be seen that the area of air moving downwards has become wider, which agrees with the prediction of linear theory.

It is important to notice the general upstream shift of the regions where the air parcels oscillate is consistent with the radiation condition aloft which implies there is westward tilting of the phase lines. This is shown in figure 3 where a cross-section of the flow is presented. The abscissa represents the zonal direction and the ordinate represents height. This figure clearly shows how phase lines tilt upstream which explains the generation of pressure perturbations that will produce a surface pressure drag. The field shown in this figure is the vertical velocity field.

In order to evaluate the surface pressure field one may combine the expression $p' = -(d\rho/dz)\eta$ with the hydrostatic form of the relation $\rho_o u\, \partial w'/\partial x = -\partial p'/\partial z - \rho' g$.

Integrating vertically yields,

$$p'(x, y, z) = -g \frac{d\overline{\rho}}{dz} \int_z^{\infty} \eta(x, y, z')dz' \qquad (16)$$

where the perturbation pressure far away from the obstacle is taken to be nil, one finds,

$$p'(x, y, 0) = -\rho_o NUh \frac{x/a}{\left(1 + r^2/a^2\right)^{3/2}} \qquad (17)$$

since $x < 0$ on the upstream side of the mountain $p' > 0$ which means high pressure is found on the windward side of the orography. On the lee side of the obstacle $x > 0$ and consequently $p' < 0$, that is, low pressure exists on the downstream side of the orography. This surface pressure dipole pattern is bound to create surface pressure drag.

It can be shown Miranda (1990) that in the hydrostatic, non-rotating, limit when $N^2 >> (Uk)^2$, then

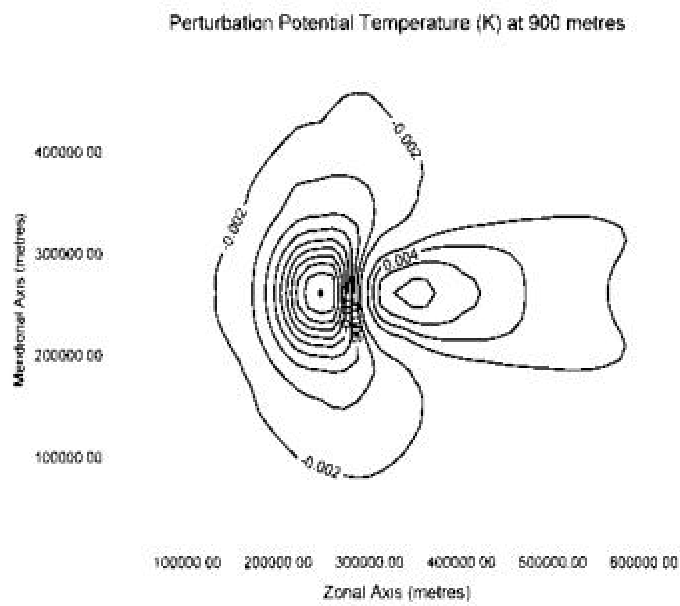


Figure 2: $\theta'$ at 1500 metres above level ground. Integration time 20 hours. Fr=200.

$$Drag_{linear} = \frac{\pi}{4} \rho_o NUah_o^2 \qquad (18)$$

This quantity can be used in order to see how good linear theory is to describe the flow investigated here. Since one can compare the evolution of the normalised surface pressure

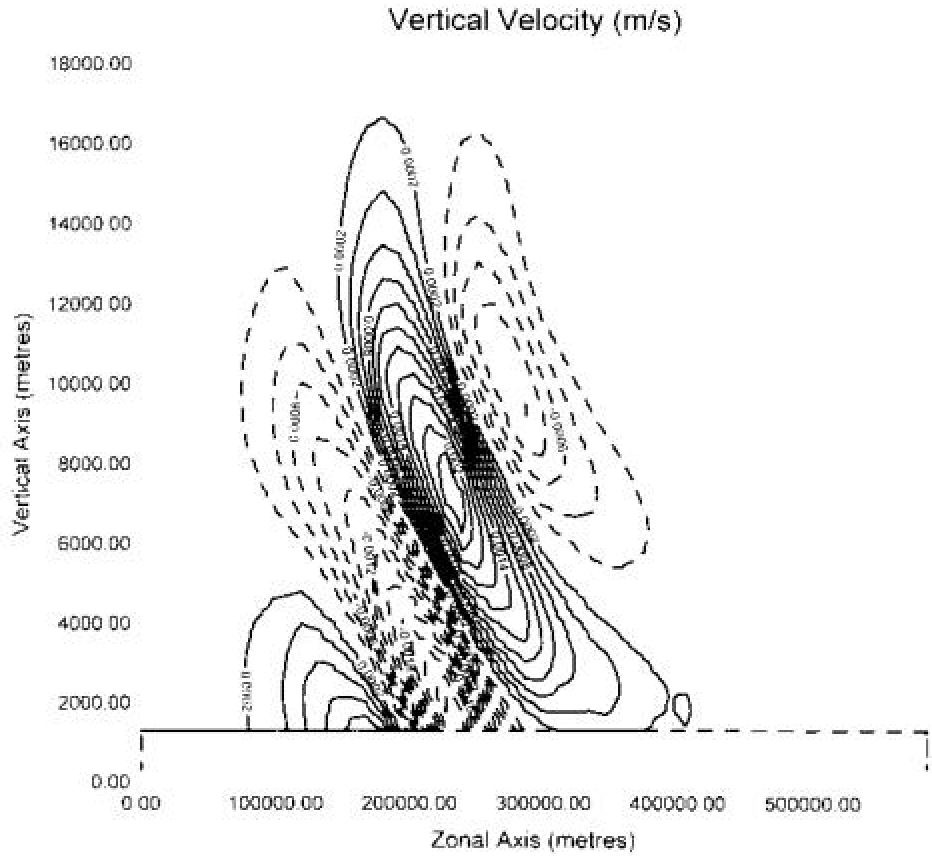


Figure 3: Zonal cross-section of the vertical velocity field passing through the centre of the orography. Integration time 20 hours. Fr=200.

drag defined as $D_{norm} = D/D_{linear}$ with respect to the evolution of the surface pressure drag predicted by linear theory. Differences between $D_{norm}$ and 1 will mean the behaviour of the flow been simulated is not purely linear.

The flow in the Reading Model is started impulsively, this causes strong fluctuations at the beginning of the simulation and after five hours of simulation a steady-state is reached in which case Dn is very close to one. This can be seen in figure 4.

In the Reading Model the drag exerted by the atmosphere on the mountain is computed by the expression,

$$\vec{D} = (D_x, D_y) = -\iint p'\vec{\nabla}h(x,y)dxdy \qquad (19)$$

where the surface integral is computed over the whole physical domain of the model (symmetric), $p'$ is the surface pressure perturbation and $h(x,y)$ is the orography.

The negative sign in (19) refers to the fact that it is the force acting on the atmosphere that is considered. Sometimes, the surface pressure drag associated with mountain ranges is also called mountain drag.

In geophysical flows, there are many mechanisms that can possibly contribute to pressure differences along the sides of a mountain and therefore to pressure drag. The responsible mechanisms are physically distinct but a combination can be possible depending on the scale of the mountain and the atmospheric conditions. These mechanisms may include, see Athanassiadou (1998),

-Flow separation
-Vertically propagating internal gravity waves
-Trapped (resonant) internal gravity waves

When the rotation of the earth is included in the study of orographic flows, there is additional pressure force acting on the mountain due to the background pressure gradient associated with the mean flow. The mountain finds itself in a pressure field, and according to Archimedes principle, experiences a force given by the mountain volume, $V_{mount}$, times the background pressure gradient:

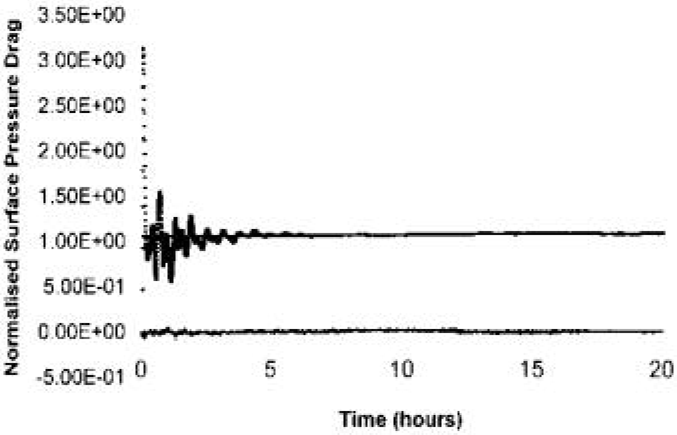


Figura 4. Normalised surface pressure drag componnts. Integration time 20 hours. Fr=200.

$$\vec{G} = V_{mount}\vec{\nabla}p = -V_{mount}(\vec{k} \times \boldsymbol{r}_o f\vec{u}_g)$$

where $p$, $\boldsymbol{r}_o, \vec{u}_g$ are the mean horizontal pressure, mean density and mean geostrophic wind respectively. This force always acts at a right angle to the mean flow and for that reason is occasionally referred to as "lift" force.

When there is no bacground rotation, $f = 0$, and hence the lift force on the mountain is nil. This is the case of the numerical experiment discussed in this paper. This result is seen in figure 4 where the meridional component of the drag is the line having a constant value of zero.

**High-level flow – Linear theory**

It is possible to show, Smith (1980), that far away from the mountain on the upstream side of the orography, $x < 0$,

$$\boldsymbol{h}(\hat{r}, \boldsymbol{q}, \hat{z}) = 0 + O\left(\frac{1}{\hat{r}^2}\right) \qquad (20)$$

where,

$\hat{z} = z\,N/U$, $\hat{x} = x/a$, $\hat{y} = y/a$, $\hat{r} = (\hat{x}^2 + \hat{y}^2)^{1/2}$, $\boldsymbol{q} = arctg(\hat{y}/\hat{x})$.

whereas downstream, $x > 0$,

$$\boldsymbol{h}(\hat{r}, \boldsymbol{q}, \hat{z}) = \frac{2h}{\hat{r}}\left|\boldsymbol{b}e^{-b}\right|\cos\left(\frac{\hat{z}}{\sin\boldsymbol{q}}\right) + O\left(\frac{1}{\hat{r}^2}\right) \qquad (21)$$

where $\boldsymbol{b} = \hat{z}\,\hat{x}/\hat{y}^2$. Written in dimensional form the previous equation becomes,

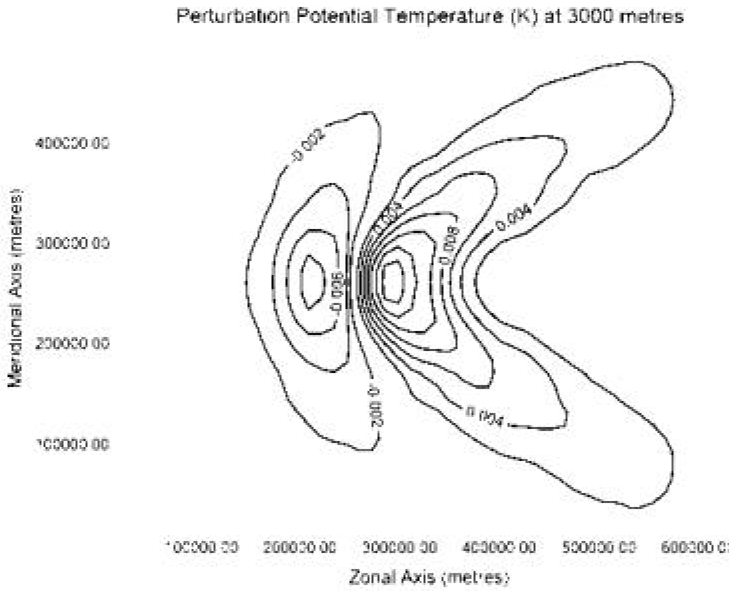


Figure 5: $\boldsymbol{q}'$ at 3000 metres above level ground. Integration time 20 hours. Fr=200.

$$\boldsymbol{h}(x, y, z) = 2h\frac{a}{r}\left|\boldsymbol{b}e^{-b}\right|\cos\left(\frac{Nz}{U\sin\boldsymbol{q}}\right) + O\left(\frac{a}{r}\right)^2 \qquad (22)$$

with $\boldsymbol{b} = \dfrac{Nzax}{Uy^2}$.

As pointed out by Smith (1980) the amplitude factor $\left|\boldsymbol{b}e^{-b}\right|$ has a maximum at

$$\boldsymbol{b} = 1 = (Nzax)/Uy^2 \qquad (23)$$

Since wave energy is proportional to the square of the amplitude of the wave this implies that wave energy will be concentrated in regions where the above condition is met or at least approximately satisfied.

For a given height $z$, equation (23) represents a parabola whose vertex is located at the origin, that is, at the centre of the orography and trails downstream. As is easily seen in (23) the parabola becomes wider as the height $z$ increases. It is possible to see this behaviour in figures 3 and 5 discussed in next section.

**High-level flow---Numerical calculations**

It is possible to confirm the behaviour predicted by linear theory discussed in the previous section by looking at the results of numerical calculations which are shown in the sequence of figures below. Figure 5 shows the perturbation potential temperature at a height of 3 km above level ground and figure 6 shows the same field at a height of 10 km above level ground. It is obvious the lee side of the flow has a parabolic shape which increases with height as predicted by equation 23.

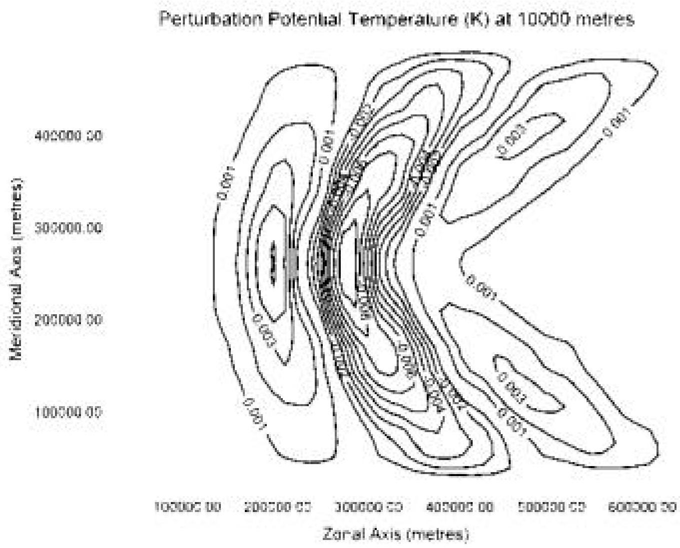


Figure 6: $q'$ at 10000 metres above level ground. Integration time 20 hours. Fr=200.

The region of ascending flow on the lee of the orography depicted in figure 6 is due to the ascending part of a mountain wave. It is easier to have an idea about the three-dimensional nature of the flow by looking at a vertical cross-section of the atmosphere. The vertical cross section shown in figure 3 corresponds to a vertical plane passing through the centre of the orography along the zonal direction.

The field shown in figure 3 is the vertical velocity. It is possible to see a set of four distinct regions in which air ascends (continuous line) and then descends (broken line). Note that the phase lines of the mountain waves are tilted toward the west which is the direction of the upcoming flow. This agrees with the results of linear theory.

It is now easy to identify the three regions of descending-ascending-descending air of figure 6 with the three parts of the mountain wave, shown in figure 3.

Figure 7 shows the vertical cross section of the vertical velocity field. This time the plane passes through the centre of the orography along the meridional direction. It depicts a cross-section of the mountain wave where the lower half of the figure shows a region of descending air (the descending part of the wave) and the upper half shows the ascending part of the mountain wave.

**Conclusions**

Smith's linear theory has been tested by means of a non-linear, three-dimensional computer model run with flow parameters such that the Froude number is equal to 200. It is found that linear theory's predictions are in good agreement with the numerical calculations for a simulation time of twenty hours.

The parabolic region located at low levels on the lee-side of the orography predicted by linear theory has been found to be generated in the simulations. It is also seen that, in accordance with linear theory the parabola opens it branches at higher levels in the flow.

Also in agreement wth linear theory consecutive regions of ascending-descending air are found at even higher levels of the flow. These patterns are seen in horizontal cross section of the flow which corresponds to an upward propagating mountain wave. Vertical cross sections of the flow along the zonal and meridional axes allow a better view of this orographic mountain wave.

Surface pressure perturbations are found to be

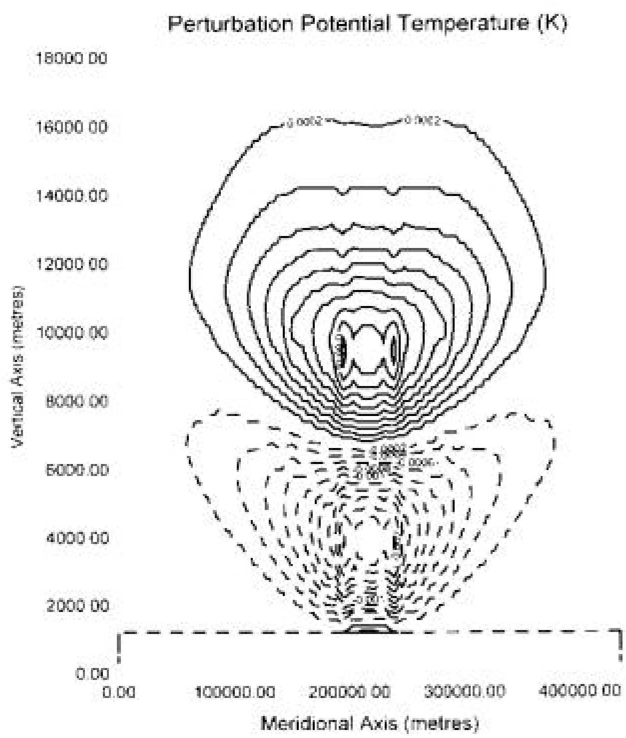


Figure 7: Meridional cross section of the vertical velocity field passing through the centre of the Orography. Integration time 20 hours. Fr=200.

generated and create a surface pressure drag whose computed value is very close to the linear prediction. In absence of background rotation in the flow the meridional component of the surface pressure drag is found to be nil.

**Resumen**

Se investiga la generación de ondas de montaña al interactuar el viento con orografía aislada en el caso de flujo con alto número de Froude. Las simulaciones numéricas fueron realizadas con un modelo numérico tridimensional y no lineal que fue corrido en una estación de trabajo *Sun* del Laboratorio de Investigaciones Geofísicas de la Universidad de Costa Rica. Se encuentra que la generación de ondas de montaña en los cálculos numéricos concuerda con la predicha por la teoría lineal de Smith. También se encuentra que el cálculo del arrastre para esta simulación da un resultado muy similar al predicho por la teoría lineal.

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