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QUADRATIC VARIATION FOR CYLINDRICAL MARTINGALE-VALUED MEASURES

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ABSTRACT. This article focuses in the definition of a quadratic variation for cylindrical orthogonal martingale-valued measures defined on Banach spaces. Sufficient and necessary conditions for the existence of such a quadratic variation are provided. Moreover, several properties of the quadratic variation are explored, as the existence of a quadratic variation operator. Our results are illustrated with numerous examples and in the case of a separable Hilbert space, we delve into the relationship between our definition of quadratic variation and the intensity measures defined by Walsh (1986) for orthogonal martingale measures with values in separable Hilbert spaces. We finalize with a construction of a quadratic covariation and we explore some of its properties.

CONTENTS

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1. Introduction

In recent years, there has been an increasing interest in the usage of cylindrical stochastic processes as models for the perturbation of infinite dimensional systems, in particular of stochastic partial differential equations (e.g. [14, 21, 23, 30, 32, 33, 35, 37]). In most of these works, the cylindrical process is a cylindrical martingale, or more generally a cylindrical semimartingale. A cylindrical martingale M on a Banach space X is a linear operator such that, for each $x^* \in X^*$, $M(x^*)$ is a (real-valued) martingale. Usually suitable continuity conditions are required for M.

Another popular alternative for the modeling of the noise of a stochastic partial differential equation is a martingale-valued measure (e.g. [2, 6, 8, 9, 10, 16]). This concept was introduced by Walsh in [38] and was motivated by the space-time Gaussian white noise. Roughly speaking, a martingale-valued measure is a family $(M(t, A) : t \geq 0, A \in \mathcal{A})$ such that $(M(t, A) : t \geq 0)$ is a real-valued square integrable martingale for each $A \in \mathcal{A}$ and $M(t, \cdot)$ is an L^2-valued finitely additive measure on $\mathcal{A}$ for each $t \geq 0$. Here $\mathcal{A}$ is a ring of Borel subsets of a topological space U. See Definition 3.3 for further details.

Motivated by the above, in this paper we introduce the concept of cylindrical martingale-valued measures and study some of its properties. This concept is a hybrid between the definitions of cylindrical martingale and of martingale-measure. Indeed, in this case we have a family $(M(t, A) : t \geq 0, A \in \mathcal{A})$ such that, for each $A \in \mathcal{A}$, $(M(t, A) : t \geq 0)$ is cylindrical square integrable martingale on a Banach space X and $M(t, \cdot)$ is an L^2-valued finitely additive measure on $\mathcal{A}$ for each $t \geq 0$. See Definition 4.1 for further details.

A particular family of cylindrical martingale-valued measures was introduced in earlier works [1, 15]. There, they developed a theory of stochastic integration and prove existence and uniqueness of solutions to certain stochastic partial differential equations with these cylindrical martingale-valued measures as noise. However, these families of cylindrical martingale-valued measures were tailor-made to provide a unifying model for cylindrical and classical Lévy processes. To the extent of our knowledge, no other work have considered the study of cylindrical martingale-valued measures.

Our treatment of cylindrical martingale-valued measures will be centered around the concepts of (predictable) quadratic variation and quadratic covariation. Our main motivation is to introduce a theory that can be used to develop stochastic calculus for cylindrical martingale-valued measures which admit such a quadratic variation.

In the literature, one can find previous attempts to define a quadratic variation for Banach space-valued martingales and for cylindrical martingales. We can cite for example [24] for Hilbert space-valued martingales, [12, 13] for local martingales with values in a separable Banach space, [26, 27] for the case of some particular classes of cylindrical local martingales, and [37] for cylindrical continuous local martingales on a Banach space. Indeed, many of the results obtained in this paper are motivated to generalize those obtained in [37] to the context of cylindrical martingale-valued measures.

Now we describe our results and the organization of this article. We introduce some preliminaries in Section 2, placing special emphasis on the concept of supremum of measures, generalizing the one that was introduced in [37]. The supremum of measures is a relevant tool for us in building a convenient definition of quadratic variation for cylindrical martingale-valued measures. However, the results in [37] need to be applied in a more general setting. For instance, we deal with the supremum for families of random measures (see for example the proofs of Theorem 3.6 and Lemma 3.9). We also need to be able to talk about the supremum of a family of sub-additive functions (Theorem 7.12). Lemmas 2.1 and 2.2, as well as Corollaries 2.3 and 2.4 establish the right generalizations we need.

In Section 3 we study the Hilbert-space valued orthogonal martingale-valued measures. Our definition is largely based on the original concept introduced by Walsh in [38] for the real-valued setting. Our main result, Theorem 3.6, shows the existence of an intensity measure for an H-valued orthogonal martingale-valued measure $(M(t,A) : t \geq 0, A \in \mathcal{A})$, that is a regularization of $\langle M(A) \rangle_t$ by means of a random predictable σ-finite measure ν on $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(U)$. We emphasize the main ideas of the original proof of [38] and take advantage of them to establish a result (Lemma 3.9) that allows us to compare two random measures on $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(U)$ by comparing them for particular values of t and $A \in \mathcal{A}$. Some applications of this result can be found in Theorem 5.8, Lemma 6.1 and Theorem 7.3. We finalize this section with a couple of examples, in particular one of a white noise measure (see Example 3.13) and the other with a construction of an H-valued orthogonal martingale-valued measure from an H-valued Lévy process (see Example 3.14).

The definition of cylindrical martingale-valued measures is introduced in Section 4. We explore the usefulness of the concept by giving some examples. In particular, in Example 4.4 we explore a finite sum of cylindrical martingales with disjoint supports, keeping orthogonality. Example 4.5 illustrates that many constructions of stochastic analysis can be modeled with the concept of martingale-valued measures. Finally, in Proposition 4.6 we see that an H-valued orthogonal martingale-valued measure induces a cylindrical orthogonal martingale-valued measure in a natural way. See Example 4.7 for a taste of the kind of new processes that this allows to handle.

The construction and study of properties of the quadratic variation for a cylindrical martingale-valued measure is carried out in Section 5. We start in Section 5.1 with the definition of a quadratic variation. Since our definition of cylindrical martingale-valued measure encloses both the concept of cylindrical martingale and of martingale-valued measure, we found convenient to formulate our definition of quadratic variation as a supremum of a family of measures. This follows what is done by Veraar and Yaroslavtsev in [37], but taking this family of measures as the intensity measures (defined in Section 3) for the real-valued martingale-valued measures $(M(t,A)(x^*) : t \geq 0, A \in \mathcal{A})$, where $x^* \in X^*$ (see Definition 5.3 for further details).

Our definition allows the existence of several quadratic variation processes. In order to establish sufficient conditions for uniqueness, we introduce the sequential boundedness property on the family of intensity measures $(\nu_{x^*} : x^* \in X^*)$ for $M(t,A)(x^*)$ (Definition 5.3). This property helps us to obtain a unique simple expression for our quadratic variation $\langle\langle M \rangle\rangle$, as a supremum of measures on a countable and dense collection of elements in the unit sphere of X^* (see Theorems 5.8 and 5.10). The sequential boundedness property is obtained in a natural way in the case of cylindrical martingales with continuous paths (see Example 5.13), thus, generalizing the work in [37] for this case. Moreover, the sequential boundedness property is a weaker condition than the uniform continuity of the family of intensity measures (see Proposition 5.7).

The characterizations obtained in the previous results allow us to explicitly construct the quadratic variation in some cases, for example, in the case of a cylindrical Lévy process (see Example 5.14) and for the cylindrical measure induced by an H-valued Lévy process (see Example 5.16). We also include some examples in which some cylindrical martingale measures do not have a quadratic variation (see Examples 5.15 and 5.17).

In Section 5.2, the main purpose is to obtain a Radon-Nikodym representation $d\alpha_M = Q_M \, d \langle\!\langle M \rangle\!\rangle$ (see Theorem 5.23), where Q_M takes values in $\mathcal{L}(X^*, X^{**})$ and the vector-valued measure α_M corresponds to a covariation operator (5.13).

Our approach is similar in nature to the one in [37], however, we use a (random) vector-valued measure instead of vector-valued stochastic process, thus implying substantial differences in the proof. This was necessary due to the irregular nature of M compared to the cylindrical continuous case in [37]. This representation will be of crucial importance in the characterization of the class of integrands for the stochastic integral with respect to M, result to appear elsewhere.

Finally in that section we compute the representation for the cylindrical martingale-valued measures corresponding to a cylindrical Lévy process and to the one induced by an H-valued Lévy process (see Examples 5.24 and 5.25). This is also carried out for the class of cylindrical martingale-valued measures introduced in [1] (see Example 5.26).

In Section 6 we study the relation between the intensity measure of an H-valued martingale-valued measure and the corresponding quadratic variation of the induced cylindrical martingale-valued measure. As main results, we obtain that the intensity measure associated to an H-valued process and the quadratic variation of the induced cylindrical process are equivalent in the sense of random measures (Theorem 6.7).

Finally, in Section 7 we address the issue of defining a vector measure for the quadratic covariation of a couple of cylindrical orthogonal martingale-valued measures M and N. In order to do that, both measures must somehow behave in a synchronized way. We introduce the concepts of mutual orthogonality and compatibility, which allows us to first define the sum $M + N$ and to compare the quadratic variation of $M + N$ with those of M and N (Theorem 7.3).

We define the vector measure $\alpha_{M,N}$ associated to the quadratic covariation in terms of α_{M+N} and α_{M-N}, by using the classical polarization identity. Here, the results of Section 2 on the supremum of a family of sub-additive non-negative set functions, are crucial to define a (positive) covariation measure $\langle\!\langle M, N \rangle\!\rangle$, which yields a corresponding quadratic covariation operator $Q_{M,N}$ which satisfies a Radon-Nikodym representation $d\alpha_{M,N} = Q_{M,N} \, d \langle\!\langle M, N \rangle\!\rangle$ (see Theorem 7.15).

2. Preliminaries and Notation

2.1. **Stochastic and cylindrical processes.** For a Banach space X we denote by X^* its (strong) dual space. In this work, the norm of the underlying Banach space will be often denoted by $\|\cdot\|$, but when it is necessary to emphasize the space, we use the notation $\|\cdot\|_X$. We will use the notation H for a Hilbert space with inner product $(\cdot,\cdot)_H$. We identify the dual of a Hilbert space with the space itself. For a Hausdorff topological space U its Borel σ-algebra will be denoted by $\mathcal{B}(U)$.

For any two Banach spaces X and Y, the Banach space of bounded linear operators form X into Y will be denoted by $\mathcal{L}(X,Y)$. We will denote the space of real-valued bounded bilinear forms on $X \times Y$ by $\mathfrak{Bil}(X,Y)$. Trace class operators on a Hilbert space H will be denoted by $\mathcal{L}_1(H)$.

Throughout this work we assume that $(\Omega,\mathcal{F},\mathbb{P})$ is a complete probability space equipped with a filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ that satisfies the *usual conditions*, i.e. it is right continuous and $\mathcal{F}_0$ contains all subsets of elements in $\mathcal{F}$ of $\mathbb{P}$-measure zero. We denote by $L^0(\Omega,\mathcal{F},\mathbb{P})$ the space of equivalence classes of real-valued random variables defined on $(\Omega,\mathcal{F},\mathbb{P})$. The space $L^0(\Omega,\mathcal{F},\mathbb{P})$ will be always equipped with the topology of convergence in probability and in this case it is a complete, metrizable, topological vector space.

Let X be a Banach space and let (S,Σ) be a measurable space. A function $\alpha:\Sigma\to X$ is called a *vector measure*, if whenever E_1 and E_2 are disjoint members of Σ then $\alpha(E_1\cup E_2) = \alpha(E_1)+\alpha(E_2)$. If, in addition, $\alpha\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty}\alpha(E_n)$ in the norm topology of X for all collection (E_n) of pairwise disjoint members of Σ, then α is called a *countably additive vector measure* or simply, α is countably additive. In any case, the *variation* of α is the extended non-negative function $|\alpha|$ whose value on a set $E\in\Sigma$ is defined by

$$|\alpha|(E) = \sup_{\Pi}\sum_{A\in\Pi}\|\alpha(A)\|,$$

where the supremum is taken over all finite partitions Π of E by members of Σ. If $|\alpha|(S) < \infty$, then α will be called a *vector measure of bounded variation.* (See [11] for further details).

A mapping $\mu:\Omega\times\Sigma\to[0,\infty]$ will be called a *random measure* if for all $E\in\Sigma$, $\omega\mapsto\mu(\omega,E)$ is measurable and for (almost) all $\omega\in\Omega$, $\mu(\omega,\cdot)$ is a measure on (S,Σ).

The predictable σ-algebra on $\Omega\times[0,\infty)$ (see Chapter 10 in [17]) is denoted by $\mathcal{P}$ and for any $T>0$ we denote by $\mathcal{P}_T$ the restriction of $\mathcal{P}$ to $\Omega\times[0,T]$. Let (S,Σ) be a measurable space and let $(\mu_t : t\geq 0)$ be a family of random measures on (S,Σ). We say that $(\mu_t : t\geq 0)$ is *predictable* if for any given $A\in S$ the mapping $(\omega,t)\mapsto\mu_t(\omega)(A)$ is predictable.

Let $I_T=[0,T]$ for $0<T<\infty$ or $I_T=[0,\infty)$ for $T=\infty$. In either case, denote by $\mathcal{M}_T^2$ the linear space of all the real-valued, càdlàg, mean-zero, square integrable martingales on the time interval I_T. The space $\mathcal{M}_T^2$ is Banach when equipped with the norm

$$\|m\|_{\mathcal{M}_T^2} = \left(\sup_{t \in I_T} \mathbb{E}\left[|m(t)|^2 \right] \right)^{1/2}.$$

Let X be a Banach space. A *cylindrical random variable* on X is a linear and continuous operator $Z : X^* \to L^0(\Omega, \mathcal{F}, \mathbb{P})$. A family of cylindrical random variables $Z = (Z_t : t \in I_T)$ on X is called a *cylindrical stochastic process* on X. A cylindrical stochastic process $M = (M_t : t \in I_T)$ is called a *cylindrical mean-zero square integrable martingale* on X is for every $x^* \in X^*$ we have $M(x^*) \in \mathcal{M}_T^2$.

Below we review some basic properties of the quadratic (co)variation. See Section 20 in [25] for proofs and further details.

Let $(H, (\cdot,\cdot)_H)$ be a separable Hilbert space and let $Y = (Y_t : t \geq 0)$ and $Z = (Z_t : t \geq 0)$ denote two H-valued square integrable martingales. There exists a unique (up to indistinguishability) predictable, real-valued process $\langle Y \, , \, Z \rangle$ with paths of finite variation such that $\langle Y \, , \, Z \rangle_0 = 0$ and $(Y,Z)_H - \langle Y \, , \, Z \rangle$ is a martingale. We call $\langle Y \, , \, Z \rangle$ the *(predictable) quadratic covariation of* Y *and* Z. We denote $\langle Y \, , \, Y \rangle$ by $\langle Y \rangle$ and call it the *(predictable) quadratic variation of* Y; this process is increasing. One can show that for any given $t \geq 0$

$$\sum_{j=1}^{\infty} \mathbb{E}\left[\left(Y_{t\wedge t^n_{j+1}} - Y_{t \wedge t^n_j}, Z_{t\wedge t^n_{j+1}} - Z_{t\wedge t^n_j} \right)_H \middle| \mathcal{F}_{t^n_j} \right] \xrightarrow{\mathbb{P}} \langle Y \, , \, Z \rangle_t, \tag{2.1}$$

where $\{0 = t^n_0 < t^n_1 < \cdots < t^n_k < \cdots\}_{n\geq 1}$ is a sequence of partitions of $[0,\infty)$ such that $t^n_j \to \infty$ as $j \to \infty$ and $\delta_n = \sup_j \left| t^n_{j+1} - t^n_j \right| \to 0$ as $n \to \infty$. For any $h \in H$, one can easily check that $Y(h) := ((Y_t, h)_H : t \geq 0)$ is a square integrable real-valued martingale, hence has (predictable) quadratic variation $\langle Y(h) \rangle$. We can relate $\langle Y \rangle$ and $\langle Y(h) \rangle$ via (2.1) as follows. Given $h, g \in H$, for any sequence of partitions $\{0 = t^n_0 < t^n_1 < \cdots < t^n_k < \cdots\}_{n \geq 1}$ of $[0, \infty)$ as described above, we have

$$\begin{aligned}\sum_{j=1}^{\infty} \mathbb{E}\left[\left(Y(h)_{t\wedge t^n_{j+1}} - Y(h)_{t\wedge t^n_j} \right) \left(Y(g)_{t\wedge t^n_{j+1}} - Y(g)_{t\wedge t^n_j} \right) \middle| \mathcal{F}_{t^n_j} \right] \\ \leq \|h\| \, \|g\| \sum_{j=1}^{\infty} \mathbb{E}\left[\left\| Y_{t\wedge t^n_{j+1}} - Y_{t\wedge t^n_j} \right\|^2 \middle| \mathcal{F}_{t^n_j} \right].\end{aligned}$$

Then taking limits as $n \to \infty$ and by the uniqueness of limits in probability we conclude from (2.1) that $\mathbb{P}$-a.e.

$$\langle Y(h) \, , \, Y(g) \rangle_t \leq \|h\| \, \|g\| \, \langle Y \rangle_t. \tag{2.2}$$

2.2. **Supremum of measures.** In this section we summarize some results on the supremum of a family of measures introduced in [37], and try to write them in a more general context to make them useful when dealing with quadratic variations. For measures μ and ν defined on a measurable space (S, Σ), we write $\nu \leq \mu$ if

$$\forall A \in \Sigma \quad \nu(A) \leq \mu(A).$$

This defines a partial ordering on the class $\mathcal{M}_+(S, \Sigma)$ of all measures on (S, Σ). Given a family $(\mu_\alpha)_{\alpha \in \Lambda}$ of measures on (S, Σ), there is always a supremum, that is, there exists $\check{\mu} \in \mathcal{M}_+(S, \Sigma)$ that satisfies

(i) For each $\alpha \in \Lambda$, $\mu_\alpha \leq \check{\mu}$

(ii) If $\mu \in \mathcal{M}_+(S, \Sigma)$ and $\mu_\alpha \leq \mu$ for each $\alpha \in \Lambda$, then $\check{\mu} \leq \mu$.

The following lemma establishes the existence we have just mentioned. We write it in the generality we need. We omit the proof, since the one in [37, Lemma 2.6] works perfectly well if we only start with a family of sub-additive set functions. We refer to [5] for a more general treatment of the concept.

Lemma 2.1. *Let $(\mu_\alpha)_{\alpha \in \Lambda}$ be a family of non-negative, finitely sub-additive set functions on a measurable space (S, Σ). For $A \in \Sigma$, let $\mathscr{P}(A)$ be the collection of finite partitions of A, by elements of Σ. Define the set function $\check{\mu}$ by*

$$\check{\mu}(A) = \sup_{\Pi \in \mathscr{P}(A)} \sum_{C \in \Pi} \sup_{\alpha \in \Lambda} \mu_\alpha(C), \qquad A \in \Sigma. \tag{2.3}$$

Then $\check{\mu}$ is the smallest (in particular, unique) finitely additive measure on (S, σ) that satisfies $\check{\mu} \geq \mu_\alpha$ for each $\alpha \in \Lambda$. If each μ_α is sub-additive, then $\check{\mu}$ is a measure on (S, Σ).

The functions μ_α might be all finite and still have $\check{\mu}(A) = \infty$ at each nonempty set; an easy example can be obtained with S finite and $\mu_n = n\mu_0$, μ_0 being the counting measure. But if there is a finite measure dominating each μ_α, it follows that $\check{\mu}$ is a finite measure.

We call $\check{\mu}$ the supremum measure of the family (μ_α) and denote it by $\sup_{\alpha \in \Lambda} \mu_\alpha$. Notice that

$$\left(\sup_{\alpha \in \Lambda} \mu_\alpha \right)(A) \geq \sup_{\alpha \in \Lambda} \mu_\alpha(A).$$

The right hand side is computed, for each A, as the classical supremum of a set of real numbers, which in general does not define a measure, not even for a family of two measures.

The following lemma is a generalization of Lemma 2.8 in [37] for the case of a family of random measures.

Lemma 2.2. *Let (S,Σ,ν) be a measure space, $(\Omega,\mathcal{F},\mathbb{P})$ a probability space. Let $\mathscr{F}$ be a family of measurable functions from $\Omega\times S$ into $[0,\infty]$ and $(f_j)_{j\in\mathbb{N}}$ a sequence in $\mathscr{F}$. Define $\overline{f}=\sup_{j\geq 1} f_j$ and assume that $\sup_{f\in\mathscr{F}} f=\overline{f}$. For each $f\in\mathscr{F}$ let μ_f be the random measure defined by*

$$\mu_f(\omega,B)=\int_B f(\omega,\cdot)d\nu,\quad B\in\Sigma.$$

If we define $\check{\mu}(\omega,\cdot):=\sup_{f\in\mathscr{F}}\mu_f(\omega,\cdot)$, then $\check{\mu}(\omega,\cdot)=\sup_{j\geq 1}\mu_{f_j}(\omega,\cdot)$ and

$$\check{\mu}(\omega,B)=\int_B \overline{f}(\omega,\cdot)d\nu. \tag{2.4}$$

In particular $\check{\mu}$ is a random measure on (S,Σ,ν).

Proof. For fixed ω we apply Lemma 2.8 in [37] to obtain both identities. Tonelli's theorem allows us to conclude that each μ_f is a random measure, so the same is true for $\check{\mu}$ because of (2.4). □

In the case of a countable number of measures, Lemma 2.9 of [37] gives us an expression for $\check{\mu}$ as the limit

$$\check{\mu}(A)=\lim_{N\to\infty}\left(\sup_{1\leq n\leq N}\mu_n\right)(A).$$

Remark 2.10 in [37] also shows that, in the case $S=\mathbb{R}$ and $\Sigma=\mathcal{B}_{\mathbb{R}}$ one has

$$\check{\mu}(a,b]=\sup_{\Pi\in\mathscr{R}}\sum_{C\in\Pi}\sup_{\alpha}\mu_\alpha(C),$$

where $\mathscr{R}$ is the family of all Riemann-type partitions of $(a,b]$, and the endpoints of the sub-intervals in such partitions can be taken rational (with the possible exception of a and b). These ideas have the following useful corollary, which we present in a bit more general setting, as will be needed afterwards.

Corollary 2.3. *The supremum of a countable family of random measures over $(\mathbb{R},\mathcal{B}_{\mathbb{R}})$ is itself a random measure. Moreover, if $\mu_n(\omega,\cdot)\leq\mu_{n+1}(\omega,\cdot)$ for each n, then for a.e. ω*

$$\left(\sup_{n\in\mathbb{N}}\mu_n(\omega,\cdot)\right)(A)=\lim_{n\to\infty}\mu_n(\omega,A),\quad A\in\mathcal{B}(\mathbb{R}).$$

The following equivalence for an infinite sum of measures is not difficult to establish.

Corollary 2.4. *The infinite sum of random measures is a random measure. Moreover*

$$\left(\sum_{n=1}^{\infty}\mu_n\right)(A)=\sum_{n=1}^{\infty}\mu_n(A)=\left(\sup_F\sum_{n\in F}\mu_n\right)(A)$$

where F runs over the finite subsets of $\mathbb{N}$.

3. Hilbert space-valued martingale-valued measures

Let H be a separable Hilbert space. In this section we recall the definition of H-valued martingale-valued measures, originally introduced by Walsh in [38] in the real-valued setting. Assume that U is a Hausdorff topological space which is Lusin in the sense that it is homeomorphic to a Borel subset of the line. Let $\mathcal{A}$ be a ring of Borel subsets of U. We also consider an underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Definition 3.1. A set function $N : \Omega \times \mathcal{A} \to H$ is called a σ-finite $L^2(\Omega, \mathcal{F}, \mathbb{P}; H)$-valued measure if it satisfies:

(i) N is L^2-valued: $\mathbb{E}[\|N(A)\|^2] < \infty$ for every $A \in \mathcal{A}$.

(ii) N is finitely additive: if $A, B \in \mathcal{A}$ are disjoint then $\mathbb{P}$-a.e.

$$N(A \cup B) = N(A) + N(B).$$

(iii) N is σ-finite: there exists a sequence $(U_n)_{n\in\mathbb{N}}$ in $\mathcal{B}(U)$ such that $U_n \uparrow U$ for each n, $\mathcal{B}(U_n) \subseteq \mathcal{A}$ and

$$\sup_{A\in\mathcal{B}(U_n)} \mathbb{E}[\|N(A)\|^2] < \infty.$$

(iv) N is *countably additive* on each U_n (as an $L^2(\Omega, \mathcal{F}, \mathbb{P}; H)$-valued function): For each sequence $(A_j)_{j\in\mathbb{N}}$ in $\mathcal{B}(U_n)$ decreasing to $\emptyset$, we have $\mathbb{E}[\|N(A_j)\|^2] \to 0$.

(v) For each $A \in \mathcal{A}$, $N(A)$ is the L^2-limit of $N(A \cap U_n)$.

As we mentioned before, this definition is based on that of [38]. We have adapted it to the Hilbert space-valued case and implemented some minor modifications. Property (v) is treated in [38] as an extension procedure, however, this could actually result in a modification[1] of N on some sets $A \in \mathcal{A}$; moreover, it is not clear in general if the limit even exists for every set $A \in \mathcal{A}$. We decided to make it part of the definition. We do observe that this condition would be superfluous if, for instance, N is countably additive on $\mathcal{A}$.

Remark 3.2. Since our space is Lusin, there exists a sequence of sets that generates $\mathcal{B}(U)$. Given a σ-finite mapping N, by taking intersections with the generating countable collection if necessary, we can assume the sequence (U_n) generates $\mathcal{B}(U)$, instead of being increasing.

[1]Consider $U = \mathbb{R}$ and $\mathcal{A} = \{A \in \mathcal{B}(\mathbb{R}) : A \text{ is bounded or } A^c \text{ is bounded}\}$. Define the deterministic set function $N(A) = 0$ for A bounded and $N(A) = 1$ for A^c bounded. This clearly satisfies properties (i)-(iv) in Definition 3.1, with $U_n = [-n, n]$. The extension suggested in [38] implies redefining $N(A) = 0$ for each $A \in \mathcal{B}(\mathbb{R})$. In the same context, define $N(A) = |A|$ for A bounded and $N(A) = 1 - |A^c|$ when A^c is bounded. In this case, the L^2-limit of $N(A \cap U_n)$ exists if, and only if, $|A| < \infty$.

Definition 3.3. An H-valued *orthogonal martingale-valued measure* is a collection $M = (M(t,A) : t \geq 0, A \in \mathcal{A})$ of H-valued random variables satisfying:

(i) For any $A \in \mathcal{A}$, $M(0,A) = 0$ $\mathbb{P}$-a.e.

(ii) For any $A \in \mathcal{A}$, $M(A) = (M(t,A))_{t\geq 0}$, is a mean-zero, square integrable H-valued martingale.

(iii) If $t > 0$, $M(t,\cdot) : \mathcal{A} \to L^2(\Omega, \mathcal{F}, \mathbb{P}; H)$ is a σ-finite $L^2(\Omega, \mathcal{F}, \mathbb{P}; H)$-valued measure.

(iv) For $t > 0$, $\langle M(A)\,,\, M(B)\rangle_t = 0$ whenever A and B in $\mathcal{A}$ are disjoint.

By the classical definition, $\langle M(A)\rangle_t$ is increasing in t for fixed A. It is also increasing in A for fixed t, according to the following.

Lemma 3.4. *If M is an H-valued orthogonal martingale-valued measure, $\langle M(\cdot)\rangle_t$ is additive on $\mathcal{A}$. More precisely, given $A, B \in \mathcal{A}$ disjoint we have $\langle M(A \cup B)\rangle_t = \langle M(A)\rangle_t + \langle M(B)\rangle_t$ $\mathbb{P}$-a.e. Moreover, for $A \subseteq B$ and $t \geq 0$ we have $\langle M(A)\rangle_t \leq \langle M(B)\rangle_t$, $\mathbb{P}$-a.e.*

Proof. If $A, B \in \mathcal{A}$ are disjoint, we have $\mathbb{P}$-a.e.

$$\langle M(A \cup B)\rangle_t = \langle M(A)\rangle_t + \langle M(B)\rangle_t + 2\langle M(A), M(B)\rangle_t = \langle M(A)\rangle_t + \langle M(B)\rangle_t$$

because of the orthogonality. The final assertion follows easily from this. □

Let us consider an H-valued orthogonal martingale-valued measure M. We fix an interval $[0,T]$ and define, following [38]:

$$\mu(A) = \mathbb{E}[\|M(T,A)\|^2] = \mathbb{E}[\langle M(A)\rangle_T], \quad \forall A \in \mathcal{A}. \tag{3.1}$$

Let (U_n) be the sequence corresponding to $M(T,\cdot)$ in (iii) of Definition 3.1. Being the map $t \mapsto \mathbb{E}[\|M_t(A)\|^2]$ increasing, the same family (U_n) works for all $M(t,\cdot)$, $0 \leq t \leq T$. In fact, by Lemma 3.4 μ is finitely additive and

$$\sup_{A\in\mathcal{B}(U_n)} \mathbb{E}[\|M(t,A)\|^2] = \mathbb{E}[\|M(t,U_n)\|^2] \leq \mathbb{E}[\|M(T,U_n)\|^2] = \mu(U_n) < \infty.$$

We summarize it in the following lemma.

Lemma 3.5. *Let M be an H-valued orthogonal martingale-valued measure. The finitely additive measure μ defined on $\mathcal{A}$ by (3.1), is σ-finite. Moreover, μ is a (countably additive) measure on each $(U_n, \mathcal{B}(U_n))$.*

The following theorem is our H-valued version of Proposition 2.7 of [38]. Besides the minor changes due to the H-valued feature, we have decided to include a self-contained proof by two reasons. First, there are details in the original proof that we consider important to emphasize and, second, the ideas of the proof will be crucial to other results in this paper.

Theorem 3.6. *Let $(M_t(A) : 0 \leq t \leq T, A \in \mathcal{A})$ be an orthogonal H-valued martingale-valued measure, where $\mathcal{A}$ is a sub-ring of $\mathcal{B}(U)$. Then there exists a family $(\nu_t, 0 \leq t \leq T)$ of random σ-finite measures on $(U, \mathcal{B}(U))$ such that:*

(i) $(\nu_t, 0 \leq t \leq T)$ *is predictable.*

(ii) For all $A \in \mathcal{A}$, $t \mapsto \nu_t(A)$ is right-continuous and increasing.

(iii) For all $0 \leq t \leq T$ and $A \in \mathcal{A}$ we have $\mathbb{P}(\nu_t(A) = \langle M(A) \rangle_t) = 1$.

Proof. Step 1. Since $(U, \mathcal{B}(U))$ is Lusin, there exists $h : U \to \hat{U}$ invertible, where $\hat{U}$ is a Borel set on the line, such that both h and h^{-1} are measurable (relative to $\mathcal{B}(U)$ and $\mathcal{B}(\hat{U})$). All the elements that define M can be copied down on $(\hat{U}, \mathcal{B}(\hat{U}))$, so it is enough to prove the theorem for $\hat{U}$.

Step 2. According to the previous step, let us assume that U is a Borel set on the line. Notice that μ is a finite measure on each $(U_n, \mathcal{B}(U_n))$. Suppose we can prove the theorem for $U = U_n$ and $\mathcal{A} = \mathcal{B}(U_n)$. That means there is a predictable family $(\nu_t^{(n)} : 0 \leq t \leq T)$ of random measures on $(U_n, \mathcal{B}(U_n))$, satisfying (i)-(iii) for $A \in \mathcal{B}(U_n)$. Suppose, moreover, that $\nu_t^n(C) \leq \nu_t^{n+1}(C)$ for each $C \in \mathcal{B}(U_n)$. We extend each ν_t^n to a measure $\hat{\nu}_t^n$ on $(U, \mathcal{B}(U))$ by

$$\hat{\nu}_t^n(C) = \nu_t^n(C \cap U_n)$$

and then define

$$\nu_t := \sup_{n \geq 1} \hat{\nu}_t^n.$$

According to Corollary 2.3, $(\nu_t : 0 \leq t \leq T)$ is a predictable family of random measures, and for each $C \in \mathcal{B}(U)$ we have

$$\nu_t(C) = \lim_{n \to \infty} \hat{\nu}_t^n(C) = \lim_{n \to \infty} \nu_t^n(C \cap U_n).$$

Since ν_t^n satisfies (iii) in $\mathcal{B}(U_n)$, given $A \in \mathcal{A}$, $\|M_t(A \cap U_n)\|^2 - \nu_t^n(A \cap U_n)$ is a martingale. Since $M_t(A \cap U_n) \to M_t(A)$ in L^2 and $\nu_t^n(A \cap U_n) \to \nu_t(A)$ in L^1 (monotone convergence) we conclude that

$$\|M_t(A \cap U_n)\|^2 - \nu_t^n(A \cap U_n) \to \|M_t(A)\|^2 - \nu_t(A)$$

in L^1. This shows that $\|M_t(A)\|^2 - \nu_t(A)$ is a martingale and then

$$\mathbb{P}(\nu_t(A) = \langle M(A) \rangle_t) = 1.$$

Step 3. By the previous step, it is enough to assume that $\mathcal{A} = \mathcal{B}(U)$ and μ is a finite measure. Because of the regularity of μ, there is an increasing family of compact sets $K_j \subseteq U$ such that $K = \cup K_j$ satisfies $\mu(U \backslash K) = \mathbb{E} \langle M(U\backslash K)\rangle_T = 0$. Since $\langle M(U\backslash K)\rangle_T$ is a non-negative random variable, we conclude $\langle (M(U\backslash K)\rangle_T = 0$ a.e. If we prove the theorem for $U = K_j$, we can apply the construction of the second step to obtain a family of random measures ν_t defined on $(U, \mathcal{B}(U))$ and concentrated on K. Since $U\backslash K$ has μ-measure zero,

$$\mathbb{P}(\nu_t(A) = \langle M(A)\rangle_t) = \mathbb{P}(\nu_t(A \cap K) = \langle M(A \cap K)\rangle_t) = 1$$

for all $A \in \mathcal{B}(U)$ and all $t \in [0, T]$.

Step 4. Following the reduction in Step 3, we will now prove the theorem for U compact on the real line, $\mathcal{A} = \mathcal{B}(U)$ and $\mu(U) < \infty$. We separate this step in several sub-steps.

Step 4.1. We extend M to $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ so that $M_t(A) = M_t(A \cap U)$. In particular, $M_t(A) = 0$ for $A \cap U = \emptyset$. For each $t \in [0, T]$ define the random function $F_t : \mathbb{R} \to \mathbb{R}$ by

$$F_t(x) = \langle M(-\infty, x]\rangle_t .$$

By Lemma 3.4, each F_t is increasing and, for $a < b$ fixed, $F_t(b) - F_t(a)$ is increasing in t. Also, for fixed t and $x \in (a, b]$ we have $\mathbb{P}$-a.e.

$$F_t(x) - F_t(a) \leq \langle M((a, b])\rangle_T ,$$

and by right-continuity this holds $\mathbb{P}$-a.e. simultaneously for all $t \in [0, T]$ and $x \in (a, b] \cap \mathbb{Q}$. Taking the expectation, we get

$$E\left[\sup_{t\in[0,T],x\in(a,b]\cap\mathbb{Q}} |F_t(x) - F_t(a)|\right] \leq \mu((a, b]). \tag{3.2}$$

Step 4.2. Define

$$\overline{F}_t(x) := \inf\{F_s(y) : x < y \in \mathbb{Q}, t < s \in \mathbb{Q}\}.$$

This function is increasing and right-continuous in both variables. The first assertion is immediate; for the second one, consider rational sequences $t_n \downarrow t$ and $x_n \downarrow x$ and note that

$$\overline{F}_t(x) \leq \overline{F}_{t_n}(x_n) \leq F_{t_{n-1}}(x_{n-1}).$$

We also observe that, for fixed x

$$\mathbb{P}\left(\overline{F}_t(x) = F_t(x) \text{ for all } t \in [0, T]\right) = 1.$$

In fact, for $t_n \downarrow t$ and $x_n \downarrow x$ we have

$$F_t(x) \le \overline{F}_t(x) \le F_{t_n}(x_n) = F_{t_n}(x) + [F_{t_n}(x_n) - F_{t_n}(x)]$$

where $F_{t_n}(x) \to F_t(x)$ by right-continuity and $F_{t_n}(x_n) - F_{t_n}(x) \to 0$ in probability by (3.2).

Step 4.3. Let ν_t be the distribution on $\mathbb{R}$ generated by $\overline{F}_t$. Note that $\nu_t(\mathbb{R}\backslash U) = 0$, for given $(\alpha, \beta] \subseteq \mathbb{R}\backslash U$, there exists $\varepsilon > 0$ such that $(\alpha, \beta+\varepsilon] \subseteq \mathbb{R}\backslash U$. This implies that $\overline{F}_t$ is constant on $(\alpha, \beta]$ and therefore $\nu_t((\alpha, \beta]) = 0$. The predictability is inherited from F_t to $\overline{F}_t$ and then to ν_t, as we can always work through rational values of x.

Step 4.4. For fixed $t < t'$, $\overline{F}_{t'} - \overline{F}_t$ is increasing. In fact, for $a < b$ the inequality

$$(\overline{F}_{t'} - \overline{F}_t)(a) < (\overline{F}_{t'} - \overline{F}_t)(b)$$

is equivalent to

$$\overline{F}_t(b) - \overline{F}_t(a) < \overline{F}_{t'}(b) - \overline{F}_{t'}(a)$$

and this is inherited from F. It follows that $\nu_t((a,b])$ is increasing and right-continuous in t. A monotone class argument allows us to deduce that $\nu_t(A)$ is increasing and right-continuous in t for any Borel set A.

Step 4.5. We now prove (iii). Let $\mathcal{G}$ be the class of those $A \in \mathcal{B}(U)$ for which $\|M_t(A)\|^2 - \nu_t(A)$ is a martingale. It is clear that $(-\infty, x] \in \mathcal{G}$, because by Step 2, $\nu_t((-\infty, x]) = F_t(x) = \langle M(-\infty, x]\rangle_t$ a.e. With $x = \sup U$ we obtain $\mathbb{R} \in \mathcal{G}$. It follows that $\mathcal{G}$ contains $\mathbb{R}$ and all finite unions of intervals of the form $(a, b]$. $\mathcal{G}$ is closed under complementation because we can write $\|M(A^c)\|_t^2 - \nu_t(A^c)$ as the sum of three martingales:

$$[\|M(\mathbb{R})\|_t^2 - \nu_t(\mathbb{R})] + [\|M(A)\|_t^2 - \nu_t(A)] - 2\left(M_t(A), M_t(A^c)\right)_H.$$

Finally, $\mathcal{G}$ is closed under monotone convergence, for if $A_n \uparrow A$ we have in L^1

$$\|M(A_n)\|_t^2 - \nu_t(A_n) \to \|M(A)\|_t^2 - \nu_t(A).$$

We conclude that $\mathcal{G} = \mathcal{B}(U)$.

To complete the proof, in order for Steps 2 and 3 to work out correctly, we need to observe that if U_1 and U_2 are compact and $U_1 \subseteq U_2$, then the constructed families ν_t^1 and ν_t^2 satisfy $\nu_t^1 \le \nu_T^2$ in U_1, in the sense that

$$\mathbb{P}\left(\nu_t^1(C) \le \nu_t^2(C) \text{ for all } C \in \mathcal{B}(U_1)\right) = 1.$$

It is enough to verify it for any interval $(a,b]$, with a,b rational, and that is straightforward following the construction of Step 4. □

Next corollary constructs a random measure ν on $\mathcal{B}(\mathbb{R}_+)\otimes\mathcal{B}(U)$, based on the family $(\nu_t : t\geq 0)$ of random random measures on $\mathcal{B}(U)$. We will say that such a random measure ν is *predictable* if the family (ν_t) is.

Corollary 3.7. *Given an H-valued orthogonal martingale-valued measure $M=(M(t,A): t\geq 0, A\in\mathcal{A})$, there exists a predictable σ-finite random measure ν on $\mathcal{B}(\mathbb{R}_+)\otimes\mathcal{B}(U)$ such that for every $t\geq 0$ and $A\in\mathcal{A}$,*

$$\mathbb{P}\left(\{\omega\in\Omega : \nu(\omega)\left([0,t]\times A\right) = \langle M(A)\rangle_t(\omega)\}\right)=1.$$

Proof. First, by a standard argument we can construct a family $(\nu_t : t\geq 0)$ of random σ-finite measures on $(U,\mathcal{B}(U))$ satisfying the statements in Theorem 3.6 for each $T>0$. We can therefore define ν by first setting $\nu([0,t]\times A)=\nu_t(A)$ for $t\geq 0$ and $A\in\mathcal{A}$ and then extending it to $\mathcal{B}(\mathbb{R}_+)\otimes\mathcal{B}(U)$, using the fact that the collection of all rectangles $[0,t]\times A$ generates the product σ-algebra. □

Remark 3.8. The previous corollary provides us with a regularization for the random mapping $(t,A)\mapsto\langle M(A)\rangle_t$, which in general, does not have to behave like a random measure. If such mapping is verified to be already predictable and countably additive on the measurable rectangles of $\mathcal{B}([0,T])\otimes\mathcal{B}(U)$, then the previous construction is not needed.

The following lemma is a consequence of the proof for the previous theorem. It receives an inequality between two random measures that is satisfied almost everywhere for each individual rectangle $(s,t]\times A$, $A\in\mathcal{A}$, and shows that it actually holds almost everywhere all over $\mathcal{B}(\mathbb{R}_+)\otimes\mathcal{B}(U)$. The result is not classic, because it is not clear at the beginning that we can pick a countable number of elements of $\mathcal{A}$ that generate $\mathcal{B}(U)$. The fact that $\mathcal{A}$ contains each $\mathcal{B}(U_n)$ fills the gap.

Lemma 3.9. *Suppose ν_1 and ν_2 are random measures on $\mathcal{B}(\mathbb{R}_+)\otimes\mathcal{B}(U)$ such that, for each $A\in\mathcal{A}$ and $0\leq s<t$*

$$\mathbb{P}(\nu_1((s,t]\times A)\leq\nu_2((s,t]\times A))=1.$$

Then $\nu_1\leq\nu_2$. More precisely

$$\mathbb{P}\left(\nu_1(C)\leq\nu_2(C), \forall C\in\mathcal{B}(\mathbb{R}_+)\otimes\mathcal{B}(U)\right)=1.$$

Proof. First, for fixed s,t define $\mu_j(B) = \nu_j((s,t] \times B)$, for $j = 1,2$, $B \in \mathcal{B}(U)$. As in the proof of Theorem 3.6, we can assume that U is a Borel set on the line. We consider the measures $\mu_{j,n}$ obtained by restricting μ_j to $(U_n, \mathcal{B}(U_n))$ and see them as defined in $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. To be precise, $\mu_{j,n}(B) := \mu_j(B \cap U_n)$. Since $\mathcal{B}(U_n) \subseteq \mathcal{A}$, for $a < b$ we have $\mathbb{P}$-a.e.

$$\mu_{1,n}((a,b]) \leq \mu_{2,n}((a,b]).$$

It follows that

$$\mathbb{P}\left(\mu_{1,n}((a,b]) \leq \mu_{2,n}((a,b]),\, \forall a,b \in \mathbb{Q}\right) = 1$$

and this is enough to conclude (working with fixed ω) that

$$\mathbb{P}\left(\mu_{1,n}(B) \leq \mu_{2,n}(B),\, \forall B \in \mathcal{B}(\mathbb{R})\right) = 1.$$

By the continuity of both measures we obtain $\mathbb{P}$-a.e.

$$\mu_1(B) = \sup_n \mu_{1,n}(B) \leq \sup_n \mu_{2,n}(B) = \mu_2(B), \quad \forall B \in \mathcal{B}(U).$$

We have shown that, for fixed s,t

$$\mathbb{P}(\nu_1((s,t] \times B) \leq \nu_2((s,t] \times B), \forall B \in \mathcal{B}(U)) = 1.$$

The remaining argument is classic using the right continuity of $\nu_\cdot$, working with fixed ω. In fact, the rectangles $(s,t] \times B$ with s,t rational and $B \in \mathcal{B}(U)$ generate $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(U)$. □

Remark 3.10. The previous lemma clearly holds with equality. More precisely, if for given s,t and $A \in \mathcal{A}$, we have $\nu_1((s,t] \times A) = \nu_2((s,t] \times A)$ $\mathbb{P}$-a.e., then $\nu_1 = \nu_2$. This, in particular, implies the uniqueness of ν in Corollary 3.7 and justifies Definition 3.12 below. It is important to recall that two random measures μ and ν are considered the same when $\mathbb{P}(\mu(C) = \nu(C)$ for all measurable $C) = 1$.

The previous lemma can also be applied to compare two signed measures, or the absolute value of a signed measure and a measure.

Lemma 3.11. *Suppose α and β are random signed measures defined on $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(U)$. If for each $A \in \mathcal{A}$ and $0 \leq s < t$*

$$\mathbb{P}(\alpha((s,t] \times A) \leq \beta((s,t] \times A)) = 1$$

then $\alpha \leq \beta$. If for each $A \in \mathcal{A}$ and $0 \leq s < t$

$$\mathbb{P}(|\alpha((s,t] \times A)| \leq \beta((s,t] \times A)) = 1$$

then $|\alpha| \leq \beta$.

Proof. If α^- is finite, the first assertion follows by applying Lemma 3.9 to the inequality

$$\alpha^+ \leq \alpha^- + \beta.$$

Otherwise, we work with

$$(\alpha^+ - \beta)^+ \leq \alpha^-.$$

The second assertion follows by applying the first one to the inequalities $-\beta \leq \alpha$ and $\alpha \leq \beta$. □

Definition 3.12. The unique random predictable σ-finite measure ν given in Corollary 3.7 will be called the intensity measure of M.

Example 3.13. Let $(U, \mathcal{B}(U), \lambda)$ be a $\sigma-$finite space. A (Gaussian) *white noise measure* based on λ is a random set function W on the sets $A \in \mathcal{U}$ of finite $\lambda-$measure, such that

(i) $W(A)$ is a $\mathcal{N}(0, \lambda(A))$ random variable.

(ii) If $A \cap B = \emptyset$, then $W(A)$ and $W(B)$ are independent and

$$W(A \cup B) = W(A) + W(B).$$

Equivalently $E[W(A)W(B)] = E[|W(A \cap B)|^2] = \lambda(A \cap B)$.

Consider a white noise measure W on $(\mathbb{R}_+ \times U, \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(U), Leb \otimes \lambda)$. This means, W is defined at least on the cylinder sets $(s,t] \times A$, with $0 \leq s \leq t$ and $A \in \mathcal{A}$. Here, $\mathcal{A}$ is the ring of finite λ-measure sets on $\mathcal{B}(U)$.

We define $M_t(A) = W([0,t] \times A)$ and observe that $M_t(A) \sim \mathcal{N}(0, t\lambda(A))$. Thus $(M_t(A) : t \geq 0, A \in \mathcal{A})$ is clearly a martingale measure with respect to its natural filtration. Besides

$$A \cap B = \emptyset \Rightarrow M_t(A) \text{ and } M_t(B) \text{ are independent },$$

hence orthogonal. We also have $\langle M(A) \rangle_t = t\lambda(A)$. Given $U_n \uparrow U$ with $\lambda(U_n) < \infty$, it is clear that $\mathcal{B}(U_n) \subseteq \mathcal{A}$ and

$$\sup_{A \in \mathcal{B}(U_n)} E|M_t(A)|^2 = t\lambda(U_n) < \infty.$$

The martingale measure M is also countably additive on each U_n. In fact, if (A_j) is a sequence on $\mathcal{B}(U_n)$ and $A_j \downarrow \emptyset$, then $\lambda(A_j) \to 0$ and consequently $E|M_t(A_j)|^2 \leq T\lambda(A_j) \to 0$. Finally, given $A \in \mathcal{A}$ we have

$$E|M_t(A) - M_t(A \cap U_n)|^2 = E|M_t(A \setminus U_n)|^2 = t\lambda(A \setminus U_n) \to 0.$$

We have shown that each $M_t(\cdot)$ satisfies Definition 3.1 and therefore, M satisfies condition (iii) in Definition 3.3. Conditions (i), (ii) and (iv) are immediate. We conclude that M is a (real-valued) orthogonal martingale-valued measure.

(a) Notice the additivity of $\langle M(\cdot)\rangle_t$ as mentioned on Lemma 3.4.

(b) Notice that the measure μ of Lemma 3.5 in this case is $\mu(A) = T\lambda(A)$.

(c) The family of measures constructed in Theorem 3.6 is given by $\nu_t(A) = t\lambda(A)$. Also, in Corollary 3.7, the intensity measure is $\nu((s,t] \times A) = (t-s)\lambda(A)$, that is $\nu = Leb \otimes \lambda$.

Example 3.14. Let H be a separable Hilbert space and let $L = (L_t : t \geq 0)$ be an H-valued càdlàg Lévy process, i.e. L has independent and stationary increments and has $\mathbb{P}$-a.e. càdlàg paths. Assume furthermore that L is $(\mathcal{F}_t)$-adapted and that $L_t - L_s$ is independent of $\mathcal{F}_s$ for all $0 \leq s < t$.

In this example we introduce an H-valued orthogonal martingale-valued measure associated to L. We will first need to recall some properties of Lévy process which can be found for example in [3, 29].

We can associate to L a Poisson random measure N on $\mathbb{R}^+ \times (H\backslash\{0\})$ given by

$$N(t,A) = \#\{0 \leq s \leq t : \Delta L_s := L_s - L_{s-} \in A\}.$$

We say that $A \in \mathcal{B}(H\backslash\{0\})$ is bounded below if $0 \notin \overline{A}$. If A is bounded below, it is well-known that $(N(t,A) : t \geq 0)$ is a Poisson process with $\mathbb{E}\left[N(t,A)\right] = t\lambda(A)$, where λ is a Lévy measure, i.e. λ is a Borel measure on H with $\lambda(\{0\}) = 0$, and $\int_H \|h\|^2 \wedge 1 \, \lambda(dh) < \infty$. Moreover, for $f : A \to H$ measurable, we may define the Poisson integral as the finite random sum:

$$\int_A f(h)\, N(t,dh) = \sum_{0 \leq s \leq t} f(\Delta L_s) \mathbb{1}_A (\Delta L_s)\,.$$

Let $\widetilde{N}(dt,dh) = N(dt,dh) - dt\lambda(dh)$ be the compensated Poisson random measure corresponding to N. For $f \in L^2\left(A, \lambda\big|_A; H\right)$ we define the compensated Poisson integral:

$$\int_A f(h)\, \widetilde{N}(t,dh) = \int_A f(h)\, N(t,dh) - t\int_A f(h)\lambda(dh).$$

It is well-known that $\left(\int_A f(h)\widetilde{N}(t,dh) : t \geq 0\right)$ is an H-valued mean-zero square integrable càdlàg martingale, and

$$\mathbb{E}\left[\left\|\int_A f(h)\widetilde{N}(t,dh)\right\|^2\right] = t\int_A \|f(h)\|^2 \lambda(dh).$$

Recall that L being a Lévy process in a separable Hilbert space, it has a Lévy-Itô decomposition

$$L_t = t\xi + W_t + \int_{\|h\|<1} h\,\widetilde{N}(t,dh) + \int_{\|h\|\geq 1} h\,N(t,dh), \tag{3.3}$$

where $\xi \in H$, $W = (W_t : t \geq 0)$ is a Wiener process in H with covariance operator Q (i.e. $\mathbb{E}[(h,W_t)_H^2] = t(h,Qh)_H$) which is positive and of trace class, W is independent of the Poisson random measure N, $\left(\int_{\|h\|\geq 1} h\,N(t,dh) : t \geq 0\right)$ is a Poisson integral as defined above, and $\left(\int_{\|h\|<1} h\,\widetilde{N}(t,dh) : t \geq 0\right)$ is an H-valued càdlàg mean-zero square integrable martingale such that

$$\int_{\|h\|<1} h\,\widetilde{N}(t,dh) = \lim_{n\to\infty}\int_{\frac{1}{n}\leq\|h\|<1} h\,\widetilde{N}(t,dh) \quad \forall t > 0 \quad (\text{limit in } L^2(\Omega,\mathcal{F},\mathbb{P};H)).$$

We can associate to the Lévy process L an H-valued orthogonal martingale-valued measure in the following way. Let $U \in \mathcal{B}(H)$ be such that $0 \in U$ and $\int_U \|u\|^2 \lambda(du) < \infty$. Take

$$\mathcal{A} = \{A \subseteq U : A - \{0\} \text{ is bounded below}\},$$

and let $\tilde{M} = (\tilde{M}(t,A) : t \geq 0, A \in \mathcal{A})$ be given by

$$\tilde{M}(t,A) = W_t\delta_0(A) + \int_{A\backslash\{0\}} h\,\widetilde{N}(t,dh), \quad \forall\, t \geq 0,\, A \in \mathcal{A}. \tag{3.4}$$

In fact, from the properties of the stochastic processes defined above it is clear that $\tilde{M}(0,A) = 0$ $\mathbb{P}$-a.e. for every $A \in \mathcal{A}$. Thus Definition 3.3(i) is satisfied. Moreover, since the Wiener process and the Poisson stochastic integrals defined above are H-valued zero-mean square integrable martingales, then for any given $A \in \mathcal{A}$ we have $\tilde{M}(A)$ is an H-valued zero-mean, square integrable martingale. This proves $\tilde{M}$ satisfies Definition 3.3(ii).

Now we check that Definition 3.3(iii)-(iv) is satisfied. Let $t \geq 0$ and $A \in \mathcal{A}$. First observe that

$$\mathbb{E}[\|\tilde{M}(t,A)\|^2] = t\left[\|Q\|_{\mathcal{L}_1(H)}\,\delta_0(A) + \int_{A\backslash\{0\}} \|u\|^2\, \lambda(du)\right], \tag{3.5}$$

and hence

$$\langle \tilde{M}(A)\rangle_t = t\,\mathbb{E}[\|\tilde{M}(1,A)\|^2] = t\,\left[\|Q\|_{\mathcal{L}_1(H)}\,\delta_0(A) + \int_{A\backslash\{0\}} \|u\|^2\, \lambda(du)\right]. \tag{3.6}$$

Let $t > 0$. If $A \cap B = \emptyset$, $A, B \in \mathcal{A}$, then $\delta_{A\cup B} = \delta_A + \delta_B$, and the Poisson stochastic integrals $\left(\int_{A\setminus\{0\}} h\, \widetilde{N}(t, dh) : t \geq 0\right)$ and $\left(\int_{B\setminus\{0\}} h\, \widetilde{N}(t, dh) : t \geq 0\right)$ are independent processes. From these facts it is easy to conclude that $\tilde{M}(t, A \cup B) = \tilde{M}(t, A) + \tilde{M}(t, B)$ and that $\langle \tilde{M}(A), \tilde{M}(B) \rangle_t = 0$. Finally, to check that each $\tilde{M}(t, \cdot)$ is a σ-finite $L^2(\Omega, \mathcal{F}, \mathbb{P}; H)$-valued function take $U_n = \{0\} \cup \{h \in U : \|h\| > \frac{1}{n}\}$ for each $n \in \mathbb{N}$.

It is clear that $U = \bigcup_{n\in\mathbb{N}} U_n$ and $\mathcal{B}(U_n) \subseteq \mathcal{A}$ for each $n \in \mathbb{N}$. Moreover, from (3.5) we have

$$\begin{aligned} \sup_{A\in\mathcal{B}(U_n)} \mathbb{E}[\|\tilde{M}(t, A)\|^2] &= t \sup_{A\in\mathcal{B}(U_n)} \left[\|Q\|_{\mathcal{L}_1(H)}\, \delta_0(A) + \int_{A\setminus\{0\}} \|u\|^2\, \lambda(du) \right] \\ &\leq t \left[\|Q\|_{\mathcal{L}_1(H)} + \int_U \|u\|^2\, \lambda(du) \right] < \infty. \end{aligned}$$

4. Cylindrical Orthogonal Martingale-Valued Measures

Let X be a Banach space with separable dual X^*. The following definition extends that of orthogonal martingale-valued measure to the cylindrical context.

Definition 4.1. A *cylindrical orthogonal martingale-valued measure* on X^* is a collection $M = (M(t, A) : t \geq 0, A \in \mathcal{A})$ of cylindrical random variables on X such that:

(i) For each $A \in \mathcal{A}$, $M(0, A)(x^*) = 0$ $\mathbb{P}$-a.e. for all $x^* \in X^*$.

(ii) For each $A \in \mathcal{A}$, $M(A) = (M(t, A) : t \geq 0)$, is a cylindrical mean-zero square integrable martingale, and for each $t > 0$ and $A \in \mathcal{A}$, the map

$$M(t, A) : X^* \to L^0 (\Omega, \mathcal{F}, \mathbb{P})$$

is continuous.

(iii) If $t > 0$ and $x^* \in X^*$, $M(t, \cdot)(x^*) : \mathcal{A} \to L^2 (\Omega, \mathcal{F}, \mathbb{P})$ is a σ-finite L^2-valued measure.

(iv) If $t > 0$ and $x^* \in X^*$, $\langle M(A)(x^*)\, , \, M(B)(x^*) \rangle_t = 0$ whenever $A, B \in \mathcal{A}$ are disjoint.

Remark 4.2. Let $t > 0$ and $A \in \mathcal{A}$. Our assumption (ii) imply that the linear mapping $M(t, A) : X^* \to L^2 (\Omega, \mathcal{F}, \mathbb{P})$ is continuous. In fact, assume that $x_n^* \to x^*$ in X^* and $M(t, A)(x_n^*) \to Y$ in $L^2 (\Omega, \mathcal{F}, \mathbb{P})$. By (ii) we have $M(t, A)(x_n^*) \to M(t, A)(x^*)$ in probability. On the other hand, since L^2 convergence implies convergence in probability we have $M(t, A)(x_n^*) \to Y$ in probability. By uniqueness of limits $Y = M(t, A)(x^*)$ $\mathbb{P}$-a.e. and the closed graph theorem finishes the work.

Remark 4.3. For any $x^* \in X^*$ and $A \in \mathcal{A}$, by (ii) the process

$$M(A)(x^*) = (M(t, A)(x^*) : t \geq 0)$$

is a real-valued square-integrable martingale, so the brackets in (iv) correspond to the (real) covariation of two real-valued processes. We will also use the notation $\langle M(A)(x^*) \rangle_t$ for the (real) quadratic variation, when it exists.

Example 4.4. Consider a finite set $U = \{a_1, \dots, a_n\}$. In this case we can take $\mathcal{A} = 2^U$, which corresponds to the discrete topology. For each $k = 1, \cdots, n$, let $Z^k = (Z^k_t : t \geq 0)$ be a cylindrical càdlàg zero-mean square integrable martingale such that for each $t \geq 0$ the mapping $Z^k_t : X^* \to L^0(\Omega, \mathcal{F}, \mathbb{P})$ is continuous. Assume moreover that for each $x^* \in X^*$, the real-valued martingales $(Z^k(x^*))_{k=1}^n$ are orthogonal.

Define a family $M = (M(t, A) : t \geq 0, A \in \mathcal{A})$ by the prescription:

$$M(t, A)(x^*) = \sum_{k=1}^{n} Z^k_t(x^*) \delta_{a_k}(A), \quad \forall\, x \in X^*,\, t \geq 0,\, A \in \mathcal{A}.$$

Since every $A \in \mathcal{A}$ is either the empty set or has a finite number of elements, it is not difficult to check that (i)-(iv) in Definition 4.1 are satisfied. Hence M defined above is a cylindrical orthogonal martingale-valued measure.

Example 4.5. Let $Z : X^* \to \mathcal{M}^2_\infty$ be a continuous linear operator, in particular a cylindrical càdlàg zero-mean square integrable martingale on X. Let $g : \mathbb{R}_+ \times \Omega \to U$ be a predictable process and let $\mathcal{A} = \mathcal{B}(U)$.

Define a family $M = (M(t, A) : t \geq 0, A \in \mathcal{A})$ by the prescription:

$$M(t, A)(x^*) = \int_0^t \mathbb{1}_A(g(s))\, dZ(x^*)_s, \quad \forall x^* \in X^*,\, t \geq 0,\, A \in \mathcal{A}.$$

It is clear that for all $A \in \mathcal{A}$ and $x^* \in X^*$ we have $M(0, A)(x^*) = 0$ $\mathbb{P}$-a.e., so Definition 4.1(i) is satisfied.

The linearity of Z shows that each $M(t, A)$ defines a cylindrical random variable on X. Since $\mathbb{1}_A(g(s, \omega))$ is a real-valued bounded predictable process, then $M(t, A)(x^*)$ is a real-valued zero-mean square integrable càdàg martingale. Furthermore, by the Itô isometry we have

$$\begin{aligned} \mathbb{E}\left[|M(t, A)(x^*)|^2\right] &= \mathbb{E}\left[\left|\int_0^t \mathbb{1}_A(g(s))\, dZ(x^*)_s\right|^2\right] \\ &= \mathbb{E}\int_0^t \mathbb{1}_A(g(s))\, d\left\langle Z(x^*)\right\rangle_s \end{aligned}$$

$$\leq \mathbb{E}\left[\langle Z(x^*)\rangle_t\right] = \mathbb{E}\left[|Z(x^*)_t|^2\right].$$

Hence showing that for each $t > 0$ and $A \in \mathcal{A}$ the mapping $M(t,A) : X^* \to L^2(\Omega, \mathcal{F}, \mathbb{P})$ is continuous. This shows Definition 4.1(ii) is satisfied.

Now we check Definition 4.1(iii). In fact, let $t > 0$ and $x^* \in X^*$. We have proved above that $\mathbb{E}\left[|M(t,A)(x^*)|^2\right] < \infty$ for each $A \in \mathcal{A}$. Moreover if $A \cap B = \emptyset$, $A, B \in \mathcal{A}$, then

$$\begin{aligned} M(t, A\cup B)(x^*) &= \int_0^t \mathbb{1}_{A\cup B}(g(s))\, dZ(x^*)_s \\ &= \int_0^t \mathbb{1}_{A}(g(s))\, dZ(x^*)_s + \int_0^t \mathbb{1}_{B}(g(s))\, dZ(x^*)_s \\ &= M(t,A)(x^*) + M(t,B)(x^*). \end{aligned}$$

Furthermore, each $M(t,\cdot)(x^*)$ is easily seem to be σ-finite L^2-valued function by taking $U_n = U$ for all $n \in \mathbb{N}$.

Finally, given $t > 0$ and $x^* \in X^*$, for $A, B \in \mathcal{A}$ satisfying $A \cap B = \emptyset$ we have, by the theory of stochastic integration, that

$$\begin{aligned} \mathbb{E}\left[\langle M(A)(x^*)\,,\, M(B)(x^*)\rangle_t\right] &= \mathbb{E}\left[M(t,A)(x^*)\cdot M(t,B)(x^*)\right] \\ &= \mathbb{E}\int_0^t \mathbb{1}_{A}(g(s))\cdot \mathbb{1}_{B}(g(s))\, d\left\langle Z(x^*)\right\rangle_s = 0. \end{aligned}$$

Hence Definition 4.1(iv) is satisfied.

We finish this section with a result that allows us to look at a Hilbert space-valued martingale-valued measure as a cylindrical orthogonal martingale-valued measure.

Proposition 4.6. *Let H be a separable Hilbert space and let $\tilde{M} = (\tilde{M}(t,A) : t \geq 0, A \in \mathcal{A})$ be an H-valued orthogonal martingale-valued measure. Assume further that for every $A, B \in \mathcal{A}$ disjoint and $h \in H$, the real-valued martingales $(\tilde{M}(A), h)_H$ and $(\tilde{M}(B), h)_H$ are orthogonal. Then $\tilde{M}$ induces a cylindrical orthogonal martingale-valued measure $M = (M(t,A) : t \geq 0, A \in \mathcal{A})$ on H by means of the prescription*

$$M(t,A)(\omega)(h) := (\tilde{M}(t,A)(\omega), h)_H, \quad \forall \omega \in \Omega, t \geq 0, A \in \mathcal{A}, h \in H. \tag{4.1}$$

Proof. It is easy to verify each of the statements of Definition 4.1. Note, in particular, that for each $t > 0$ and $A \in \mathcal{A}$, the map $M(t,A) : H \to L^0(\Omega, \mathcal{F}, \mathbb{P})$ is continuous by using a Chebyshev-type argument. Indeed for any $\epsilon > 0$, one can see that

$$\mathbb{P}(|(\tilde{M}(t,A), h_n - h)| \geq \epsilon) \leq \frac{1}{\epsilon^2}\cdot \|h_n - h\|_H^2\, \mathbb{E}[\|\tilde{M}(t,A)\|^2] \to 0,$$

whenever $h_n \to h$ in H. □

Example 4.7. Let H be a separable Hilbert space and let $L = (L_t : t \geq 0)$ be an H-valued càdlàg Lévy process. Let $\tilde{M} = (\tilde{M}(t,A) : t \geq 0, A \in \mathcal{A})$ be the H-valued orthogonal martingale-valued measure defined in (3.4).

Assume $A, B \in \mathcal{A}$ are disjoint. We have two cases. Assume first that $A = \{0\}$. Since A and B are disjoint we have $\tilde{M}(t,A) = W_t \delta_0(A)$ and $\tilde{M}(t,B) = \int_B h \, \tilde{N}(t,dh)$. In such a case, since W and N are independent so are $\tilde{M}(t,A)$ and $\tilde{M}(t,B)$. Hence, $(\tilde{M}(A),h)_H$ and $(\tilde{M}(B),h)_H$ are orthogonal for every $h \in H$.

As for the second case, assume neither A nor B is $\{0\}$. Then $\tilde{M}(t,A) = \int_A h \, \tilde{N}(t,dh)$ and $\tilde{M}(t,B) = \int_B h \, \tilde{N}(t,dh)$. Since A and B are bounded below and disjoint, these Poisson integrals are independent processes (see Theorem 2.4.6 in [3]). Hence, $(\tilde{M}(A),h)_H$ and $(\tilde{M}(B),h)_H$ are orthogonal for every $h \in H$.

Now, by Proposition 4.6 then $M = (M(t,A) : t \geq 0, A \in \mathcal{A})$ defined by (4.1) is a cylindrical orthogonal martingale-valued measure. Observe for each $h \in H$, $t \geq 0$, $A \in \mathcal{A}$,

$$\mathbb{E}\left[|M(t,A)(h)|^2\right] = t\left[(h,Qh)_H \delta_0(A) + \int_{A \setminus \{0\}} (u,h)_H^2 \, \lambda(du)\right]. \tag{4.2}$$

Remark 4.8. Let H be a separable Hilbert space and let $\tilde{M} = (\tilde{M}(t,A) : t \geq 0, A \in \mathcal{A})$ be an H-valued martingale-valued measure. Let $(h_n)_{n \geq 1}$ be a orthonormal basis in H and for every $n \geq 1$ let $\tilde{M}^n(t,A) = (\tilde{M}(t,A), h_n)_H$. Using the result of Lemma 2.2 in [34] one can show that for every $A, B \in \mathcal{A}$, $t \geq 0$, we have

$$\left\langle \tilde{M}(A), \tilde{M}(B) \right\rangle_t = \sum_{n \geq 1} \left\langle \tilde{M}^n(A), \tilde{M}^n(B) \right\rangle_t.$$

Therefore, if we assume that for every $A, B \in \mathcal{A}$ disjoint and $h \in H$, the real-valued martingales $(\tilde{M}(A),h)_H$ and $(\tilde{M}(B),h)_H$ are orthogonal, we conclude that $\tilde{M}(A)$ and $\tilde{M}(B)$ are orthogonal (as H-valued martingales). Thus $\tilde{M}$ is orthogonal.

5. Construction of the Quadratic Variation

5.1. **Definition and properties of the quadratic variation.** In this section we define the (predictable) quadratic variation of a cylindrical martingale-valued measure. Our definition is based on a extension of the definition of the quadratic variation as a supremum of measures introduced by Veraar and Yaroslavtsev in [37] in the case of cylindrical continuous local martingales. In our case, we found convenient to formulate our definition in terms of the family of intensity measures defined by the family of real-valued martingale-valued measures $(M(t,A)(x^*) : t \geq 0, A \in \mathcal{A})$. The existence of such a family of measures is guaranteed by the following result.

Lemma 5.1. *For every $x^* \in X^*$, there exists a random predictable σ-finite measure ν_{x^*} on $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(U)$ such that for every $t \geq 0$ and $A \in \mathcal{A}$,*

$$\mathbb{P}\left(\{\omega \in \Omega : \nu_{x^*}(\omega)\left([0,t] \times A\right) = \langle M(A)(x^*)\rangle_t(\omega)\}\right) = 1. \tag{5.1}$$

Proof. It is enough to apply Corollary 3.7 to the $\mathbb{R}$-valued orthogonal martingale-valued measure $M(A)(x^*)$, for each $x^* \in X^*$. □

Basic properties of $\langle M(\cdot)(x^*)\rangle$ are inherited by ν_{x^*}. For instance, given a real number c, we have $\nu_{cx^*} = c^2\nu_{x^*}$ (as random measures). In fact, given t and A we have, $\mathbb{P}$-a.e.

$$c^2\nu_{x^*}([0,t] \times A) = c^2 \langle M(A)(x^*)\rangle_t = \langle cM(A)(x^*)\rangle_t = \langle M(A)(cx^*)\rangle_t .$$

Uniqueness of the intensity measure gives the identity (Remark 3.10).

Example 5.2. Let M denote the cylindrical martingale-valued measure defined in Example 4.4. For every $x^* \in X^*$ our assumption that the real-valued martingales $(Z^k(x^*))_{k=1}^n$ are orthogonal implies that $\langle M(A)(x^*)\rangle_t = \sum_{k=1}^n \left\langle Z^k(x^*)\right\rangle_t \delta_{a_k}(A)$ for every $t > 0$ and $A \in \mathcal{A} = \mathcal{B}(U)$. Hence

$$\nu_{x^*}(ds, du) = \sum_{k=1}^n \lambda_{\left\langle Z^k(x^*)\right\rangle}(ds)\delta_{a_k}(du),$$

where $\lambda_{\left\langle Z^k(x^*)\right\rangle}$ denotes the *Lebesgue-Stieltjes* measure associated to $\left\langle Z^k(x^*)\right\rangle$.

We are ready to introduce our definition of quadratic variation for M, which is an extension of Definition 3.4 in [37]. First, we will need the following terminology: let (S, Σ) be a measurable space and let $\mathcal{M}_+(S, \Sigma)$ be the set of all positive measures on (S, Σ). For $\eta, \zeta : \Omega \to \mathcal{M}_+(S, \Sigma)$ we say that $\eta \leq \zeta$ if $\eta(\omega) \leq \zeta(\omega)$ for $\mathbb{P}$-a.e. $\omega \in \Omega$. The relation "$\leq$" defines a partial order in $\mathcal{M}_+(S, \Sigma)$.

Definition 5.3. We say that M *has a quadratic variation* if there exists a random measure $\eta : \Omega \to \mathcal{M}_+(\mathbb{R}_+ \times U, \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(U))$ such that

(i) Given $t \geq 0$ and $A \in \mathcal{A}$, for $\mathbb{P}$-a.e. $\omega \in \Omega$ we have $\eta(\omega)([0,t] \times A) < \infty$.

(ii) η is a minimal element (in the partial order "$\leq$") for the collection of all the random measures $\zeta : \Omega \to \mathcal{M}_+(\mathbb{R}_+ \times U, \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(U))$ with the property: $\forall\, x^* \in X^*$ with $\|x^*\| = 1$, $\nu_{x^*} \leq \zeta$.

We say that η is a *quadratic variation* for M.

The reader might notice that in the definition above, the possibility that M has more than one quadratic variation is left open. Later in this section we introduce a sufficient condition for the existence of a unique quadratic variation. In the following result we provide a necessary condition for the existence of a quadratic variation for M.

Proposition 5.4. *Assume that M has a quadratic variation η. Let $(x_n^*)_{n\geq 1}$ be a dense subset of the unit sphere in X^* and let $\mu := \sup_{n\geq 1} \nu_{x_n^*}$. Then $\mu \leq \eta$.*

Proof. Since η is a quadratic variation for M, for every $n \geq 1$ there exists $\Omega_n \subseteq \Omega$, with $\mathbb{P}(\Omega_n) = 1$ and $\nu_{x_n^*}(\omega) \leq \eta(\omega)$ for every $\omega \in \Omega_n$. Let $\Omega_0 = \bigcap_{n\geq 1} \Omega_n$. Then $\mathbb{P}(\Omega_0) = 1$ and by the definition of supremum of measures we have $\mu(\omega) \leq \eta(\omega)$ for every $\omega \in \Omega_0$. □

We now tackle the issue of existence and uniqueness of a quadratic variation for M. Our key property is described as follows:

Definition 5.5. We say that the family of intensity measures $(\nu_{x^*} : x^* \in X^*)$ satisfies the *sequential boundedness property* if whenever (x_n^*) is dense in the unit sphere, $\|x^*\| = 1$ and $t > 0$, there exists $\Omega_{x^*} \subseteq \Omega$ with $\mathbb{P}(\Omega_{x^*}) = 1$, such that for $0 \leq s < t$, $A \in \mathcal{A}$ and $\omega \in \Omega_{x^*}$,

$$\nu_{x^*}(\omega)((s,t] \times A) \leq \sup_{n\geq 1} \nu_{x_n^*}(\omega)((s,t] \times A). \tag{5.2}$$

Note that the supremum in (5.2) is a classical supremum of real numbers, which can be infinite in some cases.

Remark 5.6. It is enough to verify (5.2) for sequences that converge to x^* in the unit sphere. In fact, if that is true and (x_n^*) is dense in the unit sphere, there is a subsequence (x_{n_k}) converging to x^*. Then $\mathbb{P}$-a.e. we have, for $0 \leq s < t$ and $A \in \mathcal{A}$,

$$\nu_{x^*}((s,t] \times A) \leq \sup_k \nu_{x_{n_k}^*}((s,t] \times A) \leq \sup_n \nu_{x_n^*}((s,t] \times A).$$

The following result provides a sufficient condition for the sequential boundedness property to hold.

Proposition 5.7. *Assume the family of intensity measures $(\nu_{x^*} : x^* \in X^*)$ satisfies the following condition: given $x_n^* \to x^*$ in the unit sphere of X^*, for every $t > 0$ we have*

$$\sup_{A\in\mathcal{A}} \sup_{0\leq s\leq t} \left| \nu_{x_n^*}((s,t] \times A) - \nu_{x^*}((s,t] \times A) \right| \xrightarrow{\mathbb{P}} 0, \quad \text{as } n \to \infty. \tag{5.3}$$

Then $(\nu_{x^*} : x^* \in X^*)$ satisfies the sequential boundedness property.

Proof. Let $X_n = \sup_{A\in\mathcal{A}} \sup_{0\le s\le t} \left|\nu_{x_n^*}((s,t]\times A) - \nu_{x^*}((s,t]\times A)\right|$. Since $X_n \xrightarrow{\mathbb{P}} 0$, there exists a sequence of positive integers $n_k \uparrow \infty$ such that $X_{n_k} \to 0$, $\mathbb{P}$-a.e. In particular, there exists $\Omega_{x^*} \subseteq \Omega$ with $\mathbb{P}(\Omega_{x^*}) = 1$ such that, for $0 \le s < t$, $A \in \mathcal{A}$ and $\omega \in \Omega_{x^*}$

$$\nu_{x^*}(\omega)((s,t]\times A) = \lim_{k\to\infty} \nu_{x_{n_k}^*}(\omega)((s,t]\times A) \le \sup_{n\ge 1} \nu_{x_n^*}(\omega)((s,t]\times A).$$

The last remark finishes the work. □

Now we provide a sufficient condition for the existence and uniqueness of a quadratic variation for M.

Theorem 5.8. *Let $(x_n^*)_{n\ge1}$ be a dense subset of the unit sphere in X^* and let $\mu = \sup_{n\ge1} \nu_{x_n^*}$. Assume*

(i) Given $t \ge 0$ and $A \in \mathcal{A}$, $\mathbb{P}$-a.e. we have $\mu([0,t]\times A) < \infty$.

(ii) $(\nu_{x^*} : x^* \in X^*)$ satisfies the sequential boundedness property.

Then μ is a quadratic variation for M. In particular, the quadratic variation is unique (any quadratic variation equals μ $\mathbb{P}$-a.e.)

Proof. We must check that μ satisfies Definition 5.3. In fact, by the sequential boundedness property, for all $x^* \in X^*$ with $\|x^*\| = 1$ we have that $\mathbb{P}$-a.e., for $0 \le s < t$ and $A \in \mathcal{A}$

$$\nu_{x^*}((s,t]\times A) \le \sup_{n\ge1} \nu_{x_n^*}((s,t]\times A) \le \mu((s,t]\times A).$$

By Lemma 3.9 this inequality extends to $\mathcal{B}(\mathbb{R}_+)\otimes\mathcal{B}(U)$, then we have that for every x^* in the unit ball of X^*, $\nu_{x^*} \le \mu$ $\mathbb{P}$-a.e., and by (i), μ is finite on the required rectangles.

Let ζ be a random measure on $\mathcal{B}(\mathbb{R}_+)\otimes\mathcal{B}(U)$ such that $\nu_{x^*} \le \zeta$ whenever $\|x^*\| = 1$. It follows that $\mathbb{P}$-a.e. $\nu_{x_n^*} \le \zeta$ for every $n \ge 1$; by definition of supremum of measures, $\mu \le \zeta$. Therefore, μ is a quadratic variation for M. Finally, given any quadratic variation η we have $\mu \le \eta$. Since η is minimal, this implies $\eta = \mu$ $\mathbb{P}$-a.e. □

Definition 5.9. If M has a unique quadratic variation, we will denote it by $\langle\!\langle M\rangle\!\rangle$ and refer to it as *the quadratic variation of M*.

As the next result shows, in the presence of the sequential boundedness property for the family of intensity measures, if a quadratic variation exists, it is unique. Other properties of the quadratic variation are given.

Theorem 5.10. *Assume M has a quadratic variation and $(\nu_{x^*} : x^* \in X^*)$ satisfies the sequential boundedness property. Then M has a unique quadratic variation $\langle\!\langle M\rangle\!\rangle$. Moreover,* *for any dense subset $(x_n^*)_{n\geq 1}$ of the unit sphere in X^* we have $\langle\langle M\rangle\rangle = \sup_{n\geq 1} \nu_{x_n^*}$ $\mathbb{P}$-a.e. In particular $\langle\langle M\rangle\rangle$ is a predictable random measure and $\mathbb{P}$-a.e.,*

$$\langle\langle M\rangle\rangle\left([0,t]\times A\right) = \sup_{\Pi\in\mathscr{R}([0,t]\times A)} \sum_{C\in\Pi} \sup_{n\geq 1} \nu_{x_n^*}(\omega)\left(C\right), \quad \forall t\geq 0, A\in\mathcal{B}(U), \tag{5.4}$$

where $\mathscr{R}([0,t]\times A)$ is the family of partitions of $[0,t]\times A$ of the form

$$\Pi = \{(t_{i-1},t_i]\times A_j : 1\leq i\leq k, 1\leq j\leq m,\ k,m\in\mathbb{N}\}, \tag{5.5}$$

where $0=t_0<t_1<\cdots<t_k=t$ are rational (with the possible exception of t), the sets $A_1,\ldots,A_m$ form a partition of $A\in\mathcal{B}(U)$, and $(x_n^*)_{n\geq 1}$ is a dense subset of the unit sphere in X^*.

Proof. Let $(x_n^*)_{n\geq 1}$ be a dense subset of the unit sphere and let $\mu=\sup_{n\geq 1}\nu_{x_n^*}$.
Let η be a quadratic variation for M. Then, by Proposition 5.4, $\mu=\sup_{n\geq 1}\nu_{x_n^*}$ satisfies for all $t\geq 0$, $A\in\mathcal{A}$, $\mathbb{P}$-a.e. $\mu([0,t]\times A)\leq \eta([0,t]\times A)<\infty$. Hence by Theorem 5.8 μ is a quadratic variation and we have $\eta=\mu$ $\mathbb{P}$-a.e. This shows uniqueness of the quadratic variation and by definition $\langle\langle M\rangle\rangle=\mu$.
Moreover, by Lemma 2.1, $\langle\langle M\rangle\rangle$ takes the form (5.4) and it is a predictable random measure by Corollary 2.3. □

Remark 5.11. Note that any partition descibed as in (5.5) can be written in the form $\{(s_j,t_j]\times A_j : j=1,\ldots,m\}$, where the numbers s_j and t_j are rational, $A_1,\ldots,A_m$ form a partition of A, and also the following is satisfied:

$$\text{for } k\neq j,\ \text{if } (s_k,t_k]\cap(s_j,t_j]\neq\emptyset,\ \text{then } (s_k,t_k]=(s_j,t_j] \text{ and } A_k\cap A_j=\emptyset. \tag{5.6}$$

Remark 5.12. Under the assumptions in Theorem 5.10, notice that by Lemma 5.1 for any given $t\geq 0$ and $A\in\mathcal{A}$ we have $\mathbb{P}$-a.e.

$$\langle\langle M\rangle\rangle\left([0,t]\times A\right) = \sup_{\Pi}\sum_{j=1}^{m_\Pi}\sup_{n\geq 1}\left(\langle M(A_j)(x_n^*)\rangle_{t_j} - \langle M(A_j)(x_n^*)\rangle_{s_j}\right), \tag{5.7}$$

where the supremum is taken over all partitions Π in the form described in Remark 5.11.

Example 5.13. Let $Z=(Z_t : t\geq 0)$ be a cylindrical continuous zero-mean square integrable martingale. Consider a one-point set $U=\{a\}$ and let M be the corresponding cylindrical orthogonal martingale-valued measure of Example 4.4. We shall verify that the family of intensity measures $(\nu_{x^*} : x^*\in X^*)$ of M satisfies the sequential boundedness property.

Let $x^* \in X^*$. By Example 5.2,for all $0 \leq s < t$, $A \in \mathcal{A} = 2^U$ we have

$$\nu_{x^*}((s,t],A) = \lambda_{\langle Z(x^*)\rangle}((s,t])\delta_a(A) = (\langle Z(x^*)\rangle_t - \langle Z(x^*)\rangle_s)\delta_a(A). \tag{5.8}$$

Assume $x_n^* \to x^*$ and let $t > 0$. Since Z defines a continuous linear operator from X^* into the space $\mathcal{M}_t^{2,c}$ of real-valued continuous square integrable martingales on $[0,t]$, then $\langle Z(x_n^*)\rangle \to \langle Z(x^*)\rangle$ in probability uniformly on $[0,t]$ (see Proposition 18.6 in [17]). Hence

$$\begin{aligned} &\sup_{A\in\mathcal{A}} \sup_{0\leq s\leq t} \left|\nu_{x_n^*}((s,t]\times A) - \nu_{x^*}((s,t]\times A)\right| \\ &= \sup_{0\leq s\leq t} \left|(\langle Z(x_n^*)\rangle_t - \langle Z(x_n^*)\rangle_s) - (\langle Z(x^*)\rangle_t - \langle Z(x^*)\rangle_s)\right| \xrightarrow{\mathbb{P}} 0, \end{aligned}$$

as $n \to \infty$. By Proposition 5.7 the family $(\nu_{x^*} : x^* \in X^*)$ satisfies the sequential boundedness property.

Notice that by (5.8), Theorem 5.8 and Theorem 5.10 we have that M has (a necessarily unique) quadratic variation if and only if for some (equivalently for any) dense subset $(x_n^*)_{n\geq 1}$ of the unit sphere in X^* we have $\mathbb{P}$-a.e.

$$\left(\sup_{n\geq 1} \lambda_{\langle Z(x_n^*)\rangle} \otimes \delta_a\right)([0,t]\times A) < \infty, \quad \forall t \geq 0,\, A \in 2^U. \tag{5.9}$$

In such a case we have $\langle\langle M\rangle\rangle = \sup_{n\geq 1} \lambda_{\langle Z(x_n^*)\rangle} \otimes \delta_a$ $\mathbb{P}$-a.e.

Observe that since $2^U = \{\emptyset, \{a\}\}$, then given $(x_n^*)_{n\geq 1}$ as above, (5.9) holds true if and only if $\mathbb{P}$-a.e. $\left(\sup_{n\geq 1} \lambda_{\langle Z(x_n^*)\rangle}\right)([0,t]) < \infty$, $\forall t \geq 0$. By Remark 2.10 in [37] this is equivalent to the existence of a non-decreasing right-continuous process $F : \mathbb{R}_+ \times \Omega \to \mathbb{R}_+$ such that $\mathbb{P}$-a.e. $\lambda_F = \sup_{n\geq 1} \lambda_{\langle Z(x_n^*)\rangle}$.

The above observation can be thought as a generalization of Proposition 3.7 in [37] in the case of a cylindrical continuous square integrable martingale.

Example 5.14. Let $Z = (Z_t : t \geq 0)$ be a cylindrical zero-mean square integrable Lévy process in X, i.e. for every $d \in \mathbb{N}$, $x_1^*, \cdots, x_d^* \in X^*$ the $\mathbb{R}^d$-valued stochastic process $(Z_t(x_1^*), \cdots, Z_t(x_d^*) : t \geq 0)$ is a Lévy process in $\mathbb{R}^d$, and $\mathbb{E}[Z_t(x^*)] = 0$ and $\mathbb{E}[|Z_t(x^*)|^2] < \infty$ for every $t \geq 0$, $x^* \in X^*$. We will always assume for such a Z that the mapping $Z_t : X^* \to L^0(\Omega, \mathcal{F}, \mathbb{P})$ is continuous.

With the above hypothesis, it follows by Theorem 4.7 in [4] that there exists a positive symmetric operator $Q : X^* \to X^{**}$, called the the *covariance operator* of Z, defined by $(Qx^*)y^* = \mathbb{E}[Z_1(x^*)Z_1(y^*)]$ $\forall x^*, y^* \in X^*$. The operator Q is linear and continuous by Proposition III.1.1 in [36]. For every $x^* \in X^*$, observe that $\langle Z(x^*)\rangle_t = t\mathbb{E}\left[|Z_1(x^*)|^2\right] = t(Q(x^*)x^*)$.

Now let $n \in \mathbb{N}$, $U = \{a_1, \cdots, a_n\}$, and $Z^1, \cdots, Z^n$ be cylindrical zero-mean square integrable Lévy processes in X with corresponding covariance operators $Q^1, \cdots, Q^n$. Assume moreover that for each $x^* \in X^*$, the real-valued martingales $(Z^k(x^*))_{k=1}^n$ are orthogonal. Let M be as defined in Example 4.4. We will show that M has a quadratic variation.

By Example 5.2 we have for every $x^* \in X^*$

$$\nu_{x^*}(ds, du) = \sum_{k=1}^n \lambda_{\langle Z^k(x^*)\rangle}(ds)\delta_{a_k}(du) = \sum_{k=1}^n (Q^k(x^*)x^*) ds \delta_{a_k}(du).$$

From the above it is immediate that for any $x^*, y^* \in X^*$,

$$\sup_{A\in\mathcal{A}} \sup_{0\leq s\leq t} |\nu_{x^*}((s,t]\times A) - \nu_{y^*}((s,t]\times A)| \leq t \left(\sum_{k=1}^n \left\| Q^k \right\| \right) (\|x^*\| + \|y^*\|) \|x^* - y^*\|$$

and hence the family of measures ν_{x^*} satisfies (5.3), therefore it has the sequential boundedness property by Proposition 5.7.

Let $(x^*_m)_{m\geq 1}$ be a dense subset of the unit sphere in X^*. According to Theorem 5.8 we must show that $\sup_{m\geq 1} \nu_{x^*_m}$ defines a random positive measure which is finite on the rectangles $[0,t]\times A$.

In fact, let $t \geq 0$ and $A \in \mathcal{A} = 2^U$. Take a partition $\{(s_j, t_j] \times A_j\}$ of $[0,t]\times A$, that satisfies (5.6), furthermore, we can take each A_j as a singleton. Then we have

$$\begin{aligned}
\sum_{j=1}^N \sup_{m\geq 1} \nu_{x^*_m}((s_j,t_j]\times A_j) &= \sum_{j=1}^N \sup_{m\geq 1} \sum_{k=1}^n (t_j - s_j)(Q^k(x^*_m)x^*_m)\delta_{a_k}(A_j) \\
&= \sum_{j=1}^N (t_j - s_j) \sum_{k=1}^n \sup_{m\geq 1} (Q^k(x^*_m)x^*_m)\delta_{a_k}(A_j) \\
&= \sum_{j=1}^N (t_j - s_j) \sum_{k=1}^n \left\| Q^k \right\| \delta_{a_k}(A_j) \\
&= \sum_{k=1}^n \left\| Q^k \right\| \sum_{j=1}^N (t_j - s_j)\delta_{a_k}(A_j)
\end{aligned}$$

Let μ be the measure on $(\mathbb{R}_+ \times U, \mathcal{B}(\mathbb{R}_+)\otimes\mathcal{B}(U))$ given by $\mu(ds,du) = \sum_{k=1}^n \left\| Q^k \right\| ds\delta_{a_k}(du)$. Then it is clear that $\mu([0,t]\times A) < \infty$ for every $t \geq 0$ and $A \in \mathcal{B}(U)$ and from the calculations above we have $\mu = \sup_{m\geq 1} \nu_{x^*_m}$. Hence by Theorem 5.8 we conclude that M has quadratic variation and $\langle\langle M \rangle\rangle (ds, du) = \sum_{k=1}^n \left\| Q^k \right\| ds\delta_{a_k}(du)$.

As the next example shows, it is not true in general that every M defined as in Example 4.4 has quadratic variation.

Example 5.15. We adapt to our context the construction in Example 3.26 in [37].

Let H be a separable Hilbert space with an orthonormal basis $(h_k)_{k\geq 1}$ and let W be a one-dimensional Brownian motion. Let $\psi : \mathbb{R}_+ \times \Omega \to H$ given by $\psi(t,\omega) = \sum_{k=1}^{\infty} |B_k|^{-1/2} \mathbb{1}_{B_k}(t) h_k$ where $[0,1] = \bigcup_{k\geq 1} B_k$ is a partition of $[0,1]$ into pairwise disjoint sets. For every $h \in H$ one can check $\int_{\mathbb{R}_+} |(\psi(s,\omega),h)_H|^2 \, ds = \|h\|^2$ so $(\psi,h)_H$ is stochastically integrable with respect to W. Moreover by the Itô isometry $\|(\psi,h)_H \cdot W\|_{\mathcal{M}^2_\infty} = \left\| \int_0^1 (\psi,h)_H dW \right\|_{L^2(\Omega)} = \|h\|$.

Let $U = \{a\}$ and $Z : H \to \mathcal{M}^2_\infty$ be the linear continuous operator given by $Z(h) = (\psi,h)_H \cdot W$ for every $h \in H$. Observe that in this case we have $M(t,A)(h) = Z_t(h)\delta_a(A)$ and $\langle M(A)(h) \rangle_t = \langle Z(h) \rangle_t \delta_a(A)$.

Assume M has quadratic variation. Let F be the $\mathbb{Q}$-span of $(h_k)_{k\geq 1}$. Then by Theorem 5.10 and (5.7) we must have

$$\begin{aligned}
\langle\!\langle M \rangle\!\rangle ([0,1]\times U) &= \lim_{\text{mesh}\to 0} \sum_{j=1}^{N} \sup_{\|h\|=1, h\in F} \left(\langle M(U)(h) \rangle_{t_j} - \langle M(U)(h) \rangle_{t_{j-1}} \right) \\
&= \lim_{\text{mesh}\to 0} \sum_{j=1}^{N} \sup_{\|h\|=1, h\in F} \left(\langle Z(h) \rangle_{t_j} - \langle Z(h) \rangle_{t_{j-1}} \right).
\end{aligned}$$

However as shown in Example 3.26 in [37] we have

$$\lim_{\text{mesh}\to 0} \sum_{j=1}^{N} \sup_{\|h\|=1} \left(\langle Z(h) \rangle_{t_j} - \langle Z(h) \rangle_{t_{j-1}} \right) = \int_0^1 \|\psi(s)\|^2 \, ds = \sum_{n=1}^{\infty} \|h_n\|^2 = \infty,$$

therefore showing $\langle\!\langle M \rangle\!\rangle ([0,1]\times U) = \infty$. Consequently, M does not have quadratic variation.

Example 5.16. Le H be a separable Hilbert space and let $L = (L_t : t \geq 0)$ be an H-valued càdlàg Lévy process. We will check that the cylindrical martingale-valued measure M defined by L as in Example 4.7 has a quadratic variation.

First, observe that by (4.2) for every $h \in H$, $t \geq 0$ and $A \in \mathcal{A}$ we have

$$\langle M(t,A)h \rangle_t = t\mathbb{E}\left[|M(1,A)(h)|^2 \right] = t \left[(h,Qh)_H \delta_0(A) + \int_{A\setminus\{0\}} (u,h)_H^2 \, \lambda(du) \right].$$

Hence

$$\nu_h((s,t]\times A)(\omega) = (t-s) \left[(h,Qh)_H \delta_0(A) + \int_{A\setminus\{0\}} (u,h)_H^2 \, \lambda(du) \right]. \tag{5.10}$$

Our first goal is to verify that (5.3) holds true. Assume $h_n \to h$ in H. For our calculations below we will need the following inequalities which can be verified easily

$$\begin{aligned} |(h_n, Qh_n)_H - (h, Qh)_H| &\leq \|Q\| \left(\sup_{n\geq 1} \|h_n\| + \|h\| \right) \|h_n - h\| . \\ \left|(u, h_n)_H^2 - (u, h)^2\right| &\leq \|u\|^2 \left(\sup_{n\geq 1} \|h_n\| + \|h\| \right) \|h_n - h\| . \end{aligned} \tag{5.11}$$

Then by (5.10) and (5.11)

$$\begin{aligned} & \sup_{A\in\mathcal{A}} \sup_{0\leq s\leq t} |\nu_{h_n}((s,t]\times A) - \nu_h((s,t]\times A)| \\ & \leq \sup_{A\in\mathcal{A}} \sup_{0\leq s\leq t} (t-s) \left[|(h_n, Qh_n)_H - (h, Qh)_H| \, \delta_0(A) + \int_{A\setminus\{0\}} \left|(u, h_n)_H^2 - (u, h)_H^2\right| \, \lambda(du) \right] \\ & \leq t \left[|(h_n, Qh_n)_H - (h, Qh)_H| + \int_U \left|(u, h_n)_H^2 - (u, h)_H^2\right| \, \lambda(du) \right] \\ & \leq t \left(\sup_{n\geq 1} \|h_n\| + \|h\| \right) \left[\|Q\| + \int_U \|u\|^2 \, \lambda(du) \right] \|h_n - h\| \to 0. \end{aligned}$$

Then $(\nu_h : h \in H)$ satisfies the sequential boundedness property by Proposition 5.7.

Our next goal is to prove that M has a quadratic variation. We do this by setting our calculations in the terms required by Lemma 2.2.

For every $h \in H$ consider the $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(U)$-measurable function

$$f_h(r,u) = (h, Qh)_H \mathbb{1}_{\{0\}}(u) + (u,h)_H^2 \mathbb{1}_{U\setminus\{0\}}(u).$$

Let $\gamma = \delta_0 + \lambda\big|_U$. Observe that by (5.10) we have

$$\nu_h((s,t]\times A)(\omega) = \int_{(s,t]\times A} f_h(r,u) \, (\text{Leb} \otimes \gamma)(dr, du).$$

Now let $\overline{f}$ be the $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(U)$-measurable function

$$\overline{f}(r,u) = \|Q\| \, \mathbb{1}_{\{0\}}(u) + \|u\|^2 \, \mathbb{1}_{U\setminus\{0\}}(u).$$

One can easily check that $\overline{f} = \sup_{\|h\|=1} f_h$. Moreover, if $(h_n : n \geq 1)$ is a sequence dense in the unit sphere in H, a standard argument shows $\overline{f} = \sup_{n\geq 1} f_{h_n}$.

If $\mu = \sup_{\|h\|=1} \nu_h$ then by Lemma 2.2 we have $\mu = \sup_{n\geq 1} \nu_{h_n}$ and

$$\begin{aligned} \mu((s,t]\times A) &= \int_{(s,t]\times A} \overline{f}(r,u) \, (\text{Leb} \otimes \gamma)(dr, du) \\ &= (t-s) \left[\|Q\| \, \delta_0(A) + \int_{A\setminus 0} \|u\|^2 \, \lambda(du) \right]. \end{aligned}$$

Then by Theorem 5.8 we conclude that M has quadratic variation and

$$\langle\langle M\rangle\rangle((s,t]\times A)=(t-s)\left[\|Q\|\,\delta_0(A)+\int_{A\setminus\{0\}}\|u\|^2\,\lambda(du)\right]. \tag{5.12}$$

Example 5.17. For M as in Example 4.5 we have

$$|M(t,A)(x^*)|^2-\int_0^t \mathbb{1}_A(g(u))d\langle Z(x^*)\rangle_u$$

is a martingale, so

$$\langle M(A)(x^*)\rangle_t=\int_0^t \mathbb{1}_A(g(u))d\langle Z(x^*)\rangle_u.$$

As this process is predictable and σ-additive, we have

$$\nu_{x^*}((s,t]\times A)=\int_s^t \mathbb{1}_A(g(u))d\langle Z(x^*)\rangle_u.$$

We now describe a particular Z for which M does not have a quadratic variation.

Suppose $X=L^2[0,1]$. Let $\mathcal{A}=\mathcal{B}(X)$ and

$$Z(h)_t=\int_0^t h(s)dW_s$$

for some (real) standard Brownian motion W. We thus have

$$d\langle Z(h)\rangle_t=h^2(t)dt.$$

Notice that (5.3) is satisfied in this case. In fact

$$\begin{aligned}\sup_{A\in\mathcal{A}}\sup_{0\le s\le t}|\nu_{h_n}((s,t]\times A)-\nu_h((s,t]\times A)| &= \sup_{A\in\mathcal{A}}\sup_{0\le s\le t}\left|\int_s^t \mathbb{1}_A(g(u))(h_n^2(u)-h^2(u))du\right| \\ &\le \int_0^t |h_n^2(u)-h^2(u)|du\to 0\end{aligned}$$

whenever $h_n\to h$.

If M had a quadratic variation it should satisfy (5.4), however, the later does not hold in general. For example, we can fix $A=U$ and consider a dyadic partition $\mathcal{D}_k$ for $[0,1]$ of level k, that is, by 2^k subintervals of length 2^{-k}. Let $\{h_n : n\ge 1\}$ be the L^2-normalized Haar basis; this family is dense on the unit ball of $L^2[0,1]$, and since each h_n is a norm one function supported on some dyadic interval I_n, we have

$$\nu_{h_n}(I_n\times U)\ge\int_{I_n} h_n^2(u)du=1,\qquad \forall\, n\ge 1.$$

Therefore,

$$\sum_{I\in\mathcal{D}_k}\sup_{n\ge1}\nu_{h_n}(I\times U)\ge 2^k.$$

Note that this sum is over one of the partitions considered in (5.4), therefore, the right hand side of that equation should be infinite.

5.2. **The quadratic variation operator measure.** Throughout this section we assume that M has a quadratic variation and the family of intensity measures satisfies the sequential boundedness property. By Theorem 5.10 the quadratic variation of M is unique. We use the following notation, for the covariation of two real-valued processes X and Y:

$$\langle X \, , \, Y \rangle_s^t := \langle X \, , \, Y \rangle_t - \langle X \, , \, Y \rangle_s \, .$$

Set $T > 0$. For any given $0 \leq s \leq t \leq T$ and $A \in \mathcal{A}$, the mapping $(x^*, y^*) \mapsto \langle M(A)(x^*), M(A)(y^*) \rangle_s^t$ defines a bilinear form on $X^* \times X^*$ taking values in the space of real-valued random variables. Conversely, for any given $x^*, y^* \in X^*$, $(s,t] \times A \mapsto \langle M(A)(x^*), M(A)(y^*) \rangle_s^t$ defines a finitely additive random signed measure on the ring of subsets of $[0,T] \times U$ generated by the sets of the form $\{0\} \times A$, and $(s,t] \times A$ for $0 \leq s < t \leq T$, $A \in \mathcal{A}$.

Define α_M for every $x^*, y^* \in X^*$ as the random set function on $\mathcal{B}([0,T]) \otimes \mathcal{B}(U)$ given by

$$\alpha_M(\omega)(C)(x^*, y^*) = \tfrac{1}{4} \left(\nu_{x^*+y^*}(C) - \nu_{x^*-y^*}(C) \right), \quad C \in \mathcal{B}([0,T]) \otimes \mathcal{B}(U). \tag{5.13}$$

We will assume that for every $x^*, y^* \in X^*$, $\alpha_M(\omega)(\cdot)(x^*, y^*)$ is a well defined signed measure on $\mathcal{B}([0,T]) \otimes \mathcal{B}(U)$. We will show (see Theorem 5.22 below) that under mild assumptions on $\langle\langle M \rangle\rangle$ we have that α_M extends to a $\mathfrak{Bil}(X^*, X^*)$-valued measure such that for each rectangle $(s,t] \times A$, $\alpha_M(\omega)((s,t] \times A)(x^*, y^*)$ equals $\langle M(A)(x^*), M(A)(y^*) \rangle_s^t (\omega)$ $\mathbb{P}$-a.e. Our first step is the following:

Theorem 5.18. *For $\mathbb{P}$-a.e. $\omega \in \Omega$, $\alpha_M(\omega)((s,t] \times A) \in \mathfrak{Bil}(X^*, X^*)$ and*

$$\| \alpha_M(\omega)((s,t] \times A) \|_{\mathfrak{Bil}(X^*, X^*)} \leq \langle\langle M \rangle\rangle (\omega)((s,t] \times A),$$

for all $0 \leq s \leq t$, $A \in \mathcal{A}$.

The proof of Theorem 5.18 will be carried out in several steps. Fix $(x_n^*)_{n \geq 1} \subseteq X^*$ a set of linearly independent vectors such that $\text{span}(x_n^*)_{n \geq 1}$ is dense in X^*. Let $F = (y_m^*)_{m \geq 1}$ be the $\mathbb{Q}$-span of $(x_n^*)_{n \geq 1}$.

Fix $0 \leq s \leq t$, $A \in \mathcal{A}$. Denote by $\widehat{\Omega}(s,t,A)$ the set of all $\omega \in \Omega$ such that $(x^*, y^*) \mapsto \langle M(A)(x^*) \, , \, M(A)(y^*) \rangle_s^t (\omega)$ is a bilinear form on $F \times F$. By the countability of F and a.s. linearity of the (real) quadratic convariation, we have $\mathbb{P}(\widehat{\Omega}(s,t,A)) = 1$.

Similarly, let $\tilde{\Omega}(s,t,A)$ be the set of all $\omega \in \Omega$ such that for all $x^*, y^* \in F$ we have

$$\frac{1}{4}\left(\nu_{x^*+y^*}(\omega)((s,t]\times A) - \nu_{x^*-y^*}(\omega)((s,t]\times A)\right) = \langle M(A)(x^*)\,,\, M(A)(y^*)\rangle_s^t(\omega). \tag{5.14}$$

By the countability of F and Lemma 5.1, we have $\mathbb{P}(\tilde{\Omega}(s,t,A)) = 1$. Moreover, notice that $\forall \omega \in \tilde{\Omega}(s,t,A)$ we have by (5.13) and (5.14) that

$$\alpha_M(\omega)((s,t]\times A)(x^*,y^*) = \langle M(A)(x^*)\,,\, M(A)(y^*)\rangle_s^t(\omega), \quad \forall x^*, y^* \in F. \tag{5.15}$$

Let $\Lambda(s,t,A) := \{\omega : \alpha_M(\omega)((s,t]\times A) \text{ is a bilinear form on } F\times F\}$. Since $\widehat{\Omega}(s,t,A) \cap \tilde{\Omega}(s,t,A) \subseteq \Lambda(s,t,A)$ we conclude that $\mathbb{P}(\Lambda(s,t,A)) = 1$.

Lemma 5.19. *For $\mathbb{P}$-a.e. $\omega \in \Omega$, $\alpha_M(\omega)(C)$ is a bilinear form on $F\times F$ for all $C \in \mathcal{B}([0,T])\otimes\mathcal{B}(U)$.*

Proof. Fix $r \in \mathbb{Q}$ and $x^*, y^*, z^* \in F$. For $0 \leq s \leq t$, $A \in \mathcal{A}$ and $\omega \in \Lambda(s,t,A)$ we have

$$\alpha_M(\omega)((s,t]\times A)(rx^*+y^*,z^*) = r\alpha_M(\omega)((s,t]\times A)(x^*,z^*) + \alpha_M(\omega)((s,t]\times A)(y^*,z^*).$$

Then, by Lemma 3.11 we have the existence of a set $\Lambda_1(r,x^*,y^*,z^*) \subseteq \Omega$ with probability 1 such that

$$\alpha_M(\omega)(\cdot)(rx^*+y^*,z^*) = r\alpha_M(\omega)(\cdot)(x^*,z^*) + \alpha_M(\omega)(\cdot)(y^*,z^*),$$

for all $\omega \in \Lambda_1(r,x^*,y^*,z^*)$. Likewise we can show the existence of a set $\Lambda_2(r,x^*,y^*,z^*) \subseteq \Omega$ with probability 1 such that

$$\alpha_M(\omega)(\cdot)(x^*,ry^*+z^*) = r\alpha_M(\omega)(\cdot)(x^*,y^*) + \alpha_M(\omega)(\cdot)(x^*,z^*),$$

for all $\omega \in \Lambda_2(r,x^*,y^*,z^*)$. Setting

$$\Lambda = \bigcap_{(r,x^*,y^*,z^*)\in\mathbb{Q}\times F^3} \Lambda_1(r,x^*,y^*,z^*)\cap\Lambda_2(r,x^*,y^*,z^*),$$

we obtain a subset of Ω of probability 1 for which $\alpha_M(\omega)(C)$ is a bilinear form on $F\times F$ for all $C \in \mathcal{P}_T\otimes\mathcal{B}(U)$. □

Lemma 5.20. *For every $0 \leq s \leq t$ and $A \in \mathcal{A}$ we have $\mathbb{P}$-a.e.*

$$|\alpha_M(\omega)((s,t]\times A)(x^*,y^*)| \leq \langle\langle M\rangle\rangle((s,t]\times A)\,\|x^*\|\,\|y^*\|\,, \quad \forall\, x^*, y^* \in F.$$

Proof. Fix $0 \leq s \leq t$ and $A \in \mathcal{A}$. Let $\widetilde{F} = \left\{\frac{x^*}{\|x^*\|} : 0 \neq x^* \in F\right\}$, this set is countable and dense in the unit sphere of X^*. By Theorem 5.10, there is a set $\Gamma(s,t,A) \subseteq \Omega$ of probability 1 such that $\left(\sup_{x^*\in\widetilde{F}}\nu_{x^*}\right)((s,t]\times A) = \langle\langle M\rangle\rangle((s,t]\times A)$.

If $0 \neq x^* \in F$, for $\omega \in \widehat{\Omega}(s,t,A) \cap \tilde{\Omega}(s,t,A) \cap \Gamma(s,t,A)$ we have by (5.14) that

$$\begin{aligned} \langle\!\langle M \rangle\!\rangle ((s,t] \times A) &\geq \nu_{\frac{x^*}{\|x^*\|}}((s,t] \times A) = \left\langle M(A)\left(\frac{x^*}{\|x^*\|}\right)\right\rangle_s^t \\ &= \frac{1}{\|x^*\|^2} \langle M(A)(x^*) \rangle_s^t = \frac{1}{\|x^*\|^2} \nu_{x^*}((s,t] \times A), \end{aligned}$$

and thus $\nu_{x^*}((s,t] \times A) \leq \langle\!\langle M \rangle\!\rangle ((s,t] \times A) \|x^*\|^2$.

Now, for $\omega \in \widehat{\Omega}(s,t,A) \cap \tilde{\Omega}(s,t,A) \cap \Gamma(s,t,A)$, and any $0 \neq x^*, y^* \in F$ we have by (5.15), the Kunita-Watanabe inequality (e.g. see Theorem 11.4.1 in [7], p.240), and (5.14), that

$$\begin{aligned} |\alpha_M(\omega)((s,t] \times A)(x^*, y^*)| &= \left| \langle M(A)(x^*)\,,\, M(A)(y^*) \rangle_s^t \right| \\ &\leq \sqrt{\langle M(A)(x^*) \rangle_s^t} \cdot \sqrt{\langle M(A)(y^*) \rangle_s^t} \\ &= \sqrt{\nu_{x^*}((s,t] \times A)} \cdot \sqrt{\nu_{y^*}((s,t] \times A)} \\ &\leq \langle\!\langle M \rangle\!\rangle ((s,t] \times A) \|x^*\| \, \|y^*\| \,. \end{aligned}$$

This finishes the proof, since $\mathbb{P}(\widehat{\Omega}(s,t,A) \cap \tilde{\Omega}(s,t,A) \cap \Gamma(s,t,A)) = 1$. □

Lemma 5.21. *For $\mathbb{P}$-a.e. $\omega \in \Omega$,*

$$|\alpha_M(\omega)(C)(x^*, y^*)| \leq \langle\!\langle M \rangle\!\rangle (C) \|x^*\| \, \|y^*\| \,, \quad \forall\, C \in \mathcal{B}([0,T]) \otimes \mathcal{B}(U),\ x^*, y^* \in F.$$

Proof. Follows from Lemmas 3.11 and 5.20. □

Proof of Theorem 5.18. By Lemmas 5.19 and 5.21 there exists $\Omega_0 \subseteq \Omega$ with probability 1, such that for each $\omega \in \Omega_0$, we have $\alpha_M(\omega)((s,t] \times A) \in \mathfrak{Bil}(F,F)$ and

$$\|\alpha_M(\omega)((s,t] \times A)\|_{\mathfrak{Bil}(F,F)} \leq \langle\!\langle M \rangle\!\rangle (\omega)((s,t] \times A) < \infty,$$

for all $0 \leq s \leq t$, $A \in \mathcal{A}$. Since $\alpha_M(\omega)((s,t] \times A)$ is bounded on $F \times F$, it can be extended to $X^* \times X^*$. □

Theorem 5.22. *Assume that, for $\mathbb{P}$-a.e. $\omega \in \Omega$,*

$$\sup_{A \in \mathcal{A}} \langle\!\langle M \rangle\!\rangle (\omega)([0,T] \times A) < \infty. \tag{5.16}$$

Then there exists a random $\mathfrak{Bil}(X^*, X^*)$-valued measure α_M defined on $\mathcal{B}([0,T]) \otimes \mathcal{B}(U)$ such that for all $x^*, y^* \in X^*$, $0 \leq s \leq t \leq T$ and $A \in \mathcal{A}$, $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$\alpha_M(\omega)((s,t] \times A)(x^*, y^*) = \langle M(A)(x^*), M(A)(y^*) \rangle_s^t (\omega). \tag{5.17}$$

Proof. By Theorem 5.18 we have that, $\mathbb{P}$-a.e. $\alpha_M : \mathcal{R}_T \to \mathfrak{Bil}(X^*, X^*)$ is a weakly countably additive measure, where $\mathcal{R}_T$ denotes the ring of subsets of $[0,T] \times U$ generated by the sets of the form $\{0\} \times A$, and $(s,t] \times A$ for $0 \leq s \leq t \leq T$, $A \in \mathcal{A}$. By Theorem 5.18 and Equation (5.16), $\mathbb{P}$-a.e. the range of α_M on $\mathcal{R}_T$ is bounded in $\mathfrak{Bil}(X^*, X^*)$. Then by the Carathéodory-Hahn-Kluvanek extension theorem (see [20]; see also Theorem I.5.2 in [11], p.27) α_M has a unique countably additive extension to a $\mathfrak{Bil}(X^*, X^*)$-valued measure on the σ-ring $\mathcal{S}_T$ generated by the ring $\mathcal{R}_T$.

As for our final step, we will show that $\mathcal{S}_T = \mathcal{B}([0,T]) \otimes \mathcal{B}(U)$. To see why this is true, recall that by definition $\mathcal{S}_T$ is the smallest σ-ring containing $\mathcal{R}_T$, in particular $\mathcal{S}_T$ is closed under countable unions. Therefore, because $U_n \in \mathcal{A}$, $\forall n \in \mathbb{N}$ and $U = \bigcup_{n\in\mathbb{N}} U_n$, we have $[0,T] \times U \in \mathcal{S}_T$. Hence, $\mathcal{S}_T$ is a σ-algebra and consequently $\mathcal{S}_T = \mathcal{B}([0,T]) \otimes \mathcal{B}(U)$. ☐

We are ready to introduce the *quadratic variation operator measure*. The random measure α_M induces a random measure Γ_M on $\mathcal{B}([0,T]) \otimes \mathcal{B}(U)$ taking values in $\mathcal{L}(X^*, X^{**})$ by means of the prescription

$$\langle \Gamma_M(\omega)(C)x^* \, , \, y^* \rangle = \alpha_M(\omega)(C)(x^*, y^*), \quad \forall \, x^*, y^* \in X^*, \, C \in \mathcal{B}([0,T]) \otimes \mathcal{B}(U). \tag{5.18}$$

The reader must be aware that on the left hand side of (5.18), $\langle \cdot \, , \, \cdot \rangle$ corresponds to the duality relation for the pair (X^*, X^{**}).

From Theorem 5.18 we get the following useful inequality

$$\langle \Gamma_M(\omega)(\cdot)x^* \, , \, y^* \rangle \leq \|x^*\| \, \|y^*\| \, \langle\!\langle M \rangle\!\rangle \, (\omega)(\cdot) \text{ on } \mathcal{B}([0,T]) \otimes \mathcal{B}(U). \tag{5.19}$$

The proof of the existence of the operator-valued quadratic variation and some of its properties are contained in the following theorem.

Theorem 5.23. *Let $T > 0$, and assume that M satisfies (5.16). Then there exists a process $Q_M : \Omega \times [0,T] \times U \to \mathcal{L}(X^*, X^{**})$ such that $\mathbb{P}$-a.e. $\omega \in \Omega$*

$$\langle \Gamma_M(\omega)(C)x^* \, , \, y^* \rangle = \int_C \langle Q_M(\omega, r, u)x^* \, , \, y^* \rangle \; \langle\!\langle M \rangle\!\rangle \, (\omega)(dr, du) \tag{5.20}$$

for all $x^*, y^* \in X^*$, $C \in \mathcal{B}([0,T]) \otimes \mathcal{B}(U)$. Moreover the following properties hold:

(i) For every $x^*, y^* \in X^*$, the mapping $(\omega, r, u) \mapsto \langle Q_M(\omega, r, u)x^* \, , \, y^* \rangle$ is predictable, that is, $\mathcal{P}_T \otimes \mathcal{B}(U)$-measurable.

(ii) $\mathbb{P}$-a.e. $\omega \in \Omega$, $Q_M(\omega, \cdot, \cdot)$ is positive and symmetric $\langle\!\langle M \rangle\!\rangle$-a.e.

(iii) $\mathbb{P}$-a.e. $\omega \in \Omega$, $\|Q_M(\omega, \cdot, \cdot)\|_{\mathcal{L}(X^*, X^{**})} = 1$ $\langle\!\langle M \rangle\!\rangle$-a.e.

Proof. First observe that by (5.19), there is a full probability set $\Omega_0 \subseteq \Omega$ such that for $\omega \in \Omega_0$ we have $\langle \Gamma_M(\omega)x^* \, , \, y^* \rangle \ll \langle\!\langle M \rangle\!\rangle (\omega)$ on $\mathcal{B}([0,T]) \otimes \mathcal{B}(U)$ for each $x^*, y^* \in X^*$, and hence there is a Radon-Nikodyn density $q_{x^*,y^*}(\omega)$ of the real-valued measure $\langle \Gamma_M(\omega)x^* \, , \, y^* \rangle$ with respect to $\langle\!\langle M \rangle\!\rangle (\omega)$. This density is $\mathcal{B}([0,T]) \otimes \mathcal{B}(U)$-measurable and satisfies $|q_{x^*,y^*}(\omega)(r,u)| \leq \|x^*\| \, \|y^*\|$, $\langle\!\langle M \rangle\!\rangle (\omega)$-a.e.

Since X^* is separable, one can choose a modification of q_{x^*,y^*}, such that for every $x^*, y^* \in X^*$ the following holds true (see the construction in the proof of Theorem 1.2.34 in [13], p.37):

a) For every (ω, r, u) and $x^* \in X^*$, the mapping $q_{x^*}(\omega)(r,u) : y^* \mapsto q_{x^*,y^*}(\omega)(r,u)$ is a continuous linear form on X^* satisfying $\langle q_{x^*}(\omega)(r,u) \, , \, y^* \rangle = q_{x^*,y^*}(\omega)(r,u)$ and $\|q_{x^*}(\omega)(r,u)\| \leq \|x^*\|$.

b) For each (ω, r, u), the mapping $Q_M(\omega, r, u) : x^* \mapsto q_{x^*}(\omega)(r,u)$ belongs to $\mathcal{L}(X^*, X^{**})$, $\langle Q_M(\omega, r, u)x^* \, , \, y^* \rangle = q_{x^*,y^*}(\omega)(r,u)$ for each $x^*, y^* \in X^*$ and $\|Q_M\| \leq 1$.

By the properties described above, we get that the mapping $Q_M : \Omega \times \mathbb{R}_+ \times U \to \mathcal{L}(X^*, X^{**})$ satisfies (5.20).

To prove (i), we use a modification of the proof for Theorem 3 in [31]. Let $\mathcal{G} = \{G_k : k \geq 1\}$ be the collection of rectangles of the form $(s,t] \times A$, where $s < t$ are rational, and A belongs to a countable family of sets that generates $\mathcal{B}(U)$; this collection generates $\mathcal{B} = \mathcal{B}([0,T]) \otimes \mathcal{B}(U)$. For $n \in \mathbb{N}$, let Π_n denote the finest partition of $[0,T] \times U$ by sets of the (finite) algebra generated by $\mathcal{G}_n = \{G_k : 1 \leq k \leq n\}$. Note that $\Pi_n \subseteq \Pi_{n+1}$ for every n, and $\sigma(\bigcup_{n \geq 1} \Pi_n) = \bigcup_{n \geq 1} \sigma(\Pi_n) = \mathcal{B}$. Also, if $C \in \Pi_n$, then C is of the form $(s,t] \times A$, where $s, t \in \mathbb{Q}$ and $A \in \mathcal{B}(U)$.

For $\omega \in \Omega_0$ and $x^*, y^* \in X^*$, define the sequence of functions $(\phi_n : n \geq 1)$ by

$$\phi_n(\omega)(r,u) = \frac{\langle \Gamma_M(\omega)(C)x^* \, , \, y^* \rangle}{\langle\!\langle M \rangle\!\rangle (\omega)(C)}, \quad \text{if } (r,u) \in C \in \Pi_n, \; \langle\!\langle M \rangle\!\rangle (\omega)(C) > 0.$$

For fixed $\omega \in \Omega_0$ and $\lambda > 0$, the level set $\{(r,u) \in [0,T] \times U : \phi_n(\omega)(r,u) > \lambda\}$ is the (finite) union of those sets C in Π_n that satisfy

$$\langle \Gamma_M(\omega)(C)x^* \, , \, y^* \rangle > \lambda \, \langle\!\langle M \rangle\!\rangle (\omega)(C), \quad \langle\!\langle M \rangle\!\rangle (\omega)(C) > 0.$$

If $C = (s,t] \times A$, the predictability of both measures tells us that the set of possible values ω for which the previous conditions are satisfied is $\mathcal{F}_s$-measurable, which implies that the functions ϕ_n are predictable.

Finally, by Theorem 48.3 in [28], we have that the sequence $\phi_n(\omega)$ converges to $q_{x^*,y^*}(\omega)$ $\langle\langle M\rangle\rangle(\omega)$-a.e. Therefore the map $(\omega, r, u) \mapsto q_{x^*,y^*}(\omega)(r,u)$ is predictable, and so is $\langle Q_M(\cdot)x^*\,,\,y^*\rangle$.

By the construction of Q_M and the predictability proven in the previous step, (ii) holds true.

We have, by the previous construction, that for all $\omega \in \Omega_0$, $\|Q_M(\omega, r, u)\| \leq 1$ $\forall (r, u) \in [0, T] \times U$. We claim that for all $\omega \in \Omega_0$, $\|Q_M(\omega, \cdot, \cdot)\| = 1$, $\langle\langle M\rangle\rangle$-a.e. on $[0, T] \times U$.

Suppose that there is $0 < \beta < 1$ such that the set $C = \{(r, u) \in [0, T] \times U : \|Q_M(\omega, r, u)\| \leq \beta\}$ has positive $\langle\langle M\rangle\rangle$-measure. Let (x_n^*) be any dense sequence on the unitary sphere. Since by Theorem 5.10 $\langle\langle M\rangle\rangle(\omega) = \sup_{\|x_n^*\|=1} \nu_{x_n^*}(\omega) = \sup_{n\geq 1} \alpha_M(\omega)(x_n^*, x_n^*)$. We have, by (5.18) and (5.20) that

$$\begin{aligned}\alpha_M(\omega)(C)(x_n^*, x_n^*) &= \langle \Gamma_M(\omega)(C)x_n^*\,,\,x_n^*\rangle \\ &= \int_C \langle Q_M(\omega, r, u)x_n^*\,,\,x_n^*\rangle\,\langle\langle M\rangle\rangle(\omega)(dr, du) \leq \langle\langle M\rangle\rangle(C)\end{aligned}$$

Taking the supremum over n, we obtain $\langle\langle M\rangle\rangle(C) \leq \beta\,\langle\langle M\rangle\rangle(C)$, which is only possible if $\langle\langle M\rangle\rangle(C) = 0$. Therefore (iii) is satisfied. □

In many practical situations one can compute Q_M via the identity

$$\alpha_M(\omega)((s,t] \times A)(x^*, x^*) = \int_{(s,t]\times A} \langle Q_M(\omega, r, u)x^*\,,\,x^*\rangle\,\langle\langle M\rangle\rangle(\omega)(dr, du), \tag{5.21}$$

by calculating α_M and $\langle\langle M\rangle\rangle$ beforehand. This idea is explored in the following examples.

Example 5.24. Let $T > 0$ and let M denotes the cylindrical martingale-valued measure defined in Example 5.14. We have

$$\nu_{x^*}(\omega)(ds, du) = \sum_{k=1}^{n} \lambda_{\langle Z^k(\omega)(x^*)\rangle}(ds)\delta_{a_k}(du) = \sum_{k=1}^{n} (Q^k(x^*)x^*)ds\delta_{a_k}(du),$$

and

$$\langle\langle M\rangle\rangle(\omega)(ds, du) = \sum_{k=1}^{n} \left\|Q^k\right\| ds\delta_{a_k}(du).$$

Observe that (5.16) is satisfied, since

$$\sup_{A\in\mathcal{A}} \langle\langle M\rangle\rangle(\omega)([0,T] \times A) \leq T\sum_{k=1}^{n} \left\|Q^k\right\| < \infty.$$

Then there exists a process $Q_M : \Omega \times [0, T] \times U \to \mathcal{L}(X^*, X^{**})$ satisfying the conditions in Theorem 5.23.

Notice that by (5.13) we have

$$\alpha_M(\omega)((s,t]\times A)(x^*,x^*)=\tfrac{1}{4}\nu_{2x^*}(\omega)((s,t]\times A))=(t-s)\sum_{k=1}^{n}(Q^k(x^*)x^*)\delta_{a_k}(A).$$

Then by (5.21) we conclude

$$Q_M(\omega,r,u)=\sum_{k=1}^{n}\frac{Q^k}{\|Q^k\|}\mathbb{1}_{\{a_k\}}(u).$$

Example 5.25. Let $T>0$ and consider the cylindrical martingale-valued measure M defined by an H-valued Lévy process as given in Example 4.7. By Example 5.16 we have

$$\nu_h((s,t]\times A)(\omega)=(t-s)\left[(h,Qh)_H\delta_0(A)+\int_{A\setminus\{0\}}(u,h)_H^2\,\lambda(du)\right],$$

and the quadratic variation is given by

$$\langle\langle M\rangle\rangle\,((s,t]\times A)=(t-s)\left[\|Q\|\,\delta_0(A)+\int_{A\setminus\{0\}}\|u\|^2\,\lambda(du)\right].$$

Condition (5.16) is satisfied, since

$$\sup_{A\in\mathcal{A}}\langle\langle M\rangle\rangle\,(\omega)([0,T]\times A)\leq T\left[\|Q\|+\int_U\|u\|^2\,\lambda(du)\right]<\infty.$$

Then we can apply Theorem 5.23 to obtain the existence of the corresponding process $Q_M:\Omega\times[0,T]\times U\to\mathcal{L}(X^*,X^{**})$. In this case

$$\begin{aligned}\alpha_M(\omega)((s,t]\times A)(h,h)&=\tfrac{1}{4}\nu_{2h}(\omega)((s,t]\times A)\\&=(t-s)\left[(h,Qh)_H\delta_0(A)+\int_{A\setminus\{0\}}(u,h)_H^2\,\lambda(du)\right],\end{aligned}$$

Then by (5.21) we have

$$Q_M(\omega,r,u)=\frac{Q}{\|Q\|}\mathbb{1}_{\{0\}}(u)+P_u\mathbb{1}_{U\setminus\{0\}}(u),$$

where for $u\neq 0$, $P_u\in\mathcal{L}(H)$ is given by

$$P_u(h)=\frac{(u,h)_H}{\|u\|^2}u,\quad\forall h\in H.$$

Example 5.26. Let H be a separable Hilbert space. In [1] they consider cylindrical martingale-valued measures for which the following properties are satisfied:

(i) For $0\leq s<t$, $(M(t,A)-M(s,A))(h)$ is independent of $\mathcal{F}_s$, for all $A\in\mathcal{A}$, $h\in H$.

(ii) For each $A \in \mathcal{A}$ and $0 \leq s < t$,

$$\mathbb{E}\left[|(M(t,A)-M(s,A))(h)|^2\right] = \int_s^t \int_A q_{r,u}(h)^2 \gamma(du) m(dr), \quad \forall\, h \in H.$$

where

(a) m is a σ-finite measure on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$, finite on bounded intervals,

(b) γ is a σ-finite measure on $(U, \mathcal{B}(U))$ satisfying $\gamma(A) < \infty$, $\forall\, A \in \mathcal{A}$,

(c) $\{q_{r,u} : r \in \mathbb{R}_+,\, u \in U\}$ is a family of continuous Hilbertian semi-norms on H, such that for each h_1, h_2 in H, the map $(r,u) \mapsto q_{r,u}(h_1,h_2)$ is $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(U)/\mathcal{B}(\mathbb{R}_+)$-measurable and bounded on $[0,T] \times U$ for all $T > 0$. Here, $q_{r,u}(\cdot,\cdot)$ denotes the continuous positive, symmetric, bilinear form on $H \times H$ associated to $q_{r,u}(\cdot)$.

One can infer from (i) and (ii) above that the family of intensity measures of M is of the form:

$$\nu_h(\omega)(C) = \int_C q_{r,u}(h)^2 (m \otimes \gamma)(dr,du), \quad \forall\, C \in \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(U),$$

We shall prove that the family (ν_h) satisfies the sequential boundedness property. By Lemma 3.5 in [1] for any given $T > 0$ there exists $K = K(T) > 0$ such that

$$q_{r,u}(\cdot) \leq K \left\|\cdot\right\|, \quad \forall (r,u) \in [0,T] \times U. \tag{5.22}$$

Let (h_n) be a sequence on the unit sphere such that $h_n \to h$.

$$\begin{aligned} |\nu_{h_n}(\omega)(C) - \nu_h(\omega)(C)| &\leq \int_C \left| q_{r,u}(h)^2 - q_{r,u}(h_n)^2 \right| (m \otimes \gamma)(dr,du) \\ &\leq 2K^2 \left\| h_n - h \right\| (m \otimes \gamma)(C) \to 0. \end{aligned}$$

It follows that

$$\nu_h(\omega)(C) = \lim \nu_{h_n}(\omega)(C) \leq \sup_{n \geq 1} \nu_{h_n}(\omega)(C).$$

By Remark 5.6 we conclude that (ν_h) satisfies the sequential boundedness property.

Now we prove that M has a quadratic variation. First notice that for all $C \in \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(U)$,

$$\nu_h(\omega)(C) \leq K^2 \int_C \left\| h \right\|^2 (m \otimes \gamma)(dr,du) = K^2 \left\| h \right\|^2 (m \otimes \gamma)(C).$$

We can construct a measure that dominates each ν_h for $\|h\| = 1$. In fact, for each $n \in \mathbb{N}$ let K_n with the property (5.22). Define a measure κ on $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(U)$ by the prescription

$$\kappa(C) = \sum_{n \in \mathbb{N}} K_n (m \otimes \gamma)(C \cap ([n-1,n) \times U)).$$

Observe that for $t > 0$ and $A \in \mathcal{A}$, if $n-1 \leq t < n$ then

$$\kappa([0,t]\times A) \leq \left(\max_{1\leq k\leq n} K_n \right) m([0,t])\gamma(A) < \infty.$$

Moreover, from the calculations above we have $\nu_h \leq \kappa$ on $\mathcal{B}(\mathbb{R}_+)\otimes\mathcal{B}(U)$ $\forall \, \|h\|=1$. Therefore

$$\sup_{n\geq 1} \nu_{h_n} \leq \sup_{\|h\|=1} \nu_h \leq \kappa.$$

Then, by Theorem 5.8, M has a unique quadratic variation $\langle\!\langle M \rangle\!\rangle = \sup_{n\geq 1} \nu_{h_n}$. Furthermore by Lemma 2.2, we have

$$\langle\!\langle M \rangle\!\rangle (C) = \sup_{n\geq 1} \nu_{h_n}(C) = \int_C \sup_{n\geq 1} q_{r,u}(h_n)^2 (m\otimes\gamma)(dr,du),$$

for all $C \in \mathcal{B}(\mathbb{R}_+)\otimes\mathcal{B}(U)$. If we further assume that

$$\int_0^T \int_U \sup_n q_{r,u}(h_n)^2 \gamma(du) m(dr) < \infty,$$

(this holds true if, for example, $\gamma(U) < \infty$ since by (5.22) we have $\mu_M(\Omega\times[0,T]\times U) \leq Km([0,T])\gamma(U)$) then (5.16) is satisfied, and hence the conditions in Theorem 5.23. Our next goal is to calculate the process $Q_M : \Omega\times[0,T]\times U \to \mathcal{L}(H,H)$; to do this, notice that by definition we have

$$\alpha_M(\omega)((s,t]\times A)(h,h) = \tfrac{1}{4}\nu_{2h}((s,t]\times A) = \int_s^t \int_A q_{r,u}(h)^2 \gamma(du) m(dr).$$

Then by (5.21) we conclude that $Q_M(\omega,r,u) \in \mathcal{L}(H,H)$ is defined via the identity

$$(Q_M(\omega,r,u)h,g)_H = \frac{q_{r,u}(h,g)}{\|q_{r,u}(\cdot,\cdot)\|_{\mathfrak{Bil}(H,H)}}, \quad \forall h,g \in H,$$

where $q_{r,u}(\cdot,\cdot)$ denotes the continuous positive, symmetric, bilinear form on $H\times H$ associated to the continuous Hilbertian semi-norm $q_{r,u}$.

Remark 5.27. The reader can easily check from Example 5.16 that the cylindrical martingale valued-measure induced by an H-valued Lévy process (Example 4.7) is a particular case of the cylindrical martingale-valued measures introduced in Example 5.26. Indeed, with the notation introduced above we can take m as the Lebesgue measure on $(\mathbb{R}_+,\mathcal{B}(\mathbb{R}_+))$, $\gamma = \delta_0 + \lambda\big|_U$ for λ the Lévy measure of L, and the family $\{q_{r,u} : r\in\mathbb{R}_+,\, u\in U\}$ is given by

$$q_{r,u}(h)^2 = \begin{cases} (h,Qh)_H, & \text{if } u=0, \\ (u,h)_H^2, & \text{if } u\in U\setminus\{0\}. \end{cases}$$

Other examples of cylindrical martingale-valued measures with the conditions given in Example 5.26 can be found in Section 3 in [1].

6. (Cylindrical) quadratic variation for Hilbert space-valued martingale-valued measures

6.1. **Existence of (cylindrical) quadratic variation.** Let H be a separable Hilbert space. Let $\tilde{M} = (\tilde{M}(t,A) : t \geq 0, A \in \mathcal{A})$ be an H-valued orthogonal martingale-valued measure. Let $\tilde{\nu}$ be the intensity measure of $\tilde{M}$, according to Definition 3.12.

Let $M = (M(t,A) : t \geq 0, A \in \mathcal{A})$ be the cylindrical orthogonal martingale-valued measure on H corresponding to $\tilde{M}$ as in Proposition 4.6. We recall that it is necessary to assume that for every $A, B \in \mathcal{A}$ disjoint and $h \in H$, the real-valued martingales $(\tilde{M}(A), h)_H$ and $(\tilde{M}(B), h)_H$ are orthogonal.

By (2.2) for every $A \in \mathcal{A}$ and $h \in H$ we have $\mathbb{P}$-a.e.

$$\langle M(A)(h) \rangle_t \leq \|h\|^2 \langle \tilde{M}(A) \rangle_t \quad \forall t \geq 0.$$

When we replace the quadratic variations with the intensity measures, we can do better, thanks to Lemma 3.9. We write the result explicitly.

Lemma 6.1. *Let ν_h be the intensity measure of $M(h) = (\tilde{M}, h)_H$. For $\|h\| = 1$ we have $\nu_h \leq \tilde{\nu}$. More precisely*

$$\mathbb{P}\left(\nu_h\left(C\right) \leq \tilde{\nu}(C),\, \forall C \in \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(U)\right) = 1.$$

The existence of the quadratic variation of M in this case is guaranteed by the sequential boundedness property.

Theorem 6.2. *Assume the family $(\nu_h : h \in H)$ satisfies the sequential boundedness property. Then M has a unique quadratic variation and $\langle\langle M \rangle\rangle \leq \tilde{\nu}$*

Proof. Let $(g_n)_{n\geq 1}$ be a dense subset of the unit sphere of H. By Lemma 6.1, $\nu_{g_n} \leq \tilde{\nu}$ for all $n \in \mathbb{N}$. Hence $\mu := \sup_{n\geq 1} \nu_{g_n} \leq \tilde{\nu}$ and in particular, for each $t \geq 0$ and $A \in \mathcal{A}$, $\mu([0,t] \times A) \leq \tilde{\nu}([0,t] \times A) < \infty$ $\mathbb{P}$-a.e. By Theorem 5.8, M has the unique quadratic variation $\langle\langle M \rangle\rangle = \mu \leq \tilde{\nu}$. □

Our next goal is to show that $\tilde{\nu}$ has an alternative expression as a random series of intensity measures along an orthonormal basis in H (see Proposition 6.3 below).

Let $(h_n)_{n\geq 1}$ be a orthonormal basis in H and denote $\nu_n = \nu_{h_n}$. For every $N \geq 1$ let $\rho_N = \sum_{n=1}^{N} \nu_n$. Each ρ_N is a random predictable σ-finite measure on $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(U)$. Since the family $(\rho_N)_{N\geq 1}$ is increasing we have

$$\rho := \sup_{n\geq 1} \rho_n = \lim_{N\to\infty} \left(\sup_{1\leq n\leq N} \rho_n \right) = \lim_{N\to\infty} \rho_N = \sum_{n=1}^{\infty} \nu_n. \tag{6.1}$$

We conclude that ρ is a random predictable σ-finite measure on $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(U)$.

Proposition 6.3. *Let $\tilde{\nu}$ and ν_n be the respective intensity measures of $\tilde{M}$ and $M(h_n) = (\tilde{M}, h_n)_H$. As random measures on $\mathcal{B}(\mathbb{R}^+) \otimes \mathcal{B}(U)$ we have*

$$\tilde{\nu} = \sum_{n=1}^{\infty} \nu_n.$$

Proof. By uniqueness of the intensity measure, it is enough to prove that

$$\left(\|\tilde{M}(t,A)\|^2 - \rho\left([0,t] \times A\right) : t \geq 0 \right)$$

is a martingale for any $A \in \mathcal{A}$. For $0 \leq s < t$ and $A \in \mathcal{A}$ we have by (6.1) that

$$\begin{aligned}
\mathbb{E}\left[\|\tilde{M}(t,A)\|^2 - \rho\left([0,t]\times A\right) \Big| \mathcal{F}_s \right] &= \sum_{n=1}^{\infty} \mathbb{E}\left[(\tilde{M}(t,A), h_n)_H^2 - \nu_n\left([0,t]\times A\right) \Big| \mathcal{F}_s \right] \\
&= \sum_{n=1}^{\infty} \left[(\tilde{M}(s,A), h_n)_H^2 - \nu_n\left([0,s]\times A\right) \right] \\
&= \|\tilde{M}(s,A)\|^2 - \rho\left([0,s]\times A\right).
\end{aligned}$$

□

Example 6.4. Let $L = (L_t : t \geq 0)$ be an H-valued càdlàg Lévy process with corresponding H-valued martingale-valued measure $\tilde{M}$ as in Example 3.14 and cylindrical martingale-valued measure M as in Example 4.7. Given $0 \leq s < t$ and $A \in \mathcal{A}$ we have by (5.10) and Proposition 6.3 that

$$\begin{aligned}
\nu((s,t]\times A)(\omega) &= \sum_{n=1}^{\infty} \nu_{h_n}((s,t]\times A)(\omega) \\
&= (t-s) \sum_{n=1}^{\infty} \left[(h_n, Qh_n)_H \delta_0(A) + \int_{A\setminus\{0\}} (u, h_n)_H^2 \, \nu(du) \right] \\
&= (t-s) \left[\|Q\|_{L_1(H)} \, \delta_0(A) + \int_{A\setminus\{0\}} \|u\|^2 \, \nu(du) \right].
\end{aligned}$$

Here $L_1(H)$ denotes the space of trace-class operators on H equipped with the trace-norm. Observe that by (5.12) it follows $\langle\!\langle M \rangle\!\rangle < \nu$.

6.2. **The (cylindrical) quadratic variation.** In what follows, we assume that $(\nu_h : h \in H)$ satisfies the sequential boundedness property. By Theorems 6.2 and 6.3, $\langle\!\langle M \rangle\!\rangle$ exists and

$$\langle\!\langle M \rangle\!\rangle \leq \tilde{\nu} = \sum_{n=1}^{\infty} \nu_n.$$

Now we explore the existence of the α_M, Γ_M and Q_M for M.

Proposition 6.5. *The quadratic variation $\langle\!\langle M \rangle\!\rangle$ is integrable.*

Proof. For $A \in \mathcal{A}$ we have

$$\mathbb{E}\left[\langle\!\langle M \rangle\!\rangle([0,T] \times A)\right] \leq \mathbb{E}\left[\tilde{\nu}([0,T] \times A)\right] = \mathbb{E}[\langle \tilde{M}(A) \rangle_T]$$

and this is finite thanks to Lemma 3.5. □

Proposition 6.6. *If $\tilde{\nu}$ is $\mathbb{P}$-a.e. finite, then α_M, Γ_M and Q_M exist.*

Proof. In fact

$$\sup_{A \in \mathcal{A}} \langle\!\langle M \rangle\!\rangle(\omega)([0,T] \times A) \leq \sup_{A \in \mathcal{A}} \tilde{\nu}([0,T] \times A) < \infty \quad \mathbb{P}\text{-a.e.}$$

The existence of α_M, Γ_M and Q_M now follows by Theorems 5.22 and 5.23. □

Now, notice that by (5.19), for $C \in \mathcal{B}([0,T]) \otimes \mathcal{B}(U)$ and $\|h\| = \|g\| = 1$ we have, $\mathbb{P}$-a.e.

$$\alpha_M(C)(h,g) = \langle \Gamma_M(C)h, g \rangle \leq \langle\!\langle M \rangle\!\rangle(C)$$

We conclude that, for $\|h\| = \|g\| = 1$,

$$\alpha_M(\cdot)(h,g) \leq \langle\!\langle M \rangle\!\rangle \leq \tilde{\nu}.$$

Theorem 6.7. *As random measures on $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(U)$, $\langle\!\langle M \rangle\!\rangle$ and $\tilde{\nu}$ are equivalent and*

$$\frac{d\tilde{\nu}}{d\langle\!\langle M \rangle\!\rangle} = \operatorname{Tr} Q_M \quad \mathbb{P}\text{-a.e.}$$

Proof. Since $\langle\!\langle M \rangle\!\rangle \leq \tilde{\nu}$, we have in particular $\langle\!\langle M \rangle\!\rangle \ll \tilde{\nu}$ $\mathbb{P}$-a.e.

Let $(h_n : n \geq 1)$ be an orthonormal basis. Given $s < t \leq T$ and $A \in \mathcal{A}$ we have, $\mathbb{P}$-a.e.

$$\alpha_M((s,t] \times A)(h_n, h_n) = \nu_n((s,t] \times A)$$

and therefore

$$\sum_{n=1}^{\infty} \alpha_M((s,t]\times A)(h_n,h_n) = \sum_{n=1}^{\infty} \nu_n((s,t]\times A) = \tilde{\nu}((s,t]\times A) \quad \mathbb{P}\text{-a.e.}$$

The left-hand side also coincides, $\mathbb{P}$-a.e. with

$$\int_{(s,t]\times A} \sum_{n=1}^{\infty} \langle Q_M h_n, h_n\rangle \, d\,\langle\!\langle M\rangle\!\rangle = \int_{(s,t]\times A} \operatorname{Tr} Q_M \; d\,\langle\!\langle M\rangle\!\rangle\,.$$

We conclude that, $\mathbb{P}$-a.e.

$$\tilde{\nu}((s,t]\times A) = \int_{(s,t]\times A} \operatorname{Tr} Q_M \; d\,\langle\!\langle M\rangle\!\rangle\,. \tag{6.2}$$

Thanks to Lemma 3.9, this identity extends to

$$\forall C\in\mathcal{B}([0,T])\otimes\mathcal{B}(U) \quad \tilde{\nu}(C) = \int_C \operatorname{Tr} Q_M \; d\,\langle\!\langle M\rangle\!\rangle\,,$$

valid $\mathbb{P}$-a.e. This finishes the proof. □

7. Quadratic Covariation For Cylindrical Martingale-Valued Mesures

In analogy with Section 5.2, we introduce a quadratic covariation operator measure for two cylindrical orthogonal martingale-valued measures. This object is used to define a (positive) quadratic covariation which mimics the construction of quadratic variation in Section 5.1. Then we show that a Radon-Nikodym representation holds. For this, we must first take care of some technical issues concerning the way these measures are related.

7.1. **Mutually orthogonal martingale-valued measures.** The main goal of this subsection is to define addition and subtraction for cylindrical martingale-valued measures. In order for our definition to work, the martingale measures must be mutually orthogonal in the following sense.

Definition 7.1. We say that two cylindrical orthogonal martingale-valued measures M and N are *mutually orthogonal* if for any $x^*\in X^*, t\geq 0, A,B\in\mathcal{A}$ disjoint, we have

$$\mathbb{E}[M(t,A)(x^*)\cdot N(t,B)(x^*)] = 0.$$

We say that M and N are *compatible* if there is an increasing sequence $(U_n)_{n\in\mathbb{N}}$ in $\mathcal{B}(U)$ such that (iii) in Definition 3.1 is satisfied for both M and N simultaneously.

Let M and N be cylindrical orthogonal martingale-valued measures; define the collection of cylindrical random variables $M+N = ((M+N)(t,A) : t\geq 0, A\in\mathcal{A})$ by $(M+N)(t,A) = M(t,A)+N(t,A)$.

Lemma 7.2. *Let M and N be two mutually orthogonal and compatible cylindrical orthogonal martingale-valued measures, then $M+N$ is a cylindrical orthogonal martingale-valued measure.*

Proof. Properties (i) and (ii) of Definition 4.1 are clear from the definition. To see (iii), let $t \geq 0$ and $x^* \in X^*$. For every $A \in \mathcal{A}$ we have

$$\mathbb{E}[|(M+N)(t,A)(x^*)|^2] \leq 2\mathbb{E}[|M(t,A)(x^*)|^2] + 2\mathbb{E}[|N(t,A)(x^*)|^2] < \infty, \tag{7.1}$$

and since both $M(t,\cdot)(x^*)$ and $N(t,\cdot)(x^*)$ are finitely additive on $\mathcal{A}$, so is $(M+N)(t,\cdot)(x^*)$ by definition. We only need to check that for such t and x^*, the mapping $(M+N)(t,\cdot)(x^*) : \Omega \times \mathcal{A} \to \mathbb{R}$ is σ-finite. For this, let $(U_n)_{n\geq 1}$ be the family of sets corresponding to the definition of compatibility for M and N. For $A \in \mathcal{B}(U_n)$, by (7.1) we obtain

$$\sup_{A\in\mathcal{B}(U_n)} \mathbb{E}[|(M+N)(t,A)(x^*)|^2] < \infty,$$

and by considering a sequence $(A_j)_{j\geq 1}$ in $\mathcal{B}(U_n)$ that decreases to $\emptyset$, the same inequality and the countable additivity of $M(t,\cdot)(x^*)$ and $N(t,\cdot)(x^*)$ on $\mathcal{B}(U_n)$ imply that $(M+N)(t,\cdot)(x^*)$ is also countably additive on each $\mathcal{B}(U_n)$.

Finally, if $M(t,A)(x^*)$ and $N(t,A)(x^*)$ are the L^2-limits of $M(t,A\cap U_n)(x^*)$ and $N(t,A\cap U_n)(x^*)$ respectively, it is clear that $(M+N)(t,A)(x^*)$ is the L^2-limit of $(M+N)(t,A\cap U_n)(x^*)$.

To see (iv) of Definition 4.1, let $x^* \in X^*$ and choose any $A, B \in \mathcal{A}$ with $A\cap B = \emptyset$. Since M and N are both orthogonal measures, and by the mutual orthogonality, we have for $t \geq 0$

$$\begin{aligned}
&\langle (M+N)(A)(x^*)\,,\,(M+N)(B)(x^*)\rangle_t \\
&\qquad = \langle M(A)(x^*)\,,\,M(B)(x^*)\rangle_t + \langle M(A)(x^*)\,,\,N(B)(x^*)\rangle_t \\
&\qquad\quad + \langle N(A)(x^*)\,,\,M(B)(x^*)\rangle_t + \langle N(A)(x^*)\,,\,N(B)(x^*)\rangle_t = 0.
\end{aligned}$$

□

Theorem 7.3. *Let M and N be two mutually orthogonal and compatible cylindrical orthogonal martingale-valued measures that have a unique quadratic variation. If $M+N$ has a quadratic variation η, then $\eta \leq 2\langle\langle M\rangle\rangle + 2\langle\langle N\rangle\rangle$.*

If we further assume that the family of intensity measures for $M+N$ has the sequential boundedness property, then the unique quadratic variation of $M+N$ satisfies

$$\langle\langle M+N\rangle\rangle \leq 2\langle\langle M\rangle\rangle + 2\langle\langle N\rangle\rangle. \tag{7.2}$$

Proof. For any $x^* \in X^*$, denote respectively by $\nu^M_{x^*}$, $\nu^N_{x^*}$ and $\nu^{M+N}_{x^*}$ the intensity measures of M, N and $M+N$ at x^*. Observe that for any $t>0$ and $A \in \mathcal{A}$ we have $\mathbb{P}$-a.e.

$$\begin{aligned}
\nu^{M+N}_{x^*}([0,t]\times A) &= \langle (M+N)(A)(x^*)\rangle_t \\
&\leq 2\langle M(A)(x^*)\rangle_t + 2\langle N(A)(x^*)\rangle_t \\
&= 2\,\nu^M_{x^*}([0,t]\times A) + 2\,\nu^N_{x^*}([0,t]\times A).
\end{aligned}$$

By Lemma 3.9 we conclude that $\nu^{M+N}_{x^*} \leq 2\nu^M_{x^*} + 2\nu^N_{x^*}$ and therefore, by Definition 5.3

$$\nu^{M+N}_{x^*} \leq 2\langle\langle M\rangle\rangle + 2\langle\langle N\rangle\rangle. \tag{7.3}$$

Suppose that η is a quadratic variation for $M+N$. By Definition 5.3(ii) and (7.3) we have $\eta \leq 2\langle\langle M\rangle\rangle + 2\langle\langle N\rangle\rangle$. Finally, if the family of intensity measures for $M+N$ has the sequential boundedness property, by Theorem 5.10, $M+N$ has a unique quadratic variation. □

7.2. **Construction of the quadratic covariation operator measure.** The following assumptions will be required throughout this section,

Assumption 7.4.

- (i) M and N are two mutually orthogonal and compatible cylindrical orthogonal martingale-valued measures.
- (ii) The families of intensity measures for M, N, $M+N$ and $M-N$ satisfy the sequential boundedness property.
- (iii) M and N have quadratic variation. Moreover, the unique (Theorem 5.10) quadratic variations for M and N satisfy (5.16).

Several interesting conclusions arise from the assumptions above. First, Theorem 7.3 implies that $M+N$ and $M-N$ have a unique quadratic variation. Moreover, by (7.3) we have that the quadratic variations $\langle\langle M+N\rangle\rangle$ and $\langle\langle M-N\rangle\rangle$ satisfy (5.16). Therefore, Theorem 5.22 shows the existence of two random $\mathfrak{Bil}(X^*, X^*)$-valued measures α_{M+N} and α_{M-N}, defined on $\mathcal{B}([0,T]) \otimes \mathcal{B}(U)$, such that for $x^*, y^* \in X^*$, $0 \leq s \leq t \leq T$, and $A \in \mathcal{A}$, $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$\alpha_{M+N}(\omega)((s,t]\times A)(x^*,y^*) = \langle (M+N)(A)(x^*), (M+N)(A)(y^*)\rangle^t_s(\omega), \tag{7.4}$$

and

$$\alpha_{M-N}(\omega)((s,t]\times A)(x^*,y^*) = \langle (M-N)(A)(x^*), (M-N)(A)(y^*)\rangle^t_s(\omega). \tag{7.5}$$

Definition 7.5. We define a random $\mathfrak{Bil}(X^*, X^*)$-valued measure $\alpha_{M,N}$ on $\mathcal{B}([0,T])\otimes\mathcal{B}(U)$ by

$$\alpha_{M,N}(\omega)(C) := \tfrac{1}{4}\left[\alpha_{M+N}(\omega)(C) - \alpha_{M-N}(\omega)(C)\right], \quad C \in \mathcal{B}([0,T])\otimes\mathcal{B}(U). \tag{7.6}$$

Proposition 7.6. *Let $x^*, y^* \in X^*$, $0 \le s \le t \le T$, and $A \in \mathcal{A}$. For $\mathbb{P}$-a.e. $\omega \in \Omega$ we have,*

$$\begin{aligned}&\alpha_{M,N}(\omega)((s,t]\times A)(x^*, y^*)\\&= \tfrac{1}{2}\Big[\langle M(A)(x^*), N(A)(y^*)\rangle_s^t(\omega) + \langle N(A)(x^*), M(A)(y^*)\rangle_s^t(\omega)\Big].\end{aligned}$$

In particular,

$$\alpha_{M,N}(\omega)((s,t]\times A)(x^*, x^*) = \langle M(A)(x^*), N(A)(x^*)\rangle_s^t(\omega).$$

Proof. By (7.4), (7.5) and (7.6), and using that the covariation for real-valued martingales is bilinear, we have $\mathbb{P}$-a.e.

$$\begin{aligned}&\alpha_{M,N}((s,t]\times A)(x^*, y^*)\\&= \tfrac{1}{4}\Big[\langle (M+N)(A)(x^*), (M+N)(A)(y^*)\rangle_s^t - \langle (M-N)(A)(x^*), (M-N)(A)(y^*)\rangle_s^t\Big]\\&= \tfrac{1}{2}\Big[\langle M(A)(x^*), N(A)(y^*)\rangle_s^t + \langle N(A)(x^*), M(A)(y^*)\rangle_s^t\Big].\end{aligned}$$

Since $\langle M(A)(x^*), N(A)(x^*)\rangle_s^t = \langle N(A)(x^*), M(A)(x^*)\rangle_s^t$ $\mathbb{P}$-a.e., we get the last assertion. □

Corollary 7.7. *As random $\mathfrak{Bil}(X^*, X^*)$-valued measures on $\mathcal{B}([0,T])\otimes\mathcal{B}(U)$, $\alpha_{M,M} = \alpha_M$, $\alpha_{M,N} = \alpha_{N,M}$, and*

$$\alpha_{M+N} = \alpha_M + \alpha_N + 2\alpha_{M,N}, \quad \alpha_{M-N} = \alpha_M + \alpha_N - 2\alpha_{M,N}. \tag{7.7}$$

Proof. Let $x^*, y^* \in X^*$. Given $0 \le s < t \le T$ and $A \in \mathcal{A}$, by Theorem 5.22 and Proposition 7.6 we have $\mathbb{P}$-a.e. $\alpha_{M,M}(\omega)((s,t]\times A)(x^*, y^*) = \alpha_M(\omega)((s,t]\times A)(x^*, y^*)$ and $\alpha_{M,N}(\omega)((s,t]\times A)(x^*, y^*) = \alpha_{N,M}(\omega)((s,t]\times A)(x^*, y^*)$. Then by Lemma 3.11 we have $\mathbb{P}$-a.e. $\alpha_{M,M}(\omega)(C)(x^*, y^*) = \alpha_M(\omega)(C)(x^*, y^*)$ and $\alpha_{M,N}(\omega)(C)(x^*, y^*) = \alpha_{N,M}(\omega)(C)(x^*, y^*)$ for every $C \in \mathcal{B}([0,T])\otimes\mathcal{B}(U)$.

To remove the dependence on $x^*, y^* \in X^*$, we proceed as in the proof of Theorem 5.18. Fix $(x_n^*)_{n\ge1} \subseteq X^*$ a set of linearly independent vectors such that $\operatorname{span}(x_n^*)_{n\ge1}$ is dense in X^*. Let $F = (y_m^*)_{m\ge1}$ be the $\mathbb{Q}$-span of $(x_n^*)_{n\ge1}$. One can construct a subset Ω_0 of Ω with probability one such that for every $\omega \in \Omega_0$ we have $\alpha_{M,M}(\omega)(C)(x^*, y^*) = \alpha_M(\omega)(C)(x^*, y^*)$ and $\alpha_{M,N}(\omega)(C)(x^*, y^*) = \alpha_{N,M}(\omega)(C)(x^*, y^*)$ for every $C \in \mathcal{B}([0,T])\otimes\mathcal{B}(U)$, $x^*, y^* \in F$. Since $\alpha_{M,M}(\omega)(C)$, $\alpha_M(\omega)(C)$, $\alpha_{M,N}(\omega)(C)$ and $\alpha_{N,M}(\omega)(C)$ are all elements in $\mathfrak{Bil}(X^*, X^*)$, by a standard density argument we conclude that for every $\omega \in \Omega_0$, $\alpha_{M,M}(\omega)(C) = \alpha_M(\omega)(C)$ and $\alpha_{M,N}(\omega)(C) = \alpha_{N,M}(\omega)(C)$ with equality in $\mathfrak{Bil}(X^*, X^*)$.

By a similar reasoning, it is enough to show (7.7) for rectangles. We use Theorem 5.18, Proposition 7.6 and the identities (7.4) and (7.5). In fact, given $x^*, y^* \in X^*$, $0 \leq s < t \leq T$ and $A \in \mathcal{A}$, we have $\mathbb{P}$-a.e.

$$\begin{aligned}
&\alpha_M(\omega)((s,t] \times A)(x^*, y^*) + \alpha_N(\omega)((s,t] \times A)(x^*, y^*) + 2\alpha_{M,N}(\omega)((s,t] \times A)(x^*, y^*) \\
&= \langle M(A)(x^*), M(A)(y^*) \rangle_s^t (\omega) + \langle M(A)(x^*), N(A)(y^*) \rangle_s^t (\omega) \\
&\ + \langle N(A)(x^*), M(A)(y^*) \rangle_s^t (\omega) + \langle N(A)(x^*), N(A)(y^*) \rangle_s^t (\omega) \\
&= \alpha_{M+N}(\omega)((s,t] \times A)(x^*, y^*).
\end{aligned}$$

The other identity in (7.7) is similar. □

Now we prove a Cauchy-Schwarz inequality that relates $\alpha_{M,N}(C)$, $\alpha_M(C)$ and $\alpha_N(C)$. But first, we need the following result.

Lemma 7.8. *For every $x^*, y^* \in X^*$, as random measures on $\mathcal{B}([0,T]) \otimes \mathcal{B}(U)$, we have $\alpha_M(\cdot)(x^*, y^*) = \alpha_M(\cdot)(y^*, x^*)$, $\alpha_N(\cdot)(x^*, y^*) = \alpha_N(\cdot)(y^*, x^*)$, and $\alpha_{M,N}(\cdot)(x^*, y^*) = \alpha_{M,N}(\cdot)(y^*, x^*)$.*

Proof. Follows from similar arguments to those used in the proof of Corollary 7.7. □

Proposition 7.9. $\mathbb{P}$*-a.e. for every $C \in \mathcal{B}([0,T]) \otimes \mathcal{B}(U)$,*

$$\|\alpha_{M,N}(C)\|_{\mathfrak{Bil}(X^*,X^*)} \leq \sqrt{\|\alpha_M(C)\|_{\mathfrak{Bil}(X^*,X^*)}} \cdot \sqrt{\|\alpha_N(C)\|_{\mathfrak{Bil}(X^*,X^*)}}. \tag{7.8}$$

Proof. Our first goal is to show $\mathbb{P}$-a.e. for every $C \in \mathcal{B}([0,T]) \otimes \mathcal{B}(U)$, $x^* \in X^*$,

$$|\alpha_{M,N}(C)(x^*, x^*)| \leq \sqrt{\alpha_M(C)(x^*, x^*)} \cdot \sqrt{\alpha_N(C)(x^*, x^*)}. \tag{7.9}$$

First, by the properties in Corollary 7.7 we have

$$\begin{aligned}
&\alpha_{M+rN}(\omega)(C)(x^*, y^*) \\
&= \alpha_{M+rN,M+rN}(\omega)(C)(x^*, y^*) \\
&= \alpha_{M,M}(\omega)(C)(x^*, y^*) + r^2\alpha_{N,N}(\omega)(C)(x^*, y^*) + 2r\alpha_{M,N}(\omega)(C)(x^*, y^*) \\
&= \alpha_M(\omega)(C)(x^*, y^*) + r^2\alpha_N(\omega)(C)(x^*, y^*) + 2r\alpha_{M,N}(\omega)(C)(x^*, y^*).
\end{aligned}$$

Then for $x^* = y^*$, we have

$$0 \leq \alpha_M(\omega)(C)(x^*, x^*) + r^2\alpha_N(\omega)(C)(x^*, x^*) + 2r\alpha_{M,N}(\omega)(C)(x^*, x^*).$$

By considering the discriminant of the above quadratic equation we conclude that (7.9) holds true.

Our next objective shall be to show that $\mathbb{P}$-a.e. for every $C \in \mathcal{B}([0,T]) \otimes \mathcal{B}(U)$

$$\|\alpha_{M,N}(C)\|_{\mathfrak{Bil}(X^*,X^*)} = \sup_{\|x^*\|=1} |\alpha_{M,N}(C)(x^*,x^*)| \,. \tag{7.10}$$

Clearly we have $|\alpha_{M,N}(C)(x^*,x^*)| \leq \|\alpha_{M,N}(C)\|_{\mathfrak{Bil}(X^*,X^*)}$ for $\|x^*\| = 1$. For the converse inequality, let $K = \sup\{|\alpha_{M,N}(C)(x^*,x^*)| : \|x^*\| = 1\}$. We must show that $|\alpha_{M,N}(C)(x^*,y^*)| \leq K$ for all $\|x^*\| = \|y^*\| = 1$.

By Lemma 7.8 we have

$$\alpha_{M,N}(C)(x^*,y^*) = \tfrac{1}{4}\left[\alpha_{M,N}(C)(x^*+y^*,x^*+y^*) - \alpha_{M,N}(C)(x^*-y^*,x^*-y^*)\right].$$

But then,

$$\begin{aligned} |\alpha_{M,N}(C)(x^*,y^*)| &\leq \tfrac{1}{4}\max\{|\alpha_{M,N}(C)(x^*+y^*,x^*+y^*)|\,, |\alpha_{M,N}(C)(x^*-y^*,x^*-y^*)|\} \\ &\leq \tfrac{1}{4}\max\{K\,\|x^*+y^*\|^2\,, K\,\|x^*-y^*\|^2\} \\ &\leq \tfrac{K}{4}(\|x\|+\|y\|)^2 = K. \end{aligned}$$

This shows (7.10). In a similar way we can show, by Lemma 7.8, that

$$\|\alpha_{M}(C)\|_{\mathfrak{Bil}(X^*,X^*)} = \sup_{\|x^*\|=1} |\alpha_{M}(C)(x^*,x^*)| \,,$$
$$\|\alpha_{N}(C)\|_{\mathfrak{Bil}(X^*,X^*)} = \sup_{\|x^*\|=1} |\alpha_{N}(C)(x^*,x^*)| \,.$$

Finally, if we take $\sup_{\|x^*\|=1}$ in (7.9) we obtain (7.8). □

Definition 7.10. We say that M and N *have a (positive) quadratic covariation*, if there is a random measure $\eta : \Omega \to \mathcal{M}([0,T] \times U, \mathcal{B}([0,T] \otimes \mathcal{B}(U)))$ such that

(i) Given $0 \leq s < t \leq T$, $A \in \mathcal{A}$, we have $\eta((s,t] \times A) < \infty$ $\mathbb{P}$-a.e.

(ii) η is a minimal element (in the partial order "$\leq$") for the collection of all the random measures $\zeta : \Omega \to \mathcal{M}_+(\mathbb{R}_+ \times U, \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(U))$ with the property: $\forall\, x^*, y^* \in X^*$ with $\|x^*\| = \|y^*\| = 1$, $|\alpha_{M,N}(\cdot)(x^*,y^*)| \leq \zeta$.

Proposition 7.11. *Assume that M and N have a (positive) quadratic covariation η. Let $(x_n)_{n\geq 1}$ be a dense subset of the unit sphere in X^* and let $\mu := \sup\limits_{m,n\geq 1} |\alpha_{M,N}(\cdot)(x^*_m, x^*_n)|$. Then $\mu \leq \eta$.*

Proof. Basically, it is the same proof of Proposition 5.4. Indeed, since η is a quadratic covariation for M and N, we see that for every $m, n \geq 1$ there exists $\Omega_{m,n} \subseteq \Omega$, with $\mathbb{P}(\Omega_{m,n}) = 1$ and $|\alpha_{M,N}(\cdot)(x_m^*, x_n^*)| \leq \eta$. Now, Lemma 2.1 guarantees that μ is a measure. Then, by the definition of supremum of measures, it follows that

$$\mu(\omega) \leq \eta(\omega), \qquad \forall \omega \in \Omega_0,$$

where $\Omega_0 = \bigcap_{m,n=1} \Omega_{m,n}$, which is a set of probability one. □

Theorem 7.12. *Let $(x_n^*)_{n\geq 1}$ be a dense subset of the unit sphere in X^* and let $\mu = \sup_{m,n} |\alpha_{M,N}(\cdot)(x_m^*, x_n^*)|$. Then μ is a (positive) quadratic covariation for M and N. In particular, the (positive) quadratic covariation is unique (any (positive) quadratic covariation equals μ $\mathbb{P}$-a.e).*

Proof. By Theorems 5.18 and 7.3, $\mathbb{P}$-a.e. we have for all $0 \leq t$, $A \in \mathcal{A}$, $\|x^*\| = \|y^*\| = 1$,

$$\begin{aligned} |\alpha_{M,N}([0,t] \times A)(x^*, y^*)| &\leq \frac{1}{4}\left(|\alpha_{M+N}([0,t] \times A)(x^*, y^*)| + |\alpha_{M-N}([0,t] \times A)(x^*, y^*)|\right) \\ &\leq \frac{1}{4}\left(\langle\langle M+N \rangle\rangle ([0,t] \times A) + \langle\langle M-N \rangle\rangle ([0,t] \times A)\right) \\ &\leq \langle\langle M \rangle\rangle ([0,t] \times A) + \langle\langle N \rangle\rangle ([0,t] \times A). \end{aligned}$$

Since $\langle\langle M \rangle\rangle + \langle\langle N \rangle\rangle$ is a random measure on $\mathcal{B}([0,T]) \otimes \mathcal{B}(U)$, by the definition of supremum of measures for $0 \leq t$, $A \in \mathcal{A}$, we have $\mathbb{P}$-a.e.

$$\mu([0,t] \times A) \leq \langle\langle M \rangle\rangle ([0,t] \times A) + \langle\langle N \rangle\rangle ([0,t] \times A) < \infty.$$

By continuity of the bilinear form α, for all $x^*, y^* \in X^*$ with norm 1, for $0 \leq s < t \leq T$ and $A \in \mathcal{A}$, $\mathbb{P}$-a.e., we have

$$|\alpha_{x^*,y^*}((s,t] \times A)| \leq \mu((s,t] \times A).$$

By Lemma 3.11 this inequality extends to $\mathcal{B}([0,T]) \otimes \mathcal{B}(U)$. The rest of the proof follows the same arguments from that of Theorem 5.8. □

Definition 7.13. The unique (positive) quadratic covariation of M and N defined in Theorem 7.12 will be denoted by $\langle\langle M, N \rangle\rangle$.

Remark 7.14. As a direct consequence of Theorem 7.12 we have $\langle\langle M, N \rangle\rangle$ is a predictable random measure and $\mathbb{P}$-a.e.,

$$\langle\langle M, N \rangle\rangle ([0,t] \times A) = \sup_{\Pi \in \mathscr{R}([0,t]\times A)} \sum_{C \in \Pi} \sup_{m,n \geq 1} \alpha_{M,N}(x_m^*, x_n^*)(\omega)\,(C)\,, \quad \forall t \geq 0, A \in \mathcal{B}(U), \tag{7.11}$$

where $\mathscr{R}([0,t]\times A)$ is the family of partitions of $[0,t]\times A$ of the form

$$\Pi = \{(t_{i-1},t_i]\times A_j : 1\le i\le k, 1\le j\le m,\ k,m\in\mathbb{N}\}, \tag{7.12}$$

where $0=t_0<t_1<\cdots<t_k=t$ are rational (with the possible exception of t), the sets $A_1,\dots,A_m$ form a partition of $A\in\mathcal{B}(U)$, and $(x_n^*)_{n\ge1}$ is a dense subset of the unit sphere in X^*.

We are ready to introduce the *quadratic covariation operator measure.* We define an $\mathcal{L}(X^*,X^{**})$-valued measure $\Gamma_{M,N}$ on $\mathcal{B}([0,T])\otimes\mathcal{B}(U)$ by

$$\langle \Gamma_{M,N}(C)x^*\,,\,y^*\rangle = \alpha_{M,N}(C)(x^*,y^*). \tag{7.13}$$

where the left hand side, $\langle\cdot\,,\,\cdot\rangle$ corresponds to the duality relation for the pair (X^*,X^{**}).

Theorem 7.15. *There exists a process $Q_{M,N}:\Omega\times[0,T]\times U\to\mathcal{L}(X^*,X^{**})$ such that $\mathbb{P}$-a.e. $\omega\in\Omega$*

$$\langle \Gamma_{M,N}(\omega)(C)x^*\,,\,y^*\rangle = \int_C \langle Q_{M,N}(\omega,r,u)x^*\,,\,y^*\rangle\ \langle\!\langle M,N\rangle\!\rangle(\omega)(dr,du) \tag{7.14}$$

for all $x^*,y^*\in X^*$, $C\in\mathcal{B}([0,T])\otimes\mathcal{B}(U)$. Moreover the following properties hold:

- *(i) For every $x^*,y^*\in X^*$, the mapping $(\omega,r,u)\mapsto\langle Q_{M,N}(\omega,r,u)x^*\,,\,y^*\rangle$ is predictable, that is, $\mathcal{P}_T\otimes\mathcal{B}(U)$-measurable.*
- *(ii) $\mathbb{P}$-a.e. $\omega\in\Omega$, $Q_{M,N}(\omega,\cdot,\cdot)$ is positive and symmetric, $\langle\!\langle M,N\rangle\!\rangle$-a.e.*
- *(iii) $\mathbb{P}$-a.e. $\omega\in\Omega$, $\|Q_{M,N}(\omega,\cdot,\cdot)\|_{\mathcal{L}(X^*,X^{**})}=1$, $\langle\!\langle M,N\rangle\!\rangle$-a.e.*

Proof. As shown in the proof of Theorem 7.12 for any given $0\le t$ and $A\in\mathcal{A}$, we have $\mathbb{P}$-a.e.

$$\langle\!\langle M,N\rangle\!\rangle([0,t]\times A)\le\langle\!\langle M\rangle\!\rangle([0,t]\times A)+\langle\!\langle N\rangle\!\rangle([0,t]\times A).$$

By applying Lemma 3.9 we have $\mathbb{P}$-a.e. for every $C\in\mathcal{B}([0,T])\otimes\mathcal{B}(U)$

$$\langle\!\langle M,N\rangle\!\rangle(C)\le\langle\!\langle M\rangle\!\rangle(C)+\langle\!\langle N\rangle\!\rangle(C).$$

Since $\langle\!\langle M\rangle\!\rangle$ and $\langle\!\langle N\rangle\!\rangle$ satisfy (5.16), then $\langle\!\langle M,N\rangle\!\rangle$ satisfies it as well. Now, it is enough to apply similar arguments as in the proof of Theorem 5.23. ☐

Example 7.16. Let m be a real-valued orthogonal martingale-valued measure. Let H be a separable Hilbert space and let f,g be two H-valued square integrable processes with respect to m, i.e. $f,g:\Omega\times\mathbb{R}_+\to H$ are predictable and

$$\mathbb{E}\int_0^T\int_U \|f(\omega,s)\|_H^2\ \nu^m(ds,du)<\infty,\quad \mathbb{E}\int_0^T\int_U \|g(\omega,s)\|_H^2\ \nu^m(ds,du)<\infty,$$

where ν^m is the intensity measure of m.

We define two H-valued orthogonal martingale-valued measures M and N by

$$M(t,A)(\omega) = \int_0^T \int_A f(\omega,s)\, m(ds,du),$$
$$N(t,A)(\omega) = \int_0^T \int_A g(\omega,s)\, m(ds,du).$$

Observe that

$$\langle M(A)(h)\rangle_t(\omega) = \int_0^t \int_A (f(\omega,s),h)_H^2\, \nu^m(ds,du) = \nu_h^M([0,t]\times A).$$

Then by Lemma 2.2 we have

$$\langle\langle M\rangle\rangle((s,t]\times A) = \int_0^t \int_A \|f(\omega,s)\|_H^2\, \nu^m(ds,du).$$

Likewise, we have

$$\langle\langle N\rangle\rangle((s,t]\times A) = \int_0^t \int_A \|g(\omega,s)\|_H^2\, \nu^m(ds,du).$$

Observe that $\langle\langle M\rangle\rangle$ and $\langle\langle N\rangle\rangle$ satisfy (5.16). In fact, $\mathbb{P}$-a.e.

$$\sup_{A\in\mathcal{A}} \langle\langle M\rangle\rangle([0,t]\times A) \leq \int_0^t \int_U \|f(\omega,s)\|_H^2\, \nu^m(ds,du) < \infty,$$

and similarly for N.

Moreover, M and N are mutually orthogonal because

$$\langle M(A),N(B)\rangle_t = \int_0^t \int_{A\cap B} (f(\omega,s),g(\omega,s))_H\, \nu^m(ds,du),$$

which vanishes if $A\cap B=\emptyset$.

Now, observe that

$$(M+N)([0,t]\times A) = \int_0^t \int_A (f(\omega,s)+g(\omega,s))\, m(ds,du),$$

and for $h_1,h_2\in H$,

$$\begin{aligned}\langle (M+N)(A)(h_1),&(M+N)(A)(h_2)\rangle_t \\ &= \int_0^t \int_A (f(\omega,s)+g(\omega,s),h_1)_H (f(\omega,s)+g(\omega,s),h_2)_H\, \nu^m(ds,du).\end{aligned}$$

Therefore, for every $C\in\mathcal{B}([0,T])\otimes\mathcal{B}(U)$, $h_1,h_2\in H$,

$$\alpha_{M+N}(C)(h_1,h_2) = \int_C (f(\omega,s)+g(\omega,s),h_1)_H (f(\omega,s)+g(\omega,s),h_2)_H\, \nu^m(ds,du).$$

Likewise, we have

$$(M-N)([0,t]\times A)=\int_0^t\int_A (f(\omega,s)-g(\omega,s))\, m(ds,du),$$

and for for every $C\in\mathcal{B}([0,T])\otimes\mathcal{B}(U)$, $h_1,h_2\in H$,

$$\alpha_{M-N}(C)(h_1,h_2)=\int_C (f(\omega,s)-g(\omega,s),h_1)_H(f(\omega,s)-g(\omega,s),h_2)_H\, \nu^m(ds,du).$$

Hence, we have

$$\alpha_{M,N}(C)(h_1,h_2)=\int_C H(\omega,s,h_1,h_2)\, \nu^m(ds,du),$$

where $H(\omega,s,h_1,h_2)$ is equal to

$$\begin{aligned}&\tfrac{1}{4}\left[((f+g)(\omega,s),h_1)_H(f+g)(\omega,s),h_2)_H-(f-g)(\omega,s),h_1)_H(f-g)(\omega,s),h_2)_H\right]\\ =&\frac{1}{2}((f(\omega,s),h_1)_H(g(\omega,s),h_2)_H+(g(\omega,s),h_1)_H(f(\omega,s),h_2)_H).\end{aligned}$$

We know from Theorem 7.12 that the (positive) quadratic covariation of M and N exists. To compute it, observe that by Theorem 7.12, Theorem 4.9 in [5], and finally by Lemma 2.2, we have for $C\in\mathcal{B}([0,T])\otimes\mathcal{B}(U)$,

$$\begin{aligned}\langle\langle M,N\rangle\rangle(C)&=\sup_{m,n}|\alpha_{M,N}(\cdot)(x_m^*,x_n^*)|\\ &=\int_C \sup_{\|h_1\|=\|h_2\|=1} |H(\omega,s,h_1,h_2)|\ \nu^m(ds,du),\end{aligned}$$

where $H(\omega,s,h_1,h_2)$ is given above. It is easy to find an upper bound for the general case, but for particular functions f and g it is possible to give an explicit formula.

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Conflict of interest The authors have no conflicts of interest to declare that are relevant to the content of this article.

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